Quasigroups arisen by right nuclear extension

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Abstract. The aim of this paper is to prove that a quasigroup Q with right unit is isomorphic to an f-extension of a right nuclear normal subgroup G by the factor quasigroup Q/G if and only if there exists a normalized left transversal $\Sigma \subset Q$ to G in Q such that the right translations by elements of Σ commute with all right translations by elements of the subgroup G. Moreover, a loop Q is isomorphic to an f-extension of a right nuclear normal subgroup G by a loop if and only if G is middle-nuclear, and there exists a normalized left transversal to G in Q contained in the commutant of G.

Keywords: extension of quasigroups, right nucleus, quasigroup with right unit, transversal

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1. Introduction

A loop extension is called (right) nuclear, if the kernel of the corresponding homomorphism is contained in the (right) nucleus of the extension. In our previous paper [2] we made a systematic study of right nuclei of quasigroups obtained by an extension process in the category of quasigroups with right unit. The investigated extensions of quasigroups are defined by a slight modification of non-associative Schreier-type extensions of groups or loops (cf. [1]). These extensions will be determined by a triple (K, G, f), where K is a quasigroup, G is a loop and $f : K \times K \to G$ is a function, called the factor system of the extension. The main result of this paper gives a characterization of quasigroups which are isomorphic to an f-extension of a right nuclear normal subgroup by the factor quasigroup. They are precisely the quasigroups Q with a right nuclear normal subgroup G such that there exists a normalized left transversal $\Sigma \subset Q$ to G in Q such that the right translations by elements of Σ commute with all right translations by elements of the subgroup G. As an application we prove that a loop Q is isomorphic to an f-extension of a right nuclear normal subgroup G by the loop $K = Q/G$ if and only if G is also a middle-nuclear subgroup, and there exists a normalized left transversal Σ to G in Q contained in the commutant $C_Q(G)$ of G.

2. Preliminaries

A *quasigroup* Q is a set with a binary operation $(x, y) \mapsto x \cdot y$ such that the equations $a \cdot y = b$ and $x \cdot a = b$ are uniquely solvable in Q. The solutions are denoted by $y = a\backslash b$ and $x = b/a$. The element e_r is called the *right unit* of the quasigroup Q if $x \cdot e_r = x$ for all $x \in Q$. A *loop* is a quasigroup with unit element.

The *left, right* respectively *middle nucleus* of a quasigroup Q are the subgroups of Q defined by

$$\begin{aligned} N_l(Q) &= \{u;\ (u \cdot x) \cdot y = u \cdot (x \cdot y),\ x, y \in Q\}, \\ N_r(Q) &= \{u;\ (x \cdot y) \cdot u = x \cdot (y \cdot u),\ x, y \in Q\}, \\ N_m(Q) &= \{u;\ (x \cdot u) \cdot y = x \cdot (u \cdot y),\ x, y \in Q\}. \end{aligned}$$

The intersection $N(Q) = N_l(Q) \cap N_r(Q) \cap N_m(Q)$ is the *nucleus* of Q. A subgroup $G \subset Q$ of the quasigroup Q is called (*left, right, respectively middle*) *nuclear* if it is contained in the (left, right, respectively middle) nucleus of Q. If the right nucleus $N_r(Q)$ of a quasigroup Q is non-empty and e is the unit of the group $N_r(Q)$, then $xe \cdot n = x \cdot en = xn$ for any $x \in Q$, $n \in N_r(Q)$, hence e is the right unit of Q.

The *commutant* $C_Q(G)$ of a subgroup G in Q is the subset consisting of all elements $c \in Q$ such that $c \cdot x = x \cdot c$ for all $x \in G$. The *centralizer* $Z_Q(G)$ of the subgroup in Q consists of elements $z \in N(Q)$ such that $zx = xz$, for all $x \in G$. The *center* $Z(Q)$ of Q is the centralizer $Z_Q(Q)$ of Q in Q.

For any $x \in Q$ the maps $\lambda_x : y \mapsto x \cdot y$ and $\rho_x : y \mapsto y \cdot x$ are the *left* and the *right translation*s, respectively.

A subloop N of a quasigroup Q with right unit e_r is a *normal subloop* if there exists a homomorphism $\phi : Q \to Q'$ of Q onto the quasigroup Q' with right unit e'_r such that $\phi^{-1}(e'_r) = N$. In this case e_r is the unit element of N and for any $q \in Q$ one has $qN = \phi^{-1}(q')$, where $\phi(q) = q'$. Hence the map $qN \mapsto \phi(q) : Q/N \to Q'$ is bijective.

The set of left cosets $\{qN \in Q/N; q \in Q\}$ equipped with the quasigroup structure isomorphic to Q' is called the *factor quasigroup* of Q by the normal subloop N.

