On character of points in the Higson corona of a metric space

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Dedicated to the 120th birthday anniversary of Eduard Čech.

Abstract. We prove that for an unbounded metric space X, the minimal character $\mathsf{m}\chi(\check{X})$ of a point of the Higson corona $\check{X}$ of X is equal to $\mathfrak{u}$ if X has asymptotically isolated balls and to $\max\{\mathfrak{u},\mathfrak{d}\}$ otherwise. This implies that under $\mathfrak{u}<\mathfrak{d}$ a metric space X of bounded geometry is coarsely equivalent to the Cantor macrocube $2^{<\mathbb{N}}$ if and only if $\dim(\check{X})=0$ and $\mathsf{m}\chi(\check{X})=\mathfrak{d}$. This contrasts with a result of Protasov saying that under CH the coronas of any two asymptotically zero-dimensional unbounded metric separable spaces are homeomorphic.

Keywords: Higson corona, character of a point, ultrafilter number, dominating number

Classification: 03E17, 03E35, 03E50, 54D35, 54E35, 54F45

1. Introduction

In this paper we shall calculate the smallest character of a point in the corona $\check{X}$ of a metric space X and using this information we shall distinguish topologically the Higson coronas of some metric spaces of asymptotic dimension zero. There are many ways of introducing the Higson corona of a metric space. We shall follow the approach developed by I.V. Protasov in [16] and [17].

For an unbounded metric space X, let βX_d be the Stone-Čech compactification of the space X endowed with the discrete topology. The space βX_d consists of all ultrafilters on X and carries the compact Hausdorff topology generated by the sets $\bar{A}=\{p\in\beta X : A\in p\}$ where A runs over all subsets of X. In the space βX_d consider the closed subspace $X^\sharp$ consisting of all ultrafilters which extend the filter $\mathcal{F}_0=\{X\setminus B : B \text{ is a bounded subset of } X\}$ of cobounded subsets of X. Two ultrafilters $p,q\in X^\sharp$ are called *parallel* (denoted by $p\parallel q$) if for some positive real number ε we get $\{B_\varepsilon(P) : P\in p\}\subset q$ and $\{B_\varepsilon(Q) : Q\in q\}\subset p$. Here $B_\varepsilon(A)=\{x\in X : d_X(x,A)\le\varepsilon\}$ denotes the ε-neighborhood of a subset A of a metric space (X,d_X). The *corona* $\check{X}$ of X is defined as the quotient space $X^\sharp/_\sim$ of $X^\sharp$ by the smallest closed equivalence relation $\sim$ on $X^\sharp$ that contains the

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parallel relation $\|$ on $X^\sharp$. For an ultrafilter $p \in X^\sharp$ by $\check{p} \in \check{X}$ we shall denote its equivalence class in the corona $\check{X}$. For a subspace $A \subset X$ we identify the corona $\check{A}$ with the subspace $\{\check{p} : A \in p \in X^\sharp\}$ of $\check{X}$.

By Proposition 1 of [17], two ultrafilters $p, q \in X^\sharp$ belong to the same equivalence class (which means that $\check{p} = \check{q}$) if and only if for any slowly oscillating function $f : X \to [0,1]$ and its Stone-Čech extension $\beta f : \beta X_d \to [0,1]$ we get $\beta f(p) = \beta f(q)$. This allows us to define the corona $\check{X}$ of X using slowly oscillating functions. Let us recall that a function $f : X \to \mathbb{R}$ is *slowly oscillating* if for any $\varepsilon > 0$ and any $\delta < \infty$ there is a bounded subset $B \subset X$ such that for each subset $A \subset X \setminus B$ of diameter $\operatorname{diam} A \le \delta$ the image $f(A)$ has diameter $\operatorname{diam} f(A) \le \varepsilon$. It follows that for a proper metric space X the corona $\check{X}$ of X coincides with the Higson corona $\nu(X)$ defined in [19]. Let us recall that a metric space X is *proper* if each closed bounded subset of X is compact.

It is known that the coronas $\check{X}$ and $\check{Y}$ of two metric spaces (X, d_X) and (Y, d_Y) are homeomorphic if the metric spaces X, Y are *coarsely equivalent* in the sense that there are two coarse functions $f : X \to Y$ and $g : Y \to X$ such that

$$\max\{\sup_{y \in Y} d_Y(f \circ g(y), y), \sup_{x \in X} d_X(g \circ f(x), x)\} < \infty.$$

A function $f : X \to Y$ between two metric spaces (X, d_X) and (Y, d_Y) is called *coarse* if for any $\delta < \infty$ there is $\varepsilon < \infty$ such that for any points $x, x' \in X$ with $d_X(x, x') \le \delta$ we get $d_Y(f(x), f(x')) \le \varepsilon$.

