Measures of noncompactness in locally convex spaces and fixed point theory for the sum of two operators on unbounded convex sets

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Abstract. In this paper we prove a collection of new fixed point theorems for operators of the form $T+S$ on an unbounded closed convex subset of a Hausdorff topological vector space (E,Γ). We also introduce the concept of demi-τ-compact operator and τ-semi-closed operator at the origin. Moreover, a series of new fixed point theorems of Krasnosel'skii type is proved for the sum $T+S$ of two operators, where T is τ-sequentially continuous and τ-compact while S is τ-sequentially continuous (and Φ_τ-condensing, Φ_τ-nonexpansive or nonlinear contraction or nonexpansive). The main condition in our results is formulated in terms of axiomatic τ-measures of noncompactness. Apart from that we show the applicability of some our results to the theory of integral equations in the Lebesgue space.

Keywords: τ-measure of noncompactness, τ-sequential continuity, Φ_τ-condensing operator, Φ_τ-nonexpansive operator, nonlinear contraction, fixed point theorem, demi-τ-compactness, operator τ-semi-closed at origin, Lebesgue space, integral equation

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1. Introduction

Let $(E,\|\cdot\|)$ be a Banach space and let Ω be a nonempty, bounded, closed and convex subset of E. A well known theorem of Krasnosel'skii states [22] that if T is a completely continuous operator on Ω and S is a contraction of Ω (i.e. $\|Sx-Sy\|\le k\|x-y\|$ for $0<k<1$) and, if

$$Tx+Sy\in\Omega \quad \text{for all} \quad x,y\in\Omega \tag{1.1}$$

then $T+S$ has a fixed point in Ω. Krasnosel'skii theorem has been extended to abstract forms, in particular for the weak topology [6], [7], [10], [20], [26], [28] for weakly sequentially continuous operators.

The aim of this paper is to present new generalized forms of the Krasnosel'skii fixed point theorem using τ-measures of noncompactness, where T is assumed to be τ-sequentially continuous and τ-compact operator while S is assumed to be

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τ-sequentially continuous (and Φ_τ-condensing, Φ_τ-nonexpansive, nonlinear contraction or nonexpansive), defined on a convex subset of Hausdorff topological vector spaces. Moreover, other assumptions imposed on operator S are also employed (demi-τ-compactness or τ-semi-closedness at the origin; cf. Definitions 3.1 and 3.3).

In our study the set Ω will be not necessarily bounded. Apart from this condition (1.1) will be replaced with one of the following weaker conditions

$$Tx + Sx \in \Omega \quad \text{for any} \quad x \in \Omega, \tag{1.2}$$

$$[y = Tx + Sy,\ x \in \Omega] \Longrightarrow y \in \Omega. \tag{1.3}$$

In the last section of the paper we show the applicability of our result (Corollary 3.2) to the theory of nonlinear integral equations in the Lebesgue space.

It is worthwhile mentioning that the results obtained in the paper generalize and encompass several ones obtained up to now by other authors (cf. [3], [5]–[7], [10], [13], [14], [22], [26] for instance).

2. Preliminaries

Throughout this paper we assume that (E,Γ) is a Hausdorff topological vector space (HTVS, in short) with zero element θ and τ is a weaker Hausdorff locally convex vector topology on E $(\tau \le \Gamma)$. If E is a normed space, the symbol $B_r(z)$ will denote the closed ball centered at z with radius r. To denote the convergence in (E,τ) we write $\xrightarrow{\tau}$ while the symbol $\to$ denotes the convergence in (E,Γ).

In our considerations we accept the following definition of the concept of a τ-measure of noncompactness (τ-MNC, in short).

Definition 2.1. Let C be a lattice with a least element denoted by 0. A function Φ_τ defined on the family $\mathfrak{M}_E$ of all nonempty and bounded subsets of (E,Γ) with values in C will be called a τ-MNC in E if it satisfies the following conditions:

(i) $\Phi_\tau(\overline{conv}^\tau(\Omega)) \le \Phi_\tau(\Omega)$ for each $\Omega \in \mathfrak{M}_E$, where the symbol $\overline{conv}^\tau(\Omega)$ denotes the closed convex hull of Ω in (E,τ);

(ii) $\Omega_1 \subset \Omega_2 \Rightarrow \Phi_\tau(\Omega_1) \le \Phi_\tau(\Omega_2)$;

(iii) $\Phi_\tau(\{a\} \cup \Omega) = \Phi_\tau(\Omega)$ for any $a \in E$ and $\Omega \in \mathfrak{M}_E$;

(iv) $\Phi_\tau(\Omega) = 0$ if and only if Ω is relatively τ-compact in E.

Observe that (i) still holds true if we have $\Phi_\tau(\overline{conv}^\Gamma(\Omega)) \le \Phi_\tau(\Omega)$.

In the case when C has additionally the structure of a cone in a linear space over the field of real numbers, we will say that a τ-MNC Φ_τ is *positively homogeneous* provided $\Phi_\tau(\lambda\Omega) = \lambda\Phi_\tau(\Omega)$ for all $\lambda > 0$ and for $\Omega \in \mathfrak{M}_E$. Moreover, Φ_τ is referred to as *subadditive* if $\Phi_\tau(\Omega_1 + \Omega_2) \le \Phi_\tau(\Omega_1) + \Phi_\tau(\Omega_2)$ for all $\Omega_1, \Omega_2 \in \mathfrak{M}_E$.

As an example of τ-MNC we have the important and well know De Blasi measure of weak noncompactness β (see [16]) defined on $\mathfrak{M}_E$ (where E is a Banach space and τ is its weak topology) by the formula

$$\beta(\Omega) = \inf\{\varepsilon > 0 : \text{there exists a weakly compact set } D \text{ such that } \Omega \subset D + B_\varepsilon(\theta)\}.$$

It is well known that β has several useful properties. For example, it satisfies the following conditions for all $\Omega_1, \Omega_2 \in \mathfrak{M}_E$ (cf. [16]):

(v) $\beta(\Omega_1 \cup \Omega_2) = \max\{\beta(\Omega_1), \beta(\Omega_2)\}$;
(vi) $\beta(\lambda\Omega) = |\lambda|\beta(\Omega)$ for all $\lambda \in \mathbb{R}$;
(vii) $\beta(\Omega_1 + \Omega_2) \leq \beta(\Omega_1) + \beta(\Omega_2)$.

Particularly, the function β is positively homogeneous and subadditive in the sense of the above accepted definition.

