Pseudo-homotopies of the pseudo-arc

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Abstract. Let X be a continuum. Two maps $g, h : X \to X$ are said to be pseudo-homotopic provided that there exist a continuum C, points $s, t \in C$ and a continuous function $H : X \times C \to X$ such that for each $x \in X$, $H(x, s) = g(x)$ and $H(x, t) = h(x)$. In this paper we prove that if P is the pseudo-arc, g is one-to-one and h is pseudo-homotopic to g, then $g = h$. This theorem generalizes previous results by W. Lewis and M. Sobolewski.

Keywords: pseudo-arc, pseudo-contractible, pseudo-homotopy

Classification: Primary 54F15; Secondary 54B10, 54F50

1. Introduction

A *continuum* is a nondegenerate compact connected metric space. The letter P will denote the pseudo-arc. We will use the definition of the pseudo-arc as it is given in [7, 1.7]. A *map* is a continuous function. Two maps $h, g : P \to P$ are *pseudo-homotopic* provided that there exist a continuum C, points $s_0, t_0 \in C$ and a map $H : P \times C \to P$ such that $H(p, s_0) = g(p)$ and $H(p, t_0) = h(p)$ for each $p \in P$. In this case, we say that H is a *pseudo-homotopy* between g and h. The continuum X is *pseudo-contractible*, provided that the identity in X is pseudo-homotopic to a constant map. An ε*-map* between continua is a map $f : X \to Y$ such that diameter$(f^{-1}(y)) < \varepsilon$ for each $y \in f(X)$. A continuum X is *chainable* provided that for each $\varepsilon > 0$, there exists an ε-map from X into $[0, 1]$. Another way to define a chainable continuum [9, Theorem 12.11] is the following: a *chain* in a continuum X is a nonempty, finite, indexed collection $\mathcal{C} = \{U_1, \ldots, U_n\}$ of open subsets U_i of X such that $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leq 1$. The continuum X is *chainable* provided that for each $\varepsilon > 0$ there exists a chain $\mathcal{C} = \{U_1, \ldots, U_n\}$ in X such that $X = U_1 \cup \ldots \cup U_n$ and diameter$(U_i) < \varepsilon$ for each $i \in \{1, \ldots, n\}$.

The concepts of pseudo-homotopy between maps of a continuum and of a pseudo-contractible continuum were introduced by W. Kuperberg [5] and the first example of a pseudo-contractible continuum which is not contractible was also given by him. This example appears in page 2983 of [10].

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Answering a question by W. Kuperberg, in 2007 [10], M. Sobolewski proved that the pseudo-arc is not pseudo-contractible. In fact, he proved that the only pseudo-contractible chainable continuum is the arc.

There are only two known types of pseudo-homotopies for maps into the pseudo-arc, namely those pseudo-homotopies $H : P\times C \to P$ satisfying $H(P\times\{c\})$ is degenerate for each $c \in C$ or those for which there exists a map $f : P \to P$ such that $H(x, c) = f(x)$ for each $(x, c) \in X \times C$. So the following problem arises naturally.

Problem 1. *Do there exist pseudo-homotopies on the pseudo-arc different from the ones described in the paragraph above?*

In [6], W. Lewis proved that if g is a homeomorphism from the pseudo-arc onto itself and h is pseudo-homotopic to g, then $h = g$. From here, he deduced that in the space of homeomorphisms $\mathcal{H}(P)$ of the pseudo-arc there are not nondegenerate continua. It is still an open problem to determine if $\mathcal{H}(P)$ is totally disconnected [8, Question 21].

In this paper we use the technique developed by Sobolewski in [10] to prove that if $g : P \to P$ is one-to-one and h is pseudo-homotopic to g, then $g = h$.

2. Results

Given a continuum X, let $C(X)$ be the hyperspace of subcontinua of X endowed with the Hausdorff metric [2, Definition 2.1]. Given subcontinua A and B of a continuum X such that $A \subsetneq B$, an *order arc* from A to B is a map $\alpha : [0, 1] \to C(X)$ such that $\alpha(0) = A$, $\alpha(1) = B$ and, if $s < t$, then $\alpha(s) \subsetneq \alpha(t)$. The existence of order arcs is proved in [2, Theorem 14.6].