A subset $\Sigma \subset Q$ of a quasigroup Q with right unit e_r is said to be a *left transversal* to a normal subloop N in Q if it contains exactly one element from each coset of qN, $q \in Q$. If Σ contains the right unit e_r then we say that Σ is a *normalized left transversal* (cf. [3, Chapter 2]).

Let L be a loop, K a quasigroup and let f be a function $f : K \times K \to L$. The set $K \times L = \{(a, \alpha), a \in K, \alpha \in L\}$ with the operation

$$(a, \alpha) \cdot (b, \beta) := (ab, f(a, b) \cdot \alpha\beta), \tag{1}$$

is a quasigroup Q_f called the *f-extension* of the loop L by the quasigroup K. The function $f : K \times K \to L$ is the *factor system of the extension Q_f* and the map $\pi : Q_f \longrightarrow K : (a, \alpha) \mapsto a$ is the *related homomorphism of the extension Q_f*.

Assume that the right nucleus $N_r(Q_f)$ of an f-extension Q_f is a non-empty subgroup of Q_f. Then its unit $E_r \in N_r(Q_f)$ is the right unit of Q_f and its homomorphic image $e_r = \pi(E_r) \in K$ is the right unit of K. The quasigroup

Q_f is called a *right nuclear f-extension* if $\{e_r\} \times G = \{(e_r, g);\ g \in G\}$ is a right nuclear subgroup of Q_f. In this case Q_f is an f-extension of a group by a quasigroup with right unit.

In the following we focus our attention on right nuclear f-extensions of groups by quasigroups with right unit element (cf. [2, Theorem 11]).

3. Characterization

Let Q be a quasigroup with right unit and let G be a right nuclear normal subgroup of Q.

Lemma 1. *A quasigroup Q is isomorphic to an f-extension of a right nuclear normal subgroup G by the factor quasigroup Q/G with right unit if and only if Q is isomorphic to an f-extension Q_f of G by a quasigroup K with right unit e_r such that the factor system $f : K \times K \to G$ satisfies $f(x, e_r) = f(e_r, e_r) = \epsilon$, where ϵ is the unit of G.*

Proof: According to Theorem 11 in [2] an f-extension Q_f of a group G by a quasigroup K is right nuclear if and only if the factor system satisfies $f(x, e_r) = f(e_r, e_r) \in Z(G)$ for all $x \in K$. In this case for the f^*-extension Q_{f^*} of G by K defined by the factor system $f^*(x, y) = f(x, y) f(e_r, e_r)^{-1}$ the map $(x, \xi) \mapsto (x, f(e_r, e_r)\xi) : Q_f \to Q_{f^*}$ is an isomorphism. □

Lemma 2. *Let G be a group with unit ϵ, K a quasigroup with right unit e_r and let Q_f be an f-extension of G by K with factor system $f : K \times K \to G$ satisfying $f(x, e_r) = f(e_r, e_r) = \epsilon$. The subset $\Sigma = \{(x, \epsilon);\ x \in K\} \subset K \times G$ is a normalized left transversal to the normal subgroup $\bar{G} = \{(e_r, \xi);\ \xi \in G\} \subset K \times G$ of Q_f. The factor system satisfies*

$$(e_r, f(x, y)) = \sigma(xy) \backslash (\sigma(x)\sigma(y)) , \tag{2}$$

where σ is the map $x \mapsto (x, \epsilon) : K \to K \times G$. The right translation by any element of Σ commutes with all right translations by elements of $\bar{G}$, i.e.

$$\rho_{t\eta} = \rho_t \rho_\eta = \rho_\eta \rho_t \quad \text{for all} \quad t \in \Sigma,\ \eta \in G. \tag{3}$$

Proof: Clearly, $(x, \xi) = (x, \epsilon)(e_r, \xi)$ for any element $(x, \xi) \in K \times G$. Hence the subset $\Sigma = \{(x, \epsilon);\ x \in K\} \subset K \times G$ is a normalized left transversal to the subgroup $\bar{G}$. We have

$$(e_r, f(x, y)) = (xy, \epsilon) \backslash ((x, \epsilon)(y, \epsilon)) = \sigma(xy) \backslash (\sigma(x)\sigma(y)) ,$$

which is the equation (2). For any $(x, \xi) \in K \times G$ the right translation $\rho_{(y,\eta)}$ yields

$$\rho_{(y,\eta)}(x, \xi) = (xy, f(x, y)\xi\eta) = \rho_{(y,\epsilon)}\rho_{(e_r,\eta)}(x, \xi) = \rho_{(e_r,\eta)}\rho_{(y,\epsilon)}(x, \xi)$$

giving the commutation relations (3). □

Theorem 3. *If a quasigroup Q with right unit is isomorphic to an f-extension Q_f of a right nuclear normal subgroup G by the factor quasigroup Q/G then there exists a normalized left transversal Σ to G in Q satisfying the commutation relations* (3).