Definition 2.2. Let Ω be a nonempty subset of E and let Φ_τ be a τ-MNC in E with values in a lattice C which has a least element 0 and is a cone. If T maps Ω into E, we say that:

(a) T is Φ_τ-Lipschitzian if $T(D) \in \mathfrak{M}_E$ for any bounded subset D of Ω and there exists a constant $k \geq 0$ such that $\Phi_\tau(T(D)) \leq k\Phi_\tau(D)$ for $D \in \mathfrak{M}_E$, $D \subset \Omega$;
(b) T is Φ_τ-contraction if T is Φ_τ-Lipschitzian with $k < 1$;
(c) T is Φ_τ-condensing if T is Φ_τ-Lipschitzian with $k = 1$ and $\Phi_\tau(T(D)) < \Phi_\tau(D)$ for $D \in \mathfrak{M}_E$ such that $D \subset \Omega$ and $\Phi_\tau(D) > 0$;
(d) T is Φ_τ-nonexpansive if T is Φ_τ-Lipschitzian with $k = 1$.

Observe that in the formulation of points (c) and (d) of the above definition we do not need the assumption that C has cone structure.

Starting from now on we will always assume that a lattice C has cone structure (i.e. C is a lattice with a least element 0 which is a cone in a real linear space) provided we require that Φ_τ is a positively homogeneous or subadditive τ-MNC in E.

Now we formulate other definitions needed in our considerations.

Definition 2.3. A topological space X is called angelic if for every $A \subset X$ and $x \in \overline{A}$, there is a sequence $\{x_n\} \subset A$ such that $x_n \longrightarrow x$.

All metrizable locally convex spaces equipped with the weak topology are angelic (see the Eberlein-Šmulian theorem [19]).

Definition 2.4. Let Ω be a nonempty subset of E. An operator $T : \Omega \to E$ is said to be τ-compact if for any nonempty bounded subset D of Ω the set $T(D)$ is relatively τ-compact.

Definition 2.5. An operator $T : \Omega \to E$ (where Ω is a nonempty subset of E) is said to be τ-sequentially continuous on Ω if for each sequence $(x_n) \subset \Omega$ with $x_n \xrightarrow{\tau} x$ and $x \in \Omega$, we have that $Tx_n \xrightarrow{\tau} Tx$.

Remark 2.1. It is worthwhile mentioning that in several situations it is rather easy to show that a mapping between Banach spaces is weakly sequentially continuous, while the proof of weak continuity of that mapping is mostly very hard.

In many applications involving integral equation problems, one of the reasons for this difficulty is the fact that the Lebesgue dominated convergence theorem fails to work for nets.

Remark 2.2. If X is angelic, then any sequentially continuous map on a compact set is continuous.

Remark 2.3. Hereafter, by bounded sets in E, we will mean Γ-bounded sets.

3. Fixed point theory

We start with the following fixed point result.

Theorem 3.1. *Let Ω be a nonempty convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a τ-MNC in E. Assume that compact sets in (E, τ) are angelic. Then the following assertions hold, for every Φ_τ-condensing and τ-sequentially continuous map $T : \Omega \to \Omega$ with bounded range:*

(i) *T has a τ-approximate fixed point sequence, i.e. a sequence $(x_n) \subset \Omega$ such that the sequence $(x_n - Tx_n)$ converges to θ in (E, τ);*
(ii) *if Ω is τ-closed, then the set $F(T)$ of fixed points of T is nonempty and τ-compact.*

Proof: (i) Fix arbitrarily $x_0 \in \Omega$ and consider the family

$$\mathfrak{F} = \{Q \subset \Omega : Q \ \text{ is } \tau\text{-bounded, convex, } \ x_0 \in Q \ \text{ and } \ T(Q) \subset Q\}.$$

We have $\mathfrak{F} \neq \emptyset$. To see this, first note that $T(\Omega)$ is τ-bounded since $\tau \leq \Gamma$. Thus, since τ is locally convex, we get $conv(T(\Omega) \cup \{x_0\})$ is also τ-bounded (cf. [15, p. 15]). Now it is easy to see that $conv(T(\Omega) \cup \{x_0\}) \in \mathfrak{F}$. Let now $G = \bigcap_{Q \in \mathfrak{F}} Q$ and let $H = conv(T(G) \cup \{x_0\})$. We claim that $G = H$. Indeed, since $x_0 \in G$ and $T(G) \subset G$ one sees that $H \subset G$. In particular, we get $T(H) \subset T(G) \subset H$. On the other hand, since $H \subset \Omega$ and H is τ-bounded (notice $H \subset conv(T(\Omega) \cup \{x_0\})$), convex and $x_0 \in H$, we have that $H \in \mathfrak{F}$ and $G \subset H$. Therefore $G = H$ as claimed.

Now we claim that $\Phi_\tau(H) = \Phi_\tau(T(H))$. Clearly from Definition 2.1(ii), we have $\Phi_\tau(T(H)) \leq \Phi_\tau(H)$ (since $T(H) \subset H$). Now using (i)–(iii) of Definition 2.1 and the fact that $G = H$, we get

$$\Phi_\tau(H) \leq \Phi_\tau(\overline{conv}^\tau(T(G) \cup \{x_0\})) \leq \Phi_\tau((T(G) \cup \{x_0\})) = \Phi_\tau(T(G)) = \Phi_\tau(T(H)).$$

Keeping in mind that T is Φ_τ-condensing, we conclude (via Definition 2.1(iv)) that $\Phi_\tau(H) = 0$ and so $\overline{H}^\tau$ is τ-compact. Since $T(H) \subset H$ we get that $T/H : H \longrightarrow \overline{H}^\tau$ is a τ-sequentially continuous mapping. By Theorem 2.1 in [8], we get

$$\theta \in \overline{\{x - Tx, x \in H\}}.$$

Thus, there is a net $(x_\sigma) \subset H$ so that $x_\sigma - Tx_\sigma \xrightarrow{\tau} \theta$.

Claim. There exists a sequence $(x_n) \subset \{x - Tx : x \in H\}$ so that $x_n - Tx_n \overset{\tau}{\longrightarrow} \theta$.

Indeed, since $\overline{H}^{\tau}$ is compact, so is $\overline{H}^{\tau} - \overline{H}^{\tau}$. By assumption, this set is angelic. In particular, since $\theta \in \overline{\{x_\sigma - Tx_\sigma : \sigma\}}^{\tau} \subset \overline{H}^{\tau} - \overline{H}^{\tau}$, there is a sequence in (x_n) so that $x_n - Tx_n \overset{\tau}{\longrightarrow} \theta$.