Lemma 2. *Let $g_1, h_1 : P \to P$ be pseudo-homotopic maps such that g_1 is not constant and $g_1 \neq h_1$. Then there exist a pseudo-arc P_1 and pseudo-homotopic maps $h, g : P_1 \to P$ such that $\operatorname{Im} g \cap \operatorname{Im} h = \emptyset$, g is not constant and, if g_1 is one-to-one, then g is one-to-one.*

Proof: Let $H : P \times C \to P$ be a pseudo-homotopy between g_1 and h_1 and let $s_0, t_0 \in C$ be such that $H(p, s_0) = g_1(p)$ and $H(p, t_0) = h_1(p)$ for each $p \in P$. Let $p_0 \in P$ be such that $g_1(p_0) \neq h_1(p_0)$. Let D (resp., E) be the component of $g_1^{-1}(g_1(p_0))$ (resp., $h_1^{-1}(h_1(p_0)))$ containing p_0. Then $D \subset E$ or $E \subset D$. Let $D_1 = D\cap E$. Since g_1 is not constant, D_1 is a proper subcontinuum of P. Let $\alpha : [0, 1] \to C(P)$ be an order arc from D_1 to P. Since $g_1(\alpha(0)) = g_1(D_1) = \{g_1(p_0)\}$ and $h_1(\alpha(0)) = h_1(D_1) = \{h_1(p_0)\}$, we have that there exists $t > 0$ such that $g_1(\alpha(t))\cap h_1(\alpha(t)) = \emptyset$. Let $P_1 = \alpha(t)$. Then P_1 is homeomorphic to P and either $P_1 \not\subseteq g_1^{-1}(g_1(p_0))$ or $P_1 \not\subseteq h_1^{-1}(h_1(p_0))$. This implies that $g_1|P_1$ or $h_1|P_1$ is not constant. We may assume that $g_1|P_1$ is not constant. In the case that g_1 is one-to-one, we have indeed that $g_1|P_1$ is not constant. Let $H_1 = H|(P_1\times C) : P_1\times C \to P$. Then H_1 is a pseudo-homotopy between $g_1|P_1$ and $h_1|P_1$. Define $g = g_1|P_1$ and $h = h_1|P_1$. □

We consider P constructed in the plane $\mathbb{R}^2$ [7, 1.7] by using a sequence of chains $\mathcal{C}_n$, where for each $n \in \mathbb{N}$, $\mathcal{C}_{n+1}$ refines $\mathcal{C}_n$, the mesh of $\mathcal{C}_n$ is less than $\frac{1}{n}$, $\mathcal{C}_{n+1}$ is crooked in $\mathcal{C}_n$, $\mathcal{C}_n = \{U_1^{(n)}, \ldots, U_{m_n}^{(n)}\}$, the sets $U_1^{(n)}, \ldots, U_{m_n}^{(n)}$ are open in P and they cover P, and $\mathrm{cl}_P(U_i^{(n)}) \cap \mathrm{cl}_P(U_j^{(n)}) \neq \emptyset$ if and only if $|i-j| \leq 1$. Given $n \in \mathbb{N}$ and $1 \leq i \leq j \leq m_n$, let $W(i,j,n) = U_i^{(n)} \cup \ldots \cup U_j^{(n)}$. In the case that $0 \leq j < i \leq m_n$, we define $W(i,j,n) = \emptyset$.

Theorem 3. *Let $g, h : P \to P$ be pseudo-homotopic maps such that g is one-to-one. Then $g = h$.*

Proof: Let d be a metric for P. Suppose to the contrary that $g \neq h$. We are going to get a contradiction. By Lemma 2, we may assume that $\mathrm{Im}\, g \cap \mathrm{Im}\, h = \emptyset$. Let $H : P \times C \to P$ be a pseudo-homotopy between g and h and let $s_0, t_0 \in C$ be such that $H(p, s_0) = g(p)$ and $H(p, t_0) = h(p)$ for each $p \in P$.

