Remarks on sequence-covering maps

Luong Quoc Tuyen

Abstract. In this paper, we prove that each sequence-covering and boundary-compact map on g-metrizable spaces is 1-sequence-covering. Then, we give some relationships between sequence-covering maps and 1-sequence-covering maps or weak-open maps, and give an affirmative answer to the problem posed by F.C. Lin and S. Lin in [9].

Keywords: g-metrizable space, weak base, sn-network, compact map, boundary-compact map, sequence-covering map, 1-sequence-covering map, weak-open map, closed map

Classification: 54C10, 54D65, 54E40, 54E99

1. Introduction

A study of images of topological spaces under certain sequence-covering maps is an important question in general topology ([1], [2], [8]–[11], [13], [16], [21], for example). In 2000, P. Yan, S. Lin and S.L. Jiang proved that each closed sequence-covering map on metric spaces is 1-sequence-covering ([21]). Furthermore, in 2001, S. Lin and P. Yan proved that each sequence-covering and compact map on metric spaces is 1-sequence-covering ([13]). After that, T.V. An and L.Q. Tuyen proved that each sequence-covering π and s-map on metric spaces is 1-sequence-covering ([1]). Recently, F.C. Lin and S. Lin proved that each sequence-covering and boundary-compact map on metric spaces is 1-sequence-covering ([8]). Also, the authors posed the following question in [9].

Question 1.1 ([9, Question 4.6]). *Let $f : X \longrightarrow Y$ be a sequence-covering and boundary-compact map. If X is g-metrizable, then is f an 1-sequence-covering map?*

In this paper, we prove that each sequence-covering and boundary-compact map on g-metrizable spaces is 1-sequence-covering. Then, we give some relationships between sequence-covering maps and 1-sequence-covering maps or weak-open maps, and give an affirmative answer to the problem posed by F.C. Lin and S. Lin in [9].

Throughout this paper, all spaces are assumed to be Hausdorff, all maps are continuous and onto, $\mathbb{N}$ denotes the set of all natural numbers, ω denotes $\mathbb{N} \cup \{0\}$, and convergent sequence includes its limit point. Let $f : X \to Y$ be a map and $\mathcal{P}$ be a collection of subsets of X, we denote $\bigcup \mathcal{P} = \bigcup\{P : P \in \mathcal{P}\}$, $\bigcap \mathcal{P} = \bigcap\{P : P \in \mathcal{P}\}$, $f(\mathcal{P}) = \{f(P) : P \in \mathcal{P}\}$.

Definition 1.2. Let X be a space, and $P \subset X$.

(1) A sequence $\{x_n\}$ in X is called *eventually* in P, if $\{x_n\}$ converges to x, and there exists $m \in \mathbb{N}$ such that $\{x\} \bigcup \{x_n : n \geq m\} \subset P$.
(2) P is called a *sequential neighborhood* of x in X [5], if whenever $\{x_n\}$ is a sequence converging to x in X, then $\{x_n\}$ is eventually in P.

Definition 1.3. Let $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ be a cover of a space X. Assume that $\mathcal{P}$ satisfies the following conditions (a) and (b) for every $x \in X$.

(a) $\mathcal{P}_x$ is a network at x.
(b) If $P_1, P_2 \in \mathcal{P}_x$, then there exists $P \in \mathcal{P}_x$ such that $P \subset P_1 \cap P_2$.

(1) $\mathcal{P}$ is a *weak base* of X [3], if for $G \subset X$, G is open in X and every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$; $\mathcal{P}_x$ is said to be a weak neighborhood base at x.
(2) $\mathcal{P}$ is an *sn-network* for X [10], if each element of $\mathcal{P}_x$ is a sequential neighborhood of x for all $x \in X$; $\mathcal{P}_x$ is said to be an sn-network at x.