(ii) Let $C = \overline{H}^{\tau}$. By redefining the set $\mathfrak{F}$ to be

$$\mathfrak{F}_* = \{Q \subset \Omega : Q \ \text{ is } \tau\text{-bounded, } \tau\text{-closed, convex, } x_0 \in Q \ \text{ and } \ T(Q) \subset Q\}.$$

We can prove (using the same argument as in the proof of (i) and the angelicity of C) that $T(C) \subset C$. Hence $T \mid C \ : C \longrightarrow C$ is τ-sequentially continuous map on C. Again by Theorem 2.1 in [8], we get

$$\theta \in \overline{\{x - Tx, x \in C\}}^{\tau}.$$

Using this and once more the fact that C is angelic, we can find a point $x \in C$ so that $Tx = x$. So $F(T)$ is nonempty. In addition, we have $T(F(T)) = F(T)$ and $F(T)$ is τ-bounded. Hence $\Phi_\tau(F(T)) = 0$ which means that $F(T)$ is relatively τ-compact. Moreover in view of the τ-sequential continuity of T, we deduce that $F(T)$ is τ-sequentially closed. Now we show that $F(T)$ is τ-closed. To this end let $x \in \Omega$ be in $\overline{F(T)}^{\tau}$. Since $\overline{F(T)}^{\tau}$ is τ-compact, by the angelicity of $\overline{F(T)}^{\tau}$, there exists a sequence $(x_n) \subset F(T)$ such that $x_n \overset{\tau}{\longrightarrow} x$. Hence $x \in F(T)$. Thus $\overline{F(T)}^{\tau} = F(T)$ which means that $F(T)$ is τ-compact. The proof is complete. □

Remark 3.1. Example 2.4 in [8] shows that the angelicity assumption in Theorem 3.1 cannot be dropped.

Remark 3.2. Notice that Theorem 3.1 improves and generalizes Theorem 3.2 in [11], Theorem 12 in [18], Theorem 3.1 in [5], Theorem 2 in [24] and Theorem 2.2 in [25] in the context of a Banach space equipped with its weak topology and the De Blasi measure of weak noncompactness.

Corollary 3.1. *Let Ω be a nonempty convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a τ-MNC in E. Assume that compact sets in (E, τ) are angelic, Ω is τ-closed and $T : \Omega \to \Omega$ is τ-sequentially continuous and τ-compact mapping with bounded range. Then the set $F(T)$ of fixed points of T is nonempty and τ-compact.*

Indeed, the above assertion is an immediate consequence of Theorem 3.1 since T is obviously Φ_τ-condensing, where Φ_τ is an arbitrary τ-MNC in E.

Remark 3.3. Corollary 3.1 generalizes Arino, Gautier and Penot theorem [3], Theorem 2.2 and Corollary 2.3 from [28].

Corollary 3.2. *Let Ω be a nonempty convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a subadditive τ-MNC in E. Assume that compact sets in (E, τ) are angelic and Ω is τ-closed. Let $T : \Omega \to E$ and $S : \Omega \to E$ be two mappings satisfying the following conditions:*

(i) *T is τ-sequentially continuous and τ-compact;*
(ii) *S is Φ_τ-condensing and τ-sequentially continuous;*
(iii) *$(T+S)(\Omega)$ is a bounded subset of Ω.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof: Obviously $T+S$ is τ-sequentially continuous. Suppose D is a bounded subset of Ω. Then we have

$$\Phi_\tau((T+S)(D)) \leq \Phi_\tau(T(D)+S(D)) \leq \Phi_\tau(T(D)) + \Phi_\tau(S(D)) \leq \Phi_\tau(S(D)),$$

since $T(D)$ is relatively τ-compact. Thus, if $\Phi_\tau(D) > 0$, we get

$$\Phi_\tau((T+S)(D)) < \Phi_\tau(D),$$

which yields that $T+S$ is Φ_τ-condensing and we can apply Theorem 3.1 to conclude that there exists $x \in \Omega$ such that $x = Tx + Sx$. The proof is complete. □

As a consequence of the above corollary we can state the following assertion.

Corollary 3.3. *Let Ω be a nonempty, closed and convex subset of a Banach space E and let $T : \Omega \to E$, $S : E \to E$ be weakly sequentially continuous mappings satisfying the following conditions:*
(i) *T is weakly compact;*
(ii) *S is a nonlinear contraction, i.e. there exists a continuous nondecreasing function $\Psi : \mathbb{R}_+ \to \mathbb{R}_+ = [0,\infty)$ with $\Psi(z) < z$ for $z > 0$ and such that $\|Sx - Sy\| \leq \Psi(\|x-y\|)$ for $x, y \in \Omega$;*
(iii) *$(T+S)(\Omega)$ is a bounded subset of Ω.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof: Keeping in mind Corollary 3.2 it is sufficient to show that S is β-condensing. To this end take a bounded subset D of Ω. Suppose that $\beta(D) = d > 0$. Let $\varepsilon > 0$, then there exists a weakly compact set K of E with $D \subseteq K + B_{d+\varepsilon}(\theta)$. So for $x \in D$ there exist $y \in K$ and $z \in B_{d+\varepsilon}(\theta)$ such that $x = y + z$ and so

$$\|Sx - Sy\| \leq \Psi(\|x-y\|) \leq \Psi(d+\varepsilon).$$

It follows immediately, that

$$S(D) \subseteq S(K) + B_{\Psi(d+\varepsilon)}(\theta).$$

Moreover, since S is a weakly sequentially continuous mapping and K is weakly compact (see Remark 2.2) then $\overline{S(K)}^w$ is weakly compact. Therefore, $\beta(S(D)) \leq \Psi(d+\varepsilon)$. Since $\varepsilon > 0$ is arbitrary, it follows that $\beta(S(D)) \leq \Psi(d) < d = \beta(D)$. Accordingly, S is β-condensing and the proof is complete. □

Now, we formulate next results having other character than those given previously.

Theorem 3.2. *Let Ω be a nonempty convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a subadditive τ-MNC in E. Assume that compact sets in (E, τ) are angelic and Ω is τ-closed. If $T : \Omega \to E$ and $S : \Omega \to E$ are τ-sequentially continuous mappings satisfying the following conditions:*

(i) *T is τ-compact,*
(ii) *S is Φ_τ-condensing,*
(iii) *$I - S$ is invertible on $T(\Omega)$,*
(iv) *$[y = Tx + Sy,\ x \in \Omega] \Rightarrow y \in \Omega$,*
(v) *$(I - S)^{-1}T(\Omega)$ is bounded,*

then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof: At the beginning we show that the mapping $(I - S)^{-1}T$ transforms Ω into itself. In fact, by assumption (iii), for each $y \in \Omega$ the equation $z = Ty + Sz$ has a unique solution z. On the other hand assumption (iv) implies that $z = (I - S)^{-1}Ty$ is in Ω.