Let $B = \mathrm{Im}\, g$. Then B is a nondegenerate subcontinuum of P. Let $\varepsilon = \mathrm{diameter}(B)$. Let $N \in \mathbb{N}$ be such that $\frac{20}{N} < \varepsilon$ and N has the following properties: (a) if $d(g(p), g(q)) < \frac{3}{N}$, then $d(H(p,c), H(q,c)) < \frac{\varepsilon}{20}$ for each $c \in C$ (recall that g is one-to-one); and (b) $\frac{1}{N} < \min\{d(p,q) : p \in \mathrm{Im}\, g \text{ and } q \in \mathrm{Im}\, h\}$. Let $p_0, q_0 \in P$ be such that $\mathrm{diameter}(B) = d(g(p_0), g(q_0))$. Let $i_0, j_0 \in \{1, \ldots, m_N\}$ be such that $g(p_0) \in U_{i_0}^{(N)}$ and $g(q_0) \in U_{j_0}^{(N)}$. We may assume that $i_0 < j_0$. Notice that $19 < j_0 - i_0$. Let $i_1, j_1 \in \{1, \ldots, m_{N+1}\}$ be such that $g(p_0) \in U_{i_1}^{(N+1)}$ and $g(q_0) \in U_{j_1}^{(N+1)}$. Then there exist $u_0, v_0 \in \{1, \ldots, m_{N+1}\}$ such that $u_0, v_0 \in \{\min\{i_1, j_1\}, \ldots, \max\{i_1, j_1\}\}$, $U_{u_0}^{(N+1)} \cap U_{i_0}^{(N)} \neq \emptyset$, $U_{v_0}^{(N+1)} \cap U_{j_0}^{(N)} \neq \emptyset$ and $W(u_0, v_0, N+1) \subset W(i_0, j_0, N)$. We may assume that $u_0 < v_0$. Since $U_{i_1}^{(N+1)} \cap B \neq \emptyset$ and $U_{j_1}^{(N+1)} \cap B \neq \emptyset$, we have that $U_i^{(N+1)} \cap B \neq \emptyset$ for each $u_0 \leq i \leq v_0$.

By the choice of N, $\mathrm{Im}\, h$ does not intersect $W(u_0, v_0, N+1)$. Therefore, $\mathrm{Im}\, h \subset W(1, u_0 - 1, N+1)$ or $\mathrm{Im}\, h \subset W(v_0 + 1, m_{N+1}, N+1)$.

Since $\mathcal{C}_{N+1}$ is crooked in $\mathcal{C}_N$, there exist $k_0, l_0 \in \{1, \ldots, m_{N+1}\}$ such that $u_0 < k_0 < l_0 < v_0$, $U_{k_0}^{(N+1)} \cap U_{j_0-1}^{(N)} \neq \emptyset$ and $U_{l_0}^{(N+1)} \cap U_{i_0+1}^{(N)} \neq \emptyset$.

An appropriate use of Urysohn's lemma for metric continua allows us to construct a map $f_0 : P \to [-\frac{1}{2}, \frac{3}{2}]$ such that: $\mathrm{cl}_P(W(1, i_0 - 1, N)) \subset f_0^{-1}([-\frac{1}{2}, 0])$, $\mathrm{cl}_P(W(j_0 + 1, m_N, N)) \subset f_0^{-1}([1, \frac{3}{2}])$, $f_0^{-1}(0) = \mathrm{cl}_P(W(i_0, i_0 + 2, N))$, $f_0^{-1}(1) = \mathrm{cl}_P(W(j_0 - 2, j_0, N))$, $f_0^{-1}([0,1]) = \mathrm{cl}_P(W(i_0, j_0, N))$ and f_0 is a $\frac{3}{N}$-map. Again, by Urysohn's lemma, it is possible to construct a $\frac{3}{N}$-map $f : \mathrm{cl}_P\ (W(1, u_0, N+1)) \cup \mathrm{cl}_P\ (W(v_0, m_{N+1}, N+1)) \to [-\frac{1}{2}, 0] \cup [1, \frac{3}{2}]$ such that $\mathrm{cl}_P(W(1, u_0, N+1)) = f^{-1}([-\frac{1}{2}, 0])$, $\mathrm{cl}_P\ (W(v_0, m_{N+1}, N+1)) = f^{-1}([1, \frac{3}{2}])$, $\mathrm{cl}_P(U_{u_0}^{(N+1)}) = f^{-1}(0)$ and $\mathrm{cl}_P(U_{v_0}^{(N+1)}) = f^{-1}(1)$. We extend f to P, defining $f(p) = f_0(p)$ for each $p \in \mathrm{cl}_P$ $(W(u_0, v_0, N+1))$. Given $p \in \mathrm{cl}_P(U_{u_0}^{(N+1)}) \subset \mathrm{cl}_P(W(i_0, i_0 + 1, N))$, we have $f_0(p) = 0$, and given $p \in \mathrm{cl}_P(U_{v_0}^{(N+1)}) \subset \mathrm{cl}_P(W(j_0 - 1, j_0, N))$, we have $f_0(p) = 1$. This implies that f is a well-defined map from P into $[-\frac{1}{2}, \frac{3}{2}]$. It is easy to check that f is a $\frac{3}{N}$-map.