Definition 1.4. Let X be a space. Then

(1) X is *gf-countable* [3] (resp., *snf-countable* [6]), if X has a weak base (resp., sn-network) $\mathcal{P} = \bigcup \{\mathcal{P}_x : x \in X\}$ such that each $\mathcal{P}_x$ is countable;
(2) X is *g-metrizable* [17], if X is regular and has a σ-locally finite weak base;
(3) X is *sequential* [5], if whenever A is a non-closed subset of X, then there is a sequence in A converging to a point not in A.

Remark 1.5. (1) Each g-metrizable space or gf-countable space is sequential.
(2) A space X is gf-countable if and only if it is sequential and snf-countable.

Definition 1.6. Let $f : X \longrightarrow Y$ be a map. Then

(1) f is a *compact* map [4], if each $f^{-1}(y)$ is compact in X;
(2) f is a *boundary-compact* map [4], if each $\partial f^{-1}(y)$ is compact in X;
(3) f is a *weak-open* map [18], if there exists a weak base $\mathcal{P} = \bigcup \{\mathcal{P}_y : y \in Y\}$ for Y, and for $y \in Y$, there exists $x_y \in f^{-1}(y)$ such that for each open neighborhood U of x_y, $P_y \subset f(U)$ for some $P_y \in \mathcal{P}_y$;
(4) f is an *1-sequence-covering* map [10], if for each $y \in Y$, there is $x_y \in f^{-1}(y)$ such that whenever $\{y_n\}$ is a sequence converging to y in Y, there is a sequence $\{x_n\}$ converging to x_y in X with $x_n \in f^{-1}(y_n)$ for every $n \in \mathbb{N}$;
(5) f is a *sequence-covering* map [16], if every convergent sequence of Y is the image of some convergent sequence of X;
(6) f is a *quotient* map [4], if whenever $f^{-1}(U)$ is open in X, then U is open in Y.

Remark 1.7. (1) Each compact map is a compact-boundary map.
(2) Each 1-sequence-covering map is a sequence-covering map.
(3) Each closed map is a quotient map.

Definition 1.8 ([7])**.** A function $g : \mathbb{N} \times X \longrightarrow \mathcal{P}(X)$ is a `CWC`-*map*, if it satisfies the following conditions:

(1) $x \in g(n,x)$ for all $x \in X$ and $n \in \mathbb{N}$;
(2) $g(n+1,x) \subset g(n,x)$ for all $n \in \mathbb{N}$;
(3) $\{g(n,x) : n \in \mathbb{N}\}$ is a weak neighborhood base at x for all $x \in X$.

2. Main results

Theorem 2.1. *Each sequence-covering and boundary-compact map on g-metrizable spaces is* 1*-sequence-covering.*

Proof: Let $f : X \longrightarrow Y$ be a sequence-covering and boundary-compact map and X be a g-metrizable space. Firstly, we prove that Y is snf-countable. In fact, since X is g-metrizable, it follows from Theorem 2.6 in [20] that there exists a `CWC`-map g on X satisfying that $y_n \to x$ whenever $\{x_n\}$, $\{y_n\}$ are two sequences in X such that $x_n \to x$ and $y_n \in g(n,x_n)$ for all $n \in \mathbb{N}$. For each $y \in Y$ and $n \in \mathbb{N}$, we put

$$P_{y,n} = f\Big(\bigcup\{g(n,x) : x \in \partial f^{-1}(y)\}\Big), \text{ and } \mathcal{P}_y = \{P_{y,n} : n \in \mathbb{N}\}.$$

Then each $\mathcal{P}_y$ is countable and $P_{y,n+1} \subset P_{y,n}$ for all $y \in Y$ and $n \in \mathbb{N}$. Furthermore, we have