Conversely, if Σ is a normalized left transversal to the right nuclear normal subgroup G of Q satisfying the commutation relations (3) *then Q is isomorphic to the f-extension Q_f on $Q/G \times G$ determined by the factor system*

$$f(pG, qG) = \sigma(pqG) \backslash (\sigma(pG)\sigma(qG)), \quad pG, qG \in Q/G, \tag{4}$$

where $\sigma : Q/G \to Q$ is the map determined by $\sigma(qG) \in qG \cap \Sigma$ for any $q \in G$.

Proof: The first assertion follows from the previous lemma.

Now, we assume that Σ is a normalized left transversal to the right nuclear normal subgroup G of the quasigroup Q satisfying the commutation relations (3) and consider the f-extension Q_f on $Q/G \times G$ given by the factor system

$$f(pG, qG) = \sigma(pqG) \backslash (\sigma(pG)\sigma(qG)), \quad pG, qG \in Q/G.$$

Since Σ is normalized we have $f(pG, G) = \epsilon$ for any $p \in G$. We show that the bijection $\phi : Q \to Q_f$ given by $q \mapsto (qG, \sigma(qG)\backslash q)$ is an isomorphism. The elements $\sigma(pG)\backslash p$ and $\sigma(qG)\backslash q$ belong to the right nuclear subgroup G of Q, hence

$$\begin{aligned} pq &= (\sigma(pG) \cdot \sigma(pG)\backslash p) \cdot (\sigma(qG) \cdot \sigma(qG)\backslash q) \\ &= [(\sigma(pG) \cdot \sigma(pG)\backslash p) \cdot \sigma(qG)] \cdot \sigma(qG)\backslash q. \end{aligned}$$

It follows from the relations (3) and from the right nuclear property of G that

$$(\sigma(pG) \cdot \sigma(pG)\backslash p) \cdot \sigma(qG) = \sigma(pG) \cdot (\sigma(qG) \cdot \sigma(pG)\backslash p).$$

Once more using the right nuclear property we get

$$pq = \sigma(pG)\sigma(qG) \cdot (\sigma(pG)\backslash p \cdot \sigma(qG)\backslash q).$$

Hence

$$\phi(pq) = (pqG, \sigma(pqG)\backslash [\sigma(pG)\sigma(qG) \cdot (\sigma(pG)\backslash p \cdot \sigma(qG)\backslash q)]).$$

We have

$$\phi(p)\phi(q) = (pqG, f(pG, qG) \cdot [\sigma(pG)\backslash p \cdot \sigma(qG)\backslash q]),$$

where $f(pG, qG)$ is defined by (4). Hence, using the right nuclear property of G we get $\phi(pq) = \phi(p)\phi(q)$ for any $p, q \in Q$, which proves the assertion. □

For loops the previous theorem yields the following:

Theorem 4. *A loop Q is isomorphic to an f-extension Q_f of a right nuclear normal subgroup G by the factor loop Q/G if and only if*

(a) *G is a middle-nuclear subgroup,*
(b) *there exists a normalized left transversal Σ to G in Q contained in the commutant $C_Q(G)$ of G.*

In this case Q is isomorphic to the f-extension Q_f on $Q/G \times G$ determined by the factor system (4).

PROOF: Let Σ be a normalized left transversal to G in the loop Q. According to Theorem 3 the assertion is true if and only if the commutation relations (3) are satisfied:

$$x \cdot t\eta = x\eta \cdot t = xt \cdot \eta \quad \text{for all} \quad x \in Q,\ t \in \Sigma,\ \eta \in G.$$

Putting $x = e$, where e is the unit of Q, we obtain that Σ is contained in the commutant $C_Q(G)$ of the subgroup G in Q. Since G is a right nuclear subgroup we have $x \cdot t\eta = xt \cdot \eta$ for any $x \in Q$, $t \in \Sigma$, $\eta \in G$. Now, multiplying the identity $x \cdot t\eta = x\eta \cdot t$ by $\xi \in G$ we get

$$x(\eta \cdot t\xi) = x(\eta t \cdot \xi) = x(t\eta \cdot \xi) = (x \cdot t\eta)\xi = (x\eta \cdot t)\xi = x\eta \cdot t\xi.$$

Denoting $y = t\xi$ we obtain the identity

$$x(\eta \cdot y) = x\eta \cdot y.$$

Hence G is a middle-nuclear subgroup and the properties (a) and (b) are proved. Conversely, the previous arguments yield that the conditions (a) and (b) are equivalent to the commutation relations (3). □

It is well known that a group Q is isomorphic to a central extension of an abelian normal subgroup G, (i.e. G is contained in the center $Z(Q)$,) if and only if Q is isomorphic to an f-extension of G. The following assertion gives a direct generalization of this assertion to groups Q with non-necessarily abelian normal subgroup G:

Corollary 5. *A group Q is isomorphic to an f-extension Q_f of a normal subgroup G by the group $K = Q/G$ if and only if there exists a normalized left transversal Σ to G in Q contained in the centralizer $Z_Q(G)$ of the group G in Q.*

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