Further, define the mapping $F : \Omega \to \Omega$ by putting

$$Fx = (I - S)^{-1}Tx.$$

Let $D = \overline{conv}^\tau F(\Omega)$. Observe that the set D is τ-closed, convex, τ-bounded and $F(D) \subset D \subset \Omega$. Next, denote $D_1 = \overline{conv}^\tau F(D)$. Obviously, D_1 is also τ-closed, convex, τ-bounded and $F(D_1) \subset D_1 \subset D \subset \Omega$.

We claim that D_1 is τ-compact. If this is not the case, then $\Phi_\tau(D_1) > 0$. Since $F(D) \subset T(D) + SF(D)$, we obtain

$$\begin{aligned}\Phi_\tau(D_1) &\leq \Phi_\tau(F(D)) \leq \Phi_\tau(T(D) + SF(D)) \\ &\leq \Phi_\tau(T(D)) + \Phi_\tau(SF(D)).\end{aligned}$$

Since T is τ-compact, we have $\Phi_\tau(T(D)) = 0$. Thus, taking into account that S is Φ_τ-condensing, we get

$$\Phi_\tau(D_1) \leq \Phi_\tau(F(D)) \leq \Phi_\tau(S(F(D))) < \Phi_\tau(F(D)),$$

which is absurd. Hence we obtain that D_1 is τ-compact.

In view of Corollary 3.1 it remains to show that $F : D_1 \to D_1$ is τ-sequentially continuous. To do this take a sequence $(x_n) \subset D_1$ such that $x_n \xrightarrow{\tau} x$ and $x \in D_1$. Because the set $\{Fx_n\}$ is relatively τ-compact then applying the angelicity of (E, τ) and passing to a subsequence (x_{n_j}) of the sequence (x_n), we get that $Fx_{n_j} \xrightarrow{\tau} y$, $y \in D_1$. Hence we have that

$$-Tx_{n_j} + Fx_{n_j} \xrightarrow{\tau} -Tx + y.$$

On the other hand, by virtue of the τ-sequential continuity of S we deduce that $SFx_{n_j} \xrightarrow{\tau} Sy$. Combining the above established facts with the equality

$$SF = -T + F,$$

we derive that $y = Fx$.

Now we claim that $Fx_n \xrightarrow{\tau} Fx$. Suppose that this is not the case. Then there exists a subsequence (x_{n_k}) and a neighbourhood V of Fx in (E, τ) such that $Fx_{n_k} \notin V$ for all k. On the other hand we have that $x_{n_k} \xrightarrow{\tau} x$, so arguing as before we can find a subsequence $(x_{n_{k_s}})$ such that $Fx_{n_{k_s}} \xrightarrow{\tau} Fx$. Thus we obtain a contradiction. Hence it follows that F is τ-sequentially continuous.

Finally, applying Corollary 3.1 we conclude that F has a fixed point $x \in D_1$, which means that $Tx + Sx = x$. This completes the proof. □

Theorem 3.3. *Let Ω be a nonempty, closed and convex subset of a Banach space E and let $T : \Omega \to E$, $S : E \to E$ be weakly sequentially continuous mappings satisfying the following conditions:*

(i) *T is weakly compact;*
(ii) *S is a nonlinear contraction;*
(iii) *there exists a bounded subset D of E such that $T(\Omega) \subset (I - S)(D)$;*
(iv) *$[y = Tx + Sy,\ x \in \Omega] \Rightarrow y \in \Omega$.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof: First we show that $I - S$ is invertible on $T(\Omega)$. To this end fix arbitrarily $y \in \Omega$ and consider the mapping $S_y : E \to E$, defined in the following way:

$$S_y z = Ty + Sz.$$

Obviously S_y is a nonlinear contraction. Thus, by a result from [12] we infer that the operator S_y has a unique fixed point $z \in E$. Joining this statement with assumption (iv) we derive that $z \in \Omega$.

This means that $I - S$ is invertible on $T(\Omega)$.

Further observe that in view of the above facts and assumption (iv) we have that

$$(I - S)^{-1} T(\Omega) \subset D.$$

Therefore, the conclusion of our theorem follows from Theorem 3.2. The proof is complete. □

Remark 3.4. Theorem 3.3 extends and improves Theorem 2.1 from [28].

Now we define a class of operators playing an important role in our further investigations.

Definition 3.1. Let Ω be a subset of a Hausdorff topological vector space E. A mapping $T : \Omega \to E$ is said to be demi-τ-compact whenever for any sequence $(x_n) \subset \Omega$ such that the sequence $x_n - Tx_n \xrightarrow{\tau} y \in E$, there exists a τ-convergent subsequence of the sequence (x_n). In the case when $y = \theta$, we say that T is demi-τ-compact at θ.

Definition 3.2. Let Ω be a subset of a Banach space E. A mapping $T : \Omega \to E$ is said to be demi-weakly compact whenever for any sequence $(x_n) \subset \Omega$ such that the sequence $x_n - Tx_n \xrightarrow{w} y \in E$, there exists a weakly convergent subsequence of the sequence (x_n). In the case when $y = \theta$, we say that T is demi-weakly compact at θ.

Theorem 3.4. *Suppose Ω is a nonempty, closed and convex subset of a Banach space E. Next assume that the operators $T : \Omega \to E$, $S : E \to E$ are weakly sequentially continuous and satisfy the following conditions:*

(i) *T is weakly compact;*
(ii) *S is nonexpansive (i.e. $\|Sx - Sy\| \leq \|x - y\|$ for all $x, y \in E$) and demi-weakly compact;*
(iii) *there exists a bounded subset D of E and a sequence $(\lambda_n) \subset (0, 1)$ such that $\lambda_n \to 1$, $T(\Omega) \subset (I - \lambda_n S)(D)$ and $[y = \lambda_n Sy + Tx,\ x \in \Omega] \Rightarrow y \in \Omega$ for all $n = 1, 2, \ldots$.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof: Observe that from imposed assumptions it follows easily that for any natural number n the mapping $\lambda_n S$ is weakly sequentially continuous and is a nonlinear contraction. Thus, applying Theorem 3.3 to the mapping $T + \lambda_n S$ we conclude that there exists a fixed point of this mapping belonging to Ω, i.e. there exists $x_n \in \Omega$ such that

$$x_n = Tx_n + \lambda_n Sx_n \tag{3.1}$$

for $n = 1, 2, \ldots$.