(1) $\mathcal{P}_y$ is a network at y. Indeed, let $y \in U$ with U open in Y. Then there exists $n \in \mathbb{N}$ such that

$$\bigcup\{g(n,x) : x \in \partial f^{-1}(y)\} \subset f^{-1}(U).$$

If not, for each $n \in \mathbb{N}$, there exist $x_n \in \partial f^{-1}(y)$ and $z_n \in X$ such that $z_n \in g(n,x_n) - f^{-1}(U)$. Since X is g-metrizable, it follows that each compact subset of X is metrizable. Since $\{x_n\} \subset \partial f^{-1}(y)$ and f is a boundary-compact map, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x \in \partial f^{-1}(y)$. Now, for each $i \in \mathbb{N}$, we put

$$a_i = \begin{cases} x_{n_1} & \text{if } i \le n_1 \\ x_{n_{k+1}} & \text{if } n_k < i \le n_{k+1}; \end{cases}$$

$$b_i = \begin{cases} z_{n_1} & \text{if } i \le n_1 \\ z_{n_{k+1}} & \text{if } n_k < i \le n_{k+1}. \end{cases}$$

Then $a_i \to x$. Because $g(n+1,x) \subset g(n,x)$ for all $x \in X$ and $n \in \mathbb{N}$, it implies that $b_i \in g(i,a_i)$ for all $i \in \mathbb{N}$. By the property of g, it implies that $b_i \to x$. Thus, $z_{n_k} \to x$. This contradicts that $f^{-1}(U)$ is a neighborhood of x and $z_{n_k} \notin f^{-1}(U)$ for all $k \in \mathbb{N}$. Therefore, $P_{y,n} \subset U$, and $\mathcal{P}_y$ is a network at y.

(2) Let $P_{y,m}$, $P_{y,n} \in \mathcal{P}_y$. If we take $k = \max\{m,n\}$, then $P_{y,k} \subset P_{y,m} \cap P_{y,n}$.

(3) Each element of $\mathcal{P}_y$ is a sequential neighborhood of y. Let $P_{y,n} \in \mathcal{P}_y$ and $\{y_n\}$ be a sequence converging to y in Y. Since f is sequence-covering, $\{y_n\}$ is an image of some sequence converging to $x \in \partial f^{-1}(y)$. On the other hand, since $g(n,x)$ is a weak neighborhood of x, $\{y_n\}$ is eventually in $g(n,x)$. This implies that $\{y_n\}$ is eventually in $P_{y,n}$. Therefore, $P_{y,n}$ is a sequential neighborhood of y.

Therefore, $\bigcup\{\mathcal{P}_y : y \in Y\}$ is an sn-network for X, and Y is an snf-countable space.

Next, let $\mathcal{B} = \bigcup\{\mathcal{B}_x : x \in X\}$ be a σ-locally finite weak base for X. We prove that for each non-isolated point $y \in Y$, there exists $x_y \in \partial f^{-1}(y)$ such that for each $B \in \mathcal{B}_{x_y}$, there exists $P \in \mathcal{P}_y$ satisfying $P \subset f(B)$. Otherwise, there exists a non-isolated point $y \in Y$ so that for each $x \in \partial f^{-1}(y)$, there exists $B_x \in \mathcal{B}_x$ such that $P \not\subset f(B_x)$ for all $P \in \mathcal{P}_y$. Since $\mathcal{B}$ is a σ-locally finite weak base and $\partial f^{-1}(y)$ is compact, it follows that $\{B_x : x \in \partial f^{-1}(y)\}$ is countable. Assume that $\{B_x : x \in \partial f^{-1}(y)\} = \{B_m : m \in \mathbb{N}\}$. Hence, for each $m, n \in \mathbb{N}$, there exists $x_{n,m} \in P_{y,n} - f(B_m)$. For $n \geq m$, we denote $y_k = x_{n,m}$ with $k = m + n(n-1)/2$. Since $\mathcal{P}_y$ is a network at y and $P_{y,n+1} \subset P_{y,n}$ for all $n \in \mathbb{N}$, $\{y_k\}$ is a sequence converging to y in Y. On the other hand, because f is a sequence-covering map, $\{y_k\}$ is an image of some sequence $\{x_n\}$ converging to $x \in \partial f^{-1}(y)$ in X. Furthermore, since $B_x \in \{B_m : m \in \mathbb{N}\}$, there exists $m_0 \in \mathbb{N}$ such that $B_x = B_{m_0}$. Because B_{m_0} is a weak neighborhood of x, $\{x\} \bigcup\{x_k : k \geq k_0\} \subset B_{m_0}$ for some $k_0 \in \mathbb{N}$. Thus, $\{y\} \bigcup\{y_k : k \geq k_0\} \subset f(B_{m_0})$. But if we take $k \geq k_0$, then there exists $n \geq m_0$ such that $y_k = x_{n,m_0}$, and it implies that $x_{n,m_0} \in f(B_{m_0})$. This contradicts to $x_{n,m_0} \in P_{y,n} - f(B_{m_0})$.

