A note on loops of square-free order

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Abstract. Let Q be a loop such that $|Q|$ is square-free and the inner mapping group $I(Q)$ is nilpotent. We show that Q is centrally nilpotent of class at most two.

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1. Introduction

A positive integer n is said to be square-free if n is divisible by no (perfect) square except 1. If n is square-free, then in the prime factorization of n, no prime occurs more than once. In this short note we show that if a loop Q has square-free order and the inner mapping group $I(Q)$ is nilpotent, then Q is centrally nilpotent of class at most two. Our notation in loop theory and group theory is standard. If $H \leq G$, then by H_G we denote the core of H in G (the largest normal subgroup of G contained in H). By $H^G = \langle H^g \mid g \in G \rangle$ we denote the normal closure of H in G.

2. Loops and connected transversals

If Q is a loop, then the two mappings $L_a(x) = ax$ and $R_a(x) = xa$ are permutations on Q for every $a \in Q$. We write $M(Q) = \langle L_a, R_a \mid a \in Q \rangle$ and say that $M(Q)$ is the multiplication group of Q. The stabilizer of the neutral element $e \in Q$ is denoted by $I(Q)$ and it is said to be the inner mapping group of Q. If we write $A = \{L_a \mid a \in Q\}$ and $B = \{R_a \mid a \in Q\}$, then the commutator subgroup $[A, B] \leq I(Q)$ and A, B are left transversals to $I(Q)$ in $M(Q)$. Furthermore, $I(Q)$ is core-free in $M(Q)$. A loop Q is centrally nilpotent of class at most two if and only if $M(Q)' \leq N_{M(Q)}(I(Q))$ (for the details, see [1, p. 281]).

If we replace $M(Q)$ by G and $I(Q)$ by H, then we have the following situation: $H \leq G$ and A and B are two left transversals to H in G. As $[A, B] \leq H$, we say that A and B are H-connected transversals in G. If $G' \leq N_G(H)$, then Q is centrally nilpotent of class at most two. In 1990 Kepka and Niemenmaa proved the following [3, Theorem 4.1]

Theorem 2.1. *A group G is isomorphic to the multiplication group of a loop if and only if there exists a subgroup H which is core-free in G and H-connected transversals A and B such that $G = \langle A, B \rangle$.*

In the following two lemmas, we assume that A and B are H-connected transversals in G.

Lemma 2.2. *Let N be a normal subgroup of G and let L be the core of HN in G. Then AL/L and BL/L are HL/L-connected transversals in G/L.*

Lemma 2.3. *If G is a finite group, H is nilpotent and $G = \langle A, B\rangle$, then H is subnormal in G.*

For the proofs, see [3, Lemma 2.8] and [4, Theorem 2.8].

Lemma 2.4. *If $H \leq G$ is nilpotent and subnormal in G, then the normal closure H^G is nilpotent.*

For the proof, see [2, Theorem 8.8, p. 29].

3. Main theorems

Theorem 3.1. *Let H be a nilpotent proper subgroup of a finite group G and let A and B be H-connected transversals in G such that $G = \langle A, B\rangle$. If $[G : H]$ is square-free, then $G' \leq N_G(H)$.*

Proof: Clearly, we can assume that $H \neq 1$. Let G be a minimal counterexample. By Lemma 2.3, H is subnormal in G and then, by Lemma 2.4, H^G is nilpotent. Let $H^G = P \times Q$, where $P > 1$ is a Sylow p-subgroup of H^G and Q is a Hall π-subgroup of H^G with $p \notin \pi$ (here π is a — possibly empty — set of prime numbers).

Let us first assume that $Q > 1$. Then P and Q are both nontrivial characteristic subgroups of H^G, hence they are nontrivial normal subgroups of G. Let K be the core of HP in G. By Lemma 2.2, AK/K and BK/K are HP/K-connected transversals in G/K. As $[G/K : HP/K]$ is square-free, we may conclude that $(G/K)' \leq N_{G/K}(HP/K)$, which means that $G' \leq N_G(HP)$. In a similar manner we can show that $G' \leq N_G(HQ)$. Thus $G' \leq N_G(HP) \cap N_G(HQ) \leq N_G(HP \cap HQ) = N_G(H)$.

Then assume that $Q = 1$. It follows that $H \leq H^G = P$. If $x, y \in G$, then $x = ah$ and $y = bk$, where $a \in A$, $b \in B$ and $h, k \in H$. Since A and B are H-connected transversals in G, it follows that $[xP, yP] = [aP, bP] = [a, b]P = P$. We conclude that G/P is an abelian group and thus $G' \leq P$. As $[G : H]$ is square-free, we see that $[P : H] = p$ or $P = H$. But then $N_G(H) \geq P$, whence $G' \leq N_G(H)$. $\square$

We can slightly loosen the condition on $[G : H]$ if we assume that H is a p-group.

Theorem 3.2. *Let $H < G$ be a p-group and let A and B be H-connected transversals in G such that $G = \langle A, B\rangle$. If $[G : H] = pk$ and $p \nmid k$, then $G' \leq N_G(H)$.*

The proof is similar to the proof of Theorem 3.1. By using Theorems 3.1, 3.2 and 2.1, we get

Corollary 3.3. *If Q is a loop of square-free order and $I(Q)$ is nilpotent, then Q is centrally nilpotent of class at most two.*

Corollary 3.4. *Let Q be a loop and $|Q| = pk$, where p is a prime number and $p \nmid k$. If $I(Q)$ is a p-group, then Q is centrally nilpotent of class at most two.*

Remark. If G is a finite group and $|G|$ is square-free, then G is solvable (this is an easy consequence of the Burnside normal complement theorem). If $|G|$ is square-free and $I(G) = Inn(G)$ (the group of inner automorphisms of G) is nilpotent, then G is cyclic.

It is easy to construct loops of order six which are not solvable. Now, consider the following loop Q of order six:

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \\ 3 & 4 & 5 & 6 & 2 & 1 \\ 4 & 3 & 6 & 5 & 1 & 2 \\ 5 & 6 & 1 & 2 & 3 & 4 \\ 6 & 5 & 2 & 1 & 4 & 3 \end{array}$$

Here $|Q|$ is square-free, $M(Q)$ has order 24, $I(Q)$ is isomorphic to Klein's four group and Q is centrally nilpotent of class two.

References

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