Next, notice that (x_n) is a bounded sequence in the set D mentioned in assumption (iii).

Indeed, this statement is a consequence of the fact that $I - \lambda_n S$ is invertible on $T(\Omega)$ (to show this it is sufficient to adopt a suitable part of Theorem 3.3), assumption (iii) and the equalities:

$$(I - \lambda_n S)x_n = Tx_n \in (I - \lambda_n S)(D),$$

$$x_n = (I - \lambda_n S)^{-1} Tx_n \in D$$

for $n = 1, 2, \ldots$.

Now, in view of assumption (i), without loss of generality we can assume that $Tx_n \xrightarrow{w} y$, $y \in E$. Since the sequence (Sx_n) is bounded and $\lambda_n \to 1$, in the light of (3.1) we deduce that

$$x_n - Sx_n = Tx_n + (\lambda_n - 1)Sx_n \xrightarrow{w} y,$$

where $y \in E$.

Further, taking into account the demi-weak compactness of the operator S, we derive that there exists a weakly convergent subsequence (x_{n_k}) of the sequence (x_n), i.e. $x_{n_k} \xrightarrow{w} x$, $x \in \Omega$.

Obviously, we have

$$x_{n_k} = Tx_{n_k} + \lambda_{n_k} Sx_{n_k}.$$

Hence, using the weak sequential continuity of T and S, we conclude that $x = Tx + Sx$. Thus x is a fixed point of the operator $T + S$ belonging to Ω.

The proof is complete. □

Our next result refines some above presented ones.

Theorem 3.5. *Let Ω be a nonempty convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a positively homogeneous and subadditive τ-MNC in E. Assume that compact sets in (E, τ) are angelic and Ω is τ-closed. In addition, assume that $T : \Omega \to E$, $S : \Omega \to E$ are τ-sequentially continuous operators satisfying the following conditions:*

(i) *T is τ-compact;*
(ii) *S is Φ_τ-nonexpansive and demi-τ-compact;*
(iii) *there exists a bounded subset Ω_0 of Ω and a sequence $(\lambda_n) \subset (0, 1)$ such that $\lambda_n \to 1$ and $(T + \lambda_n S)(\Omega) \subset \Omega_0$.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof: Define the sequence of operators by putting $G_n = T + \lambda_n S$ for $n = 1, 2, \ldots$. Assumption (iii) implies that $G_n(\Omega)$ is bounded for $n = 1, 2, \ldots$.

Further, take an arbitrary bounded subset D of Ω. Then we obtain

$$\begin{aligned}\Phi_\tau(G_n(D)) &\le \Phi_\tau(T(D) + \lambda_n S(D)) \\ &\le \Phi_\tau(T(D)) + \lambda_n \Phi_\tau(S(D)) = \lambda_n \Phi_\tau(S(D)).\end{aligned}$$

Hence, if $\Phi_\tau(D) > 0$, we get

$$\Phi_\tau(G_n(D)) < \Phi_\tau(D).$$

Thus G_n is Φ_τ-condensing on Ω. Obviously G_n is τ-sequentially continuous, so by Theorem 3.1 we infer that G_n has a fixed point x_n in Ω, for any $n = 1, 2, \ldots$.

Now, repeating a suitable part of the proof of the preceding theorem we get the desired conclusion. This completes the proof. □

Corollary 3.4. *Let Ω be a nonempty, closed and convex subset of E and let $T : \Omega \to E$, $S : E \to E$ be weakly sequentially continuous mappings satisfying the conditions listed below:*

(i) *T is weakly compact;*
(ii) *S is nonexpansive and demi-weakly compact;*
(iii) *there exists a bounded subset Ω_0 of Ω and a sequence $(\lambda_n) \subset (0, 1)$ such that $\lambda_n \to 1$ and $(T + \lambda_n S)(\Omega) \subset \Omega_0$ for $n = 1, 2, \ldots$.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof: The proof follows immediately from Theorem 3.5, provided we show that S is β-nonexpansive, where β is the De Blasi measure of weak noncompactness in E.

To do this take D being a bounded subset of Ω and put $d = \beta(D)$. Fix $\varepsilon > 0$. Then there exists a weakly compact set K with $D \subset K + B_{d+\varepsilon}(\theta)$. This yields that for each $x \in D$ there exist $y \in K$ and $z \in B_{d+\varepsilon}(\theta)$ such that $x = y + z$. Moreover, we have

$$\|Sx - Sy\| \leq \|x - y\| \leq d + \varepsilon.$$

Hence we obtain

$$S(D) \subset S(K) + B_{d+\varepsilon}(\theta).$$

Since S is weakly sequentially continuous and K is weakly compact, then $S(K)$ is weakly compact (see Remark 2.2). This implies that $\beta(S(D)) \leq d + \varepsilon$. In view of the arbitrariness of ε we get that $\beta(S(D)) \leq d = \beta(D)$. Thus, S is β-nonexpansive, which completes the proof. □

In what follows we will utilize the concept defined below.

Definition 3.3. Let Ω be a nonempty τ-closed subset of a Hausdorff topological vector space E and let $T : \Omega \to E$ be a τ-sequentially continuous operator. T will be called a τ-semi-closed operator at θ (τ-sc, in short) if the conditions $x_n \in \Omega$, $x_n - Tx_n \to \theta$ imply that there exists $x \in \Omega$ such that $Tx = x$.

It is worthwhile mentioning that the class of weakly semi-closed operators at θ (wsc, in short) includes, as special cases, weakly sequentially continuous operators which are weakly compact, Φ-condensing operators (where Φ is a positively homogeneous and subadditive weakly MNC) or those operators T for which the set $(I - T)(\Omega)$ is weakly closed, among others (see [12]).

Now, we prove a result allowing us to indicate certain class of τ-semi-closed mappings.

Lemma 3.1. *Let Ω be a τ-closed subset of a Hausdorff topological vector space E and let $T : \Omega \to E$ be a τ-sequentially continuous mapping being demi-τ-compact at θ. Then T is a τ-semi-closed mapping at θ.*

Proof: Suppose (x_n) is a sequence in Ω such that $x_n - Tx_n \to \theta$. Since T is demi-τ-compact we infer that there exists a subsequence (x_{n_k}) of (x_n) and an element $x \in E$ such that $x_{n_k} \xrightarrow{\tau} x$.