Let $\varphi : P \to [\frac{1}{2}, \frac{9}{2}]$ be given by

$$\varphi(p) = \begin{cases} f(p) + 1, & \text{if } p \in \mathrm{cl}_P(W(1, k_0, N+1)), \\ 3 - f(p), & \text{if } p \in \mathrm{cl}_P(W(k_0, l_0, N+1)), \\ 3 + f(p), & \text{if } p \in \mathrm{cl}_P(W(l_0, m_{N+1}, N+1)). \end{cases}$$

If $p \in \mathrm{cl}_P(W(1, k_0, N+1)) \subset \mathrm{cl}_P(W(1, u_0, N+1) \cup \mathrm{cl}_P(W(i_0, j_0, N)$, then $f(p) \in [-\frac{1}{2}, 1]$ and $\varphi(p) \in [\frac{1}{2}, 2]$. If $p \in \mathrm{cl}_P(U_{u_0}^{(N+1)}) \subset \mathrm{cl}_P(W(i_0, i_0+1, N))$, then $f(p) = 0$ and $\varphi(p) = 1$. If $p \in \mathrm{cl}_P(U_{k_0}^{(N+1)}) \subset \mathrm{cl}_P(W(j_0-2, j_0, N))$, then $f(p) = 1$ and $\varphi(p) = 2$. If $p \in \mathrm{cl}_P(U_{l_0}^{(N+1)}) \subset \mathrm{cl}_P(W(i_0, i_0+2, N))$, then $f(p) = 0$ and $\varphi(p) = 3$. If $p \in \mathrm{cl}_P(U_{v_0}^{(N+1)}) \subset \mathrm{cl}_P(W(j_0-1, j_0, N))$, then $f(p) = 1$ and $\varphi(p) = 4$. If $p \in \mathrm{cl}_P\ (W(k_0, l_0, N+1)) \subset \mathrm{cl}_P(W(i_0, j_0, N))$, then $f(p) \in [0, 1]$ and $\varphi(p) \in [2, 3]$. If $p \in \mathrm{cl}_P(W(l_0, m_{N+1}, N+1)) \subset \mathrm{cl}_P(W(i_0, j_0, N) \cup \mathrm{cl}_P(W(v_0, m_{N+1}, N+1)$, then $f(p) \in [0, \frac{3}{2}]$, so $\varphi(p) \in [3, \frac{9}{2}]$. These relations in particular imply that φ is well-defined and continuous.

Since $U_{u_0}^{(N+1)} \cap B \neq \emptyset$ and $\mathrm{cl}_P(U_{u_0}^{(N+1)}) \subset f^{-1}(0)$, we have that $0 \in f(B)$ and $1 \in \varphi(B)$, similarly, $4 \in \varphi(B)$. Thus, $\varphi(g(P))$ is a closed interval containing $[1, 4]$. Consider the map $\eta = (\varphi \times \varphi) \circ (g \times g) : P \times P \to [\frac{1}{2}, \frac{9}{2}]^2$ and let $D = \mathrm{Im}\, \eta = \varphi(g(P)) \times \varphi(g(P))$. Then D is a 2-cell containing $[1, 4]^2$. By [3] and [4, Proposition 1.2], η is a *universal* map and hence essential. Recall that a map between continua $\gamma : X \to Y$ is universal provided that for each map $\lambda : X \to Y$, there exists a point $x \in X$ such that $\gamma(x) = \lambda(x)$. Moreover, a map $\gamma : X \to D$, where D is a 2-cell is *essential* provided that each map $\lambda : X \to D$ such that $\gamma(x) = \lambda(x)$ for each $x \in \gamma^{-1}(\partial D)$ is surjective.