We now prove that f is an 1-sequence-covering map. Suppose $y \in Y$. By the above proof there is $x_y \in \partial f^{-1}(y)$ such that whenever $B \in \mathcal{B}_{x_y}$, there exists $P \in \mathcal{P}_y$ satisfying $P \subset f(B)$. Let $\{y_n\}$ be a sequence in Y, which converges to y. Since $\mathcal{B}_{x_y}$ is a weak neighborhood base at x_y, we can choose a decreasing countable network $\{B_{y,n} : n \in \mathbb{N}\} \subset \mathcal{B}_{x_y}$ at x_y. We choose a sequence $\{z_n\}$ in X as follows.

Since $B_{y,n} \in \mathcal{B}_{x_y}$, by the above argument, there exists $P_{y,k_n} \in \mathcal{P}_y$ satisfying $P_{y,k_n} \subset f(B_{y,n})$ for all $n \in \mathbb{N}$. On the other hand, since each element of $\mathcal{P}_y$ is a sequential neighborhood of y, it follows that for each $n \in \mathbb{N}$, $f(B_{y,n})$ is a sequential neighborhood of y in Y. Hence, for each $n \in \mathbb{N}$, there exists $i_n \in \mathbb{N}$ such that $y_i \in f(B_{y,n})$ for every $i \geq i_n$. Assume that $1 < i_n < i_{n+1}$ for each $n \in \mathbb{N}$. Then for each $j \in \mathbb{N}$, we take

$$z_j = \begin{cases} z_j \in f^{-1}(y_j) & \text{if } \ j < i_1 \\ z_{j,n} \in f^{-1}(y_j) \cap B_{y,n} & \text{if } \ i_n \leq j < i_{n+1}. \end{cases}$$

If we put $S = \{z_j : j \geq 1\}$, then S converges to x_y in X, and $f(S) = \{y_n\}$. Therefore, f is 1-sequence-covering. □

Remark 2.2. By Theorem 2.1, we get an affirmative answer to Question 1.1.

Corollary 2.3. *Each sequence-covering quotient and boundary-compact map on g-metrizable spaces is weak-open.*

Proof: Let $f : X \longrightarrow Y$ be a sequence-covering quotient and boundary-compact map and X be a g-metrizable space. By Theorem 2.1, f is 1-sequence-covering. Since f is quotient and X is sequential, f is weak-open by Corollary 3.5 in [18]. □

By Theorem 2.1 and Remark 1.7(1), the following corollaries hold.

Corollary 2.4. *Each sequence-covering and compact map on g-metrizable spaces is* 1*-sequence-covering.*

Corollary 2.5 ([8, Theorem 2.1])**.** *Each sequence-covering and boundary-compact map on metric spaces is* 1*-sequence-covering.*

Corollary 2.6 ([9, Theorem 4.5])**.** *Each closed sequence-covering map on g-metrizable spaces is* 1*-sequence-covering.*

PROOF: Let $f : X \longrightarrow Y$ be a closed sequence-covering map and X be a g-metrizable space. By Lemma 3.1 in [15], Y is gf-countable. Furthermore, since Y is gf-countable and f is a closed map, it follows from Corollary 8 in [14] and Corollary 10 in [19] that Y contains no closed copy of S_ω. By Lemma 3.2 in [15], f is a boundary-compact map. Therefore, f is 1-sequence-covering by Theorem 2.1. □

By Corollary 2.6, we have the following corollary.