We claim that $x \in \Omega$ and $Tx = x$. Indeed, since Ω is τ-closed, so $x \in \Omega$. Moreover, the τ-sequential continuity of T implies that $Tx_{n_k} \xrightarrow{\tau} Tx$. On the other hand, we have

$$x_{n_k} - Tx = (x_{n_k} - Tx_{n_k}) + (Tx_{n_k} - Tx) \xrightarrow{\tau} \theta.$$

This yields that $x_{n_k} \xrightarrow{\tau} Tx$. Hence we infer that $Tx = x$ and the proof is complete. □

Theorem 3.6. *Let Ω be a nonempty convex subset of a Hausdorff topological vector space (E, Γ) and let Φ_τ be a positively homogeneous and subadditive τ-MNC in E. Assume that compact sets in (E, τ) are angelic and Ω is τ-closed. Further, assume that $T : \Omega \to E$, $S : \Omega \to E$ are τ-sequentially mappings satisfying the following conditions:*

(i) *T is τ-compact;*
(ii) *S is Φ_τ-nonexpansive;*
(iii) *T is τ-semi-closed at θ;*
(iv) *$(T+S)(\Omega)$ is a bounded subset of Ω.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

Proof: Fix $z \in \Omega$ and define $G_n = \lambda_n(T+S) + (1-\lambda_n)z$ $(n = 1, 2, \dots)$, where (λ_n) is a sequence in $(0,1)$ such that $\lambda_n \to 1$. Since Ω is convex and $z \in \Omega$, in view of assumption (iv) we deduce that G_n maps Ω into itself. Moreover, $G_n(\Omega)$ is bounded for any $n = 1, 2, \dots$. Obviously G_n is τ-sequentially continuous.

Now, assume that D is an arbitrary bounded subset of Ω. Then we have

$$\begin{aligned}
\Phi_\tau(G_n(D)) &= \Phi_\tau(\{\lambda_n(T+S)(D)\} + \{(1-\lambda_n)z\}) \\
&\le \lambda_n \Phi_\tau((T+S)(D)) \\
&\le \lambda_n \Phi_\tau(T(D)) + \lambda_n \Phi_\tau(S(D)) \\
&= \lambda_n \Phi_\tau(S(D)) \\
&\le \lambda_n \Phi_\tau(D).
\end{aligned}$$

Thus, if $\Phi_\tau(D) > 0$, we get

$$\Phi_\tau(G_n(D)) < \Phi_\tau(D).$$

Therefore, G_n is Φ_τ-condensing on Ω and we can apply Theorem 3.1 to obtain a sequence (x_n) such that $(x_n) \subset \Omega$ and $G_n x_n = x_n$ for $n = 1, 2, \dots$. Consequently, we obtain

$$x_n - (T+S)x_n = (\lambda_n - 1)[(T+S)x_n - z] \to 0,$$

since $\lambda_n \to 1$ as $n \to \infty$ and $(T+S)(\Omega)$ is bounded. Finally, keeping in mind assumption (iii) we conclude that there exists $x \in \Omega$ such that $Tx + Sx = x$. The proof is complete. □

Corollary 3.5. *Let Ω be a nonempty, closed and convex subset of a Banach space E. Let $T : \Omega \to E$ and $S : E \to E$ be weakly sequentially continuous mappings satisfying the following conditions:*

(i) *T is weakly compact;*
(ii) *S is nonexpansive;*
(iii) *$T + S$ is wsc;*
(iv) *$(T+S)(\Omega)$ is a bounded subset of Ω.*

Then there exists $x \in \Omega$ such that $x = Tx + Sx$.

In order to obtain the conclusion of the above corollary it is sufficient to take into account the fact that S is β-nonexpansive and apply Theorem 3.6.

4. Application to the theory of integral equations in the Lebesgue space

This section shows the applicability of our results proved in the preceding section. More precisely, we are going to show that our main result contained in Corollary 3.2 can be applied to the theory of nonlinear integral equations in Lebesgue space.

At the beginning we recall notation, definitions and auxiliary facts which will be used in our considerations (cf. [4], [5]).

Suppose that I is a bounded interval in $\mathbb{R}$. For simplicity, we will assume that $I = [0, 1]$.

Denote by $L^1 = L^1(I)$ the space of Lebesgue integrable real functions on the interval I with the standard norm

$$\|x\| = \int_0^1 |x(t)|\, dt.$$

The space L^1 is also called the Lebesgue space.

Further, denote by $S = S(I)$ the set of all real functions defined and Lebesgue measurable on I. Let $m(A)$ stand for the Lebesgue measure of a measurable subset A of $\mathbb{R}$. If we introduce in S the metric ρ by the formula

$$\rho(x, y) = \inf\{a + m(\{s \in I : |x(s) - y(s)| \geq a\}) : \ a > 0\},$$

then S becomes a complete metric space [16]. Moreover, it is known that the convergence in measure coincides with the convergence generated by the metric ρ.

Notice that convergence in measure of a sequence $\{x_n\}$ in L^1 does not imply the weak convergence of $\{x_n\}$ and conversely. However, we have the following results [23].

Lemma 4.1. *If a sequence $\{x_n\} \subset L^1$ converges weakly to $x \in L^1$ and is compact in measure then it converges in measure to x.*

Lemma 4.2. *A sequence $\{x_n\} \subset L^1$ converges strongly to $x \in L^1$ (i.e. converges in norm of L^1 to x) if and only if $\{x_n\}$ converges in measure to x and is weakly compact.*

For our further purposes the following result, which is a consequence of Lemmas 4.1 and 4.2, will be very useful.

Lemma 4.3. *Let X be a bounded subset of the Lebesgue space L^1 which is compact in measure. If an operator $T : X \to L^1$ is continuous then it is also weakly sequentially continuous.*

Now, we provide a few basic facts concerning the so-called superposition operator (cf. [2]).

Namely, assume that $f(t, x) = f : \ I \times \mathbb{R} \to \mathbb{R}$ is a given function. For an arbitrary function $x : I \to \mathbb{R}$ denote by Fx the function defined on I by the formula $(Fx)(t) = f(t, x(t))$. The operator F defined in such a way is said to be the *superposition operator* generated by the function f.

We say that the function $f = f(t, x)$ satisfies *Carathéodory conditions* if it is measurable in t for each $x \in \mathbb{R}$ and is continuous in x for almost all $t \in I$.