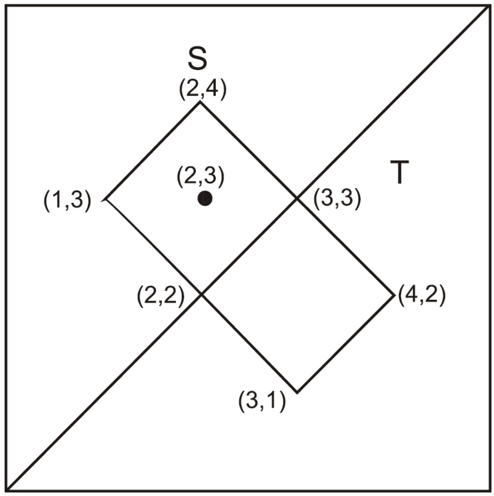
Let $\psi : [0, 5] \to [-1, 2]$ be given by

$$\psi(x) = \begin{cases} x - 1, & \text{if } x \in [0, 2], \\ 3 - x, & \text{if } x \in [2, 3], \\ x - 3, & \text{if } x \in [3, 5]. \end{cases}$$

Clearly, ψ is a continuous function such that for each $p \in P$, $\psi(\varphi(p)) = f(p)$. Let $T = \{(x, y) \in [0, 5]^2 : \psi(x) = \psi(y)\}$. Then T is the union of the diagonal of $[0, 5]^2$ and a rectangle (see Figure 1). Let S be the rhombus in the plane with vertices $(1, 3)$, $(2, 4)$, $(3, 3)$ and $(2, 2)$ and let $r : [0, 5]^2 \setminus \{(2, 3)\} \to S$ be the radial retraction with the point $(2, 3)$ as the center. Let $A = \eta^{-1}(S)$. Since η is essential, by [3, p. 225], $\eta|A : A \to S$ is not homotopic to a constant map.

Given $(p, q) \in A$, we have $\psi(\varphi(g(p))) = \psi(\varphi(g(q)))$, so $f(g(p)) = f(g(q))$. Hence, $d(g(p), g(q)) < \frac{3}{N}$. Let $c \in C$. By the choice of N, $d(H(p, c), H(q, c)) < \frac{\varepsilon}{20}$. We claim that $(\varphi(H(p, c)), \varphi(H(q, c))) \neq (2, 3)$. Suppose to the contrary that $\varphi(H(p, c)) = 2$ and $\varphi(H(q, c)) = 3$. Considering the options in the definition of φ, we obtain that $f(H(p, c)) = 1$ and $f(H(q, c)) = 0$. Thus, $H(p, c) \in \mathrm{cl}_P(W(j_0-2, j_0, N))$ and $H(q, c) \in \mathrm{cl}_P(W(i_0, i_0+2, N))$. This implies that

$\max\{d(H(p,c),g(q_0)),d(H(q,c),g(p_0))\} < \frac{3}{N}$. Hence, $d(g(p_0),g(q_0)) < \frac{\varepsilon}{20} + \frac{6}{N} < \frac{\varepsilon}{20} + \frac{3\varepsilon}{10} < \varepsilon$, a contradiction. We have shown that for every $(p,q) \in A$ and $c \in C$, $(\varphi(H(p,c)),\varphi(H(q,c))) \neq (2,3)$.