Corollary 2.7 ([11, Theorem 3.4.6])**.** *Each closed sequence-covering map on metric spaces is* 1*-sequence-covering.*

Corollary 2.8. *Each closed sequence-covering map on g-metrizable spaces is weak-open.*

PROOF: Let $f : X \longrightarrow Y$ be a closed sequence-covering map and X be a g-metrizable space. It follows from Corollary 2.6 that f is 1-sequence-covering. Since f is closed and X is sequential, f is weak-open by Corollary 3.5 in [18]. □

Acknowledgment. The author would like to express their thanks to referee for his/her helpful comments and valuable suggestions.

References

[1] An T.V., Tuyen L.Q., *Further properties of* 1*-sequence-covering maps*, Comment. Math. Univ. Carolin. **49** (2008), no. 3, 477–484.
[2] An T.V., Tuyen L.Q., *On π-images of separable metric spaces and a problem of Shou Lin*, Mat. Vesnik, (2011), to appear.
[3] Arhangel'skii A.V., *Mappings and spaces*, Russian Math. Surveys **21** (1966), no. 4, 115–162.
[4] Engelking R., *General Topology* (*revised and completed edition*), Heldermann Verlag, Berlin, 1989.
[5] Franklin S.P., *Spaces in which sequences suffice*, Fund. Math. **57** (1965), 107–115.
[6] Ge Y., *Characterizations of sn-metrizable spaces*, Publ. Inst. Math. (Beograd) (N.S) **74** (**88**) (2003), 121–128.
[7] Lee K.B., *On certain g-first countable spaces*, Pacific J. Math. **65** (1976), no. 1, 113–118.
[8] Lin F.C., Lin S., *On sequence-covering boundary compact maps of metric spaces*, Adv. Math. (China) **39** (2010), no. 1, 71–78.
[9] Lin F.C., Lin S., *Sequence-covering maps on generalized metric spaces*, arXiv: 1106.3806.
[10] Lin S., *On sequence-covering s-mappings*, Adv. Math. (China) **25** (1996), no. 6, 548–551.
[11] Lin S., *Point-Countable Covers and Sequence-Covering Mappings*, Chinese Science Press, Beijing, 2002.

[12] Lin S., Liu C., *On spaces with point-countable cs-networks*, Topology Appl. **74** (1996), 51–60.
[13] Lin S., Yan P., *Sequence-covering maps of metric spaces*, Topology Appl. **109** (2001), 301–314.
[14] Lin S., Tanaka Y., *Point-countable k-networks, closed maps, and related results*, Topology Appl. **59** (1994), 79–86.
[15] Liu C., *On weak bases*, Topology Appl. **150** (2005), 91–99.
[16] Siwiec F., *Sequence-covering and countably bi-quotient maps*, General Topology Appl. **1** (1971), 143–154.
[17] Siwiec F., *On defining a space by a weak base*, Pacific J. Math. **52** (1974), 233–245.
[18] Xia S., *Characterizations of certain g-first countable spaces*, Adv. Math. **29** (2000), 61–64.
[19] Yan P., Lin S., *Point-countable k-networks, cs^*-network and α_4-spaces*, Topology Proc. **24** (1999), 345–354.
[20] Yan P., Lin S., CWC-*mappings and metrization theorems*, Adv. Math. (China) **36** (2007), no. 2, 153–158.
[21] Yan P.F., Lin S., Jiang S.L., *Metrizability is preserved by closed sequence-covering maps*, Acta Math. Sinica **47** (2004), no. 1, 87–90.

Department of Mathematics, Da Nang University, Vietnam

E-mail: luongtuyench12@yahoo.com

(*Received* February 11, 2012)