It is well known [2] that the superposition operator F generated by a function f satisfying Carathéodory conditions transforms the metric space $S(I)$ into itself.

The fundamental property of the superposition operator defined on the space L^1 is contained in the following theorem [19].

Theorem 4.1. *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then the superposition operator F generated by f transforms the space L^1 into itself if and only if $|f(t, x)| \leq a(t) + b|x|$ for $t \in I$ and $x \in \mathbb{R}$, where $a(t)$ is a function from the space L^1 and b is a nonnegative constant. Moreover, the operator F is continuous on the space L^1.*

It is worthwhile mentioning that under assumptions of the above theorem the superposition operator F has not to be weakly sequentially continuous on the space L^1 or on a ball of L^1, for example. Indeed this fact is a consequence of the following old result due to Shragin [27].

Theorem 4.2. *Assume that $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then the superposition operator F generated by f is weakly sequentially continuous on L^1 if and only if the generating function f has the form*

$$f(t, x) = \alpha(t) + \beta(t)x,$$

where $\alpha \in L^1(I)$ and $\beta \in L^\infty(I)$.

The above theorem shows that the class of functions generating weakly sequentially continuous superposition operators is rather narrow. In order to extend this class we are forced to consider superpositions operators on some subsets of the space L^1.

Now, we are going to present very convenient and handy formula expressing the De Blasi measure of weak noncompactness (cf. Section 2) in the space $L^1 = L^1(I)$. This formula is based on the following criterion for weak noncompactness in L^1 due to Dunford and Pettis [15].

Theorem 4.3. *Let X be a bounded subset of L^1. The set X is weakly compact if and only if it has equiabsolutely continuous integrals, which means that for each $\varepsilon > 0$ there is $\delta > 0$ such that $\int_D |x(t)|\, dt \leq \varepsilon$ for any $x \in X$ and for any measurable subset D of the interval I with $m(D) \leq \delta$.*

On the base of the above theorem Appell and De Pascale [1] showed that the De Blasi Measure of weak noncompactness β in L^1 can be expressed by the formula

$$\beta(X) = \lim_{\varepsilon \to 0} \left\{ \sup_{x \in X} \left\{ \sup \left[\int_D |x(t)|\, dt : D \subset I,\ m(D) \leq \varepsilon \right] \right\} \right\}.$$

Further, we pay our attention to the description of compactness in measure. The complete description of that compactness was given by Fréchet [18] but for our purposes the following sufficient condition will be useful [23].

Theorem 4.4. *Let X be a bounded subset of the space L^1. Suppose there is a family of measurable subsets $\{\Omega_c\}_{0\le c\le 1}$ of the interval I such that $m(\Omega_c)=c$. If for any $c\in I$ and for any $x\in X$ we have*

$$x(t_1)\le x(t_2)$$

for $t_1\in\Omega_c$ and for $t_2\notin\Omega_c$, then the set X is compact in measure.

Now, let us fix $r>0$ and denote by Q_r the subset of the ball $B_r(\theta)$ in L^1 consisting of functions being a.e. nondecreasing (or a.e. nonincreasing) on the interval I in the sense that there exists a subset P of I with $m(P)=0$ and such that each function $x\in Q_r$ is nondecreasing on the set $I\setminus P$ (or nonincreasing on $I\setminus P$).

Keeping in mind Theorem 4.4 it is easily seen that the set Q_r is compact in measure.

In what follows we will consider the nonlinear integral equation of the form

$$x(t)=a(t)+\int_0^1 k(t,s)f(s,x(s))\,ds+\int_0^1 u(t,s,x(s))\,ds, \tag{4.1}$$

for $t\in I$.

Observe that Equation (4.1) contains both the component being a counterpart of Hammerstein integral equation and a counterpart corresponding to Urysohn equation. Indeed, if we define on the space L^1 the operators H and U in the following way

$$(Hx)(t)=\int_0^1 k(t,s)f(s,x(s))\,ds, \tag{4.2}$$

$$(Ux)(t)=\int_0^1 u(t,s,x(s))\,ds, \tag{4.3}$$

for $x\in L^1$ and for $t\in I$, then H is the Hammerstein integral operator while U represents the Urysohn one.

Henceforth we will assume that the functions involved in Equation (4.1) satisfy the following conditions.

(i) $a\in L^1$ is nonnegative and nondecreasing on the interval I.

(ii) $f: I\times\mathbb{R}\to\mathbb{R}$ satisfies Carathéodory conditions and there exists a function $p\in L^1$ such that

$$|f(t,x)|\le p(t)$$

for $t\in I$ and $x\in\mathbb{R}$. Moreover, $f: I\times\mathbb{R}_+\to\mathbb{R}_+$.

(iii) $k : I \times I \to \mathbb{R}_+$ is measurable with respect to both variables and such that the integral operator K defined on the space L^1 by the formula

$$(Kx)(t) = \int_0^1 k(t,s)x(s)\,ds$$

maps the space L^1 into itself.

For further purposes let us recall that the above assumption implies [23] that the operator K maps continuously the space L^1 into itself.

In what follows we will denote by $\|K\|$ the norm of the linear operator K.

Further, we formulate our remaining assumptions.

(iv) The function $t \to k(t,s)$ is a.e. nondecreasing on the interval I for almost all $s \in I$.

(v) $u(t,s,x) = u : \ I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions, that is, u is measurable with respect to (t,s) for any $x \in \mathbb{R}$ and is continuous in x for almost all $(t,s) \in I^2$.

(vi) $u(t,s,x) \geq 0$ for $(t,s) \in I^2$ and for $x \geq 0$.

(vii) The function $t \to u(t,s,x)$ is a.e. nondecreasing on the interval I for almost all $s \in I$ and for each $x \in \mathbb{R}$.

(viii) $|u(t,s,x)| \leq k_1(t,s)(q(t) + b|x|)$ for $(t,s) \in I^2$ and for $x \in \mathbb{R}$, where q is a nonnegative member of L^1, $0 \leq b = \text{const.}$ and a function $k_1 : I^2 \to \mathbb{R}_+$ is measurable and such that the linear operator K_1 generated by k_1 maps L^1 into itself.

(ix) $b\|K_1\| < 1$.

Then we can formulate our existence result concerning Equation (4.1).