Notice that $S \subset T$. By the paragraph above, the map $\sigma : A \times C \to S$ given by $\sigma((p,q),c) = r(\varphi(H(p,c)),\varphi(H(q,c)))$ is well-defined. Since for each $(p,q) \in A$, $\sigma((p,q),s_0) = r(\varphi(H(p,s_0)),\varphi(H(q,s_0))) = r(\varphi(g(p),\varphi(g(q)))$ and $\sigma((p,q),t_0) = r(\varphi(h(p),\varphi(h(q)))$, the maps $\sigma_0,\sigma_1 : A \to S$ given by $\sigma_0(p,q) = r(\varphi(g(p),\varphi(g(q)))$ and $\sigma_1(p,q) = r(\varphi(h(p),\varphi(h(q)))$ are pseudo-homotopic. Since S is an ANR, σ_0 and σ_1 are homotopic (see Claim 1 in [10]).

Notice that $\operatorname{Im} h \subset \varphi^{-1}([\frac{1}{2},1])$ or $\operatorname{Im} h \subset \varphi^{-1}([4,\frac{9}{2}])$. In the first case, for each $(p,q) \in A$, $(\varphi(h(p),\varphi(h(q))) \in [\frac{1}{2},1]^2$, so $\sigma_1(p,q)$ lies on the side of S that joins the points $(2,2)$ and $(1,3)$. This implies that σ_1 is homotopic to a constant map. The second case is similar. We conclude that σ_0 is homotopic to a constant map.

Given $(p,q) \in A$, $\eta(p,q) = (\varphi(g(p)),\varphi(g(q))) \in S$, so $\eta(p,q) = r(\eta(p,q)) = \sigma_0(p,q)$. Hence, $\sigma_0 = \eta|A$ is not homotopic to a constant map. This contradiction completes the proof of the theorem. □

3. Conclusions

Corollary 4. *Let $g,h : P \to P$ be pseudo-homotopic maps. Suppose that A is a nondegenerate subcontinuum of P such that $g|A : A \to P$ is one-to-one. Then $g|A = h|A$.*

Corollary 5. *Let $H : P \times C \to P$ be a pseudo-homotopy between the maps g and h. If $g \neq h$, then for each $c \in C$, $\bigcup\{A \in C(P) : A$ is nondegenerate and $H|A \times \{c\}$ is one-to-one$\}$ is not dense in P.*

Corollary 5 shows that if there is a pseudo-homotopy between two different non-constant maps, all the “levels” of the pseudo-homotopy must have a complicated behavior. On the other hand, a negative answer to Problem 1 would lead to answer other open problems on the pseudo-arc. Next we recall some of them.

Problem 6 ([1, Problem 6]). *Let $e : P_1 \times \ldots \times P_m \to P_1 \times \ldots \times P_m$ be an embedding of a finite product of pseudo-arcs into itself. Must e be a product of embeddings composed with a permutation of coordinates? Recently in* [1] *this problem has been solved for the product of two pseudo-arcs.*

Problem 7 ([8, Question 14]). *Does there exist a continuum X with the fixed point property such that $X \times P$ does not have the fixed point property?*

Problem 8 ([8, Question 20]). *Assume that $r : P \times P \to \Delta = \{(x, x) \in P \times P : x \in P\}$ is a continuous retraction. Must r be of the form $r(x, y) = (x, x)$ for all (x, y) or $r(x, y) = (y, y)$ for all (x, y)?*

Problem 9. *Does $E(P)$, the space of all continuous functions from the pseudo-arc into itself, contain any nondegenerate compact connected sets other than collections of constant maps?*

Notice that Theorem 3 implies that if $\mathcal{A}$ is a nondegenerate continuum contained in $E(P)$, then $\mathcal{A}$ does not contain a one-to-one element of $E(P)$. This extends the result in [6] that says that in the space of homeomorphisms $\mathcal{H}(P)$ of the pseudo-arc there are not nondegenerate continua. Notice also that Theorem 3 implies that P is not pseudo-contractible. Related to Problem 9, we can mention the following two important problems.

Problem 10 ([8, Question 22]). *Does $E(P)$, the space of all continuous functions from the pseudo-arc into itself, contain any nondegenerate connected sets other than collections of constant maps?*

Problem 11 ([8, Question 21]). *Is $\mathcal{H}(P)$, the topological group of all self-homeomorphisms of the pseudo-arc P, totally disconnected?*

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