Theorem 4.5. *Under assumptions* (i)–(ix) *Equation* (4.1) *has at least one solution $x \in L^1$ such that x is a.e. nondecreasing on the interval I.*

Proof: Observe that in view of (4.2) and (4.3) we can write Equation (4.1) in the form

$$x = a + Hx + Ux.$$

In order to show that Corollary 3.2 can be applied in our situation let us denote by T the operator defined on L^1 by the formula

$$Tx = a + Hx. \tag{4.4}$$

Further, observe that the Hammerstein operator H defined by (4.2) can be written as the product $H = KF$ of the superposition operator

$$(Fx)(t) = f(t, x(t))$$

and the linear operator

$$(Kx)(t) = \int_0^1 k(t,s)x(s)\,ds.$$

Next, let us take an arbitrary function $x \in L^1$. Then, in view of assumptions (i)–(iii) and Theorem 4.1 we infer that $Tx \in L^1$, where T is defined by (4.4).

On the other hand, keeping in mind assumptions (v) and (viii) and the majorant principle (cf. [29]) we deduce that the Urysohn operator U transforms the space L^1 into itself and is continuous.

Further, let us consider the subset Ω of the space L^1 consisting of all functions $x = x(t)$ being a.e. nonnegative and nondecreasing on the interval I. It is easily seen that the operators T and U transform the set Ω into itself. In fact, this statement is an easy consequence of assumptions (i), (ii), (iv), (vi) and (vii).

This allows us to infer that the sum $T + U$ of these operators transforms the set Ω into Ω.

Next, for an arbitrarily fixed $x \in \Omega$, in view of imposed assumptions we obtain

$$|((T + U)(x))(t)| \le a(t) + (Hx)(t) + (Ux)(t), \quad \forall t.$$

Thus

$$\begin{aligned}
\|(T + U)(x)\| &\le \|a\| + \|Hx\| + \int_0^1 u(t, s, x(s))\, ds \\
&\le \|a\| + \|KFx\| + \int_0^1 k_1(t, s)(q(s) + bx(s))\, ds \\
&\le \|a\| + \|K\|\|Fx\| + \int_0^1 k(t, s)q(s)\, ds + b\int_0^1 k_1(t, s)x(s)\, ds \\
&\le \|a\| + \|K\| \int_0^1 f(s, x(s))\, ds + \|K_1\|\|q\| + b\|K_1\|\|x\| \\
&\le \|a\| + \|K\| \int_0^1 p(s)\, ds + \|K_1\|\|q\| + b\|K_1\|\|x\| \\
&\le \|a\| + \|K\|\|p\| + \|K_1\|\|q\| + b\|K_1\|\|x\| \\
&= A + b\|K_1\|\|x\|
\end{aligned}$$

where we denoted $A = \|a\| + \|K\|\|p\| + \|K_1\|\|q\|$. This implies

$$\|(T + U)(x)\| \le A + b\|K_1\|\|x\|. \tag{4.5}$$

Further, denote by Ω_r the set consisting of all functions x belonging to Ω and such that $\|x\| \le r$, when $r = A/(1 - b\|K_1\|)$. Obviously the set Ω_r is nonempty, convex, closed and bounded. Moreover, linking (4.5) with the fact that $T + U$ is a self-mapping of the set Ω and taking into account assumption (ix) we deduce that the operator $T + U$ transforms the set Ω_r into itself. Notice also that both the operator T and U transforms the set Ω_r into itself.

Now, by assumptions (ii) and (iii) (cf. also the fact stated after assumption (iii) which asserts that the operator K is a continuous self-mapping of the space L^1) and taking into account Theorem 4.1 we infer that T transforms continuously the set Ω_r into Ω_r.

Apart from this, based on the facts established above, we deduce that the Urysohn operator U transforms continuously the set Ω_r into itself.

Thus, in virtue of the fact that Ω_r is compact in measure (cf. Theorem 4.4 and remarks made before that theorem) we infer that the operators T and U transform weakly continuously the set Ω_r into itself.

Now, we show that the operator T is weakly compact on the set Ω_r. Furthermore, the operator T is also weakly compact on the set Ω.

To prove this assertion let us take an arbitrary function $x \in \Omega$. Then, for a fixed $t \in I$ we get:

$$
\begin{aligned}
|(Tx)(t)| &\le a(t) + \left| \int_0^1 k(t,s) f(s,x(s))\,ds \right| \\
&\le a(t) + \int_0^1 k(t,s)|f(s,x(s))|\,ds \le a(t) + \int_0^1 k(t,s)p(s)\,ds.
\end{aligned} \tag{4.6}
$$

Hence, taking into account that the function $t \to \int_0^1 k(t,s)p(s)\,ds$ is an element of the space L^1, from estimate (4.6) and Theorem 4.3 we infer that the set $T(\Omega)$ is weakly compact.

Thus we showed that the operator T is weakly compact on the set Ω.

In what follows take a nonempty set $X \subset \Omega_r$ and fix $\varepsilon > 0$. Further, let D be a measurable subset of the interval I such that $m(D) \le \varepsilon$. Then, for an arbitrary $x \in X$, in view of assumption (viii) we obtain

$$
\begin{aligned}
\int_D |(Ux)(t)|\,dt &\le \int_D \left(\int_0^1 k_1(t,s)q(s)\,ds \right) dt + b \int_D \left(\int_0^1 k_1(t,s)x(s)\,ds \right) dt \\
&= \|K_1 q\|_{L^1(D)} + b\|K_1 x\|_{L^1(D)},
\end{aligned}
$$

where by $L^1(D)$ we denoted the Lebesgue space of real functions defined on the set D.

Now, taking into account that the operator K_1 maps the space $L^1(D)$ into itself and is continuous, we get

$$
\begin{aligned}
\int_D |(Ux)(x)|\,dt &\le \|K_1\|_D \|q\|_{L^1(D)} + b\|K_1\|_D \|x\|_{L^1(D)} \\
&= \|K_1\|_D \int_D q(t)\,dt + b\|K_1\|_D \int_D x(t)\,dt \\
&\le \|K_1\| \int_D q(t)\,dt + b\|K_1\| \int_D x(t)\,dt,
\end{aligned}
$$

where the symbol $\|K_1\|_D$ stands for the norm of the linear operator K_1 acting from the space $L^1(D)$ into itself.

Further, keeping in mind the fact that any singleton is weakly compact in the space L^1, in view of Theorem 4.3 we derive the following inequality

$$\beta(UX) \leq b\|K_1\|\beta(X),$$

where β denotes the De Blasi measure of weak noncompactness. Particularly, in virtue of assumption (ix) this statement means that the operator U is condensing with respect to β.

Finally, combining all the above established facts and applying Corollary 3.2 we complete the proof. □

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