The topological structure of the corona $\check{X}$ reflects certain asymptotic properties of the metric space X, in particular, the asymptotic dimension of X. Let us recall that a metric space X has asymptotic dimension $\operatorname{asdim}(X) \le n$ if for any $\varepsilon < \infty$ there is a cover $\mathcal{U}$ of X such that $\sup_{U \in \mathcal{U}} \operatorname{diam}(U) < \infty$ and each ε-ball $B_\varepsilon(x)$, $x \in X$, meets at most $(n+1)$ sets of the cover $\mathcal{U}$. The finite or infinite number

$$\operatorname{asdim}(X) = \min\{n \in \mathbb{N} \cup \{\infty\} : \operatorname{asdim}(X) \le n\}$$

is called the *asymptotic dimension* of X, see [5].

By [10] or [5, §5], for a proper metric space X of finite asymptotic dimension $\operatorname{asdim}(X)$, the corona $\check{X}$ has topological dimension $\dim(\check{X}) = \operatorname{asdim}(X)$. However it is not known if the asymptotic dimension $\operatorname{asdim}(X)$ is finite provided that the topological dimension $\dim(\check{X})$ of the corona $\check{X}$ is finite (cf. [5, §5]). In Theorem 3.1 we shall give an affirmative answer to this problem for metric spaces X with zero-dimensional corona $\check{X}$.

It follows that for two proper metric spaces X, Y with different finite asymptotic dimensions the coronas $\check{X}$ and $\check{Y}$ are not homeomorphic as they have different topological dimensions. On the other hand, for metric spaces of asymptotic dimension zero I.V. Protasov [18] proved the following striking consistency result.

Theorem 1.1 (Protasov)**.** *Under Continuum Hypothesis the corona $\check{X}$ of any asymptotically zero-dimensional unbounded separable metric space X is homeomorphic to the Stone-Čech remainder $\omega^* = \beta\omega \setminus \omega$ of the countable discrete space ω.*

In a private communication with the first author, I.V. Protasov asked if his Theorem 1.1 remains true in ZFC. In this paper we shall give a negative answer to this question of Protasov, calculating the minimal character $\mathsf{m}\chi(\check{X})$ of the corona $\check{X}$ for a metric space X.

By definition, the *minimal character* $\mathsf{m}\chi(X)$ of a topological space X is the smallest character $\min_{x\in X}\chi(x;X)$ of a point x in X, where the *character* $\chi(x;X)$ of x in X is equal to the smallest cardinality of a neighborhood base at x. The minimal character $\mathsf{m}\chi(\omega^*)$ of the Stone-Čech remainder $\omega^* = \beta\omega\setminus\omega$ is denoted by $\mathfrak{u}$ and is one of important small uncountable cardinals, see [9], [20], [7]. Another small uncountable cardinal that will appear in our considerations is the dominating number $\mathfrak{d}$, equal to the cofinality of the partially ordered set $(\omega^\omega, \le)$, see [9], [20], [7].

The cardinals $\mathfrak{u}$ and $\mathfrak{d}$ both are equal to the continuum $\mathfrak{c}$ under Continuum Hypothesis and more generally under Martin's Axiom, see [20], [13]. On the other hand, the strict inequalities $\mathfrak{u} < \mathfrak{d}$ and $\mathfrak{u} > \mathfrak{d}$ also are consistent with ZFC, see [7, p. 480].

Following [1], we shall say that a metric space (X,d) has *asymptotically isolated balls* if there is $\varepsilon < \infty$ such that for any finite $\delta \ge \varepsilon$ there is $x \in X$ such that the ε-ball $B_\varepsilon(x)$ centered at x coincides with the δ-ball $B_\delta(x)$.

The principal result of this paper is the following theorem that shows that the conclusion of Protasov's Theorem 1.1 is not true under $\mathfrak{u} < \mathfrak{d}$:

Theorem 1.2. *The corona $\check{X}$ of an unbounded metric space X has minimal character*

$$\mathsf{m}\chi(\check{X}) = \begin{cases} \mathfrak{u} & \text{if } X \text{ contains asymptotically isolated balls,} \\ \max\{\mathfrak{u}, \mathfrak{d}\} & \text{otherwise.} \end{cases}$$

This theorem will be proved in Section 5. Now we shall derive from Theorem 1.2 a corona characterization of the Cantor macro-cube.

The *Cantor macro-cube* $2^{<\mathbb{N}}$ is the metric space

$$2^{<\mathbb{N}} = \{(x_i)_{i=1}^\infty \in \{0,1\}^{\mathbb{N}} : \exists n \in \mathbb{N} \;\; \forall m \ge n \;\; x_m = 0\}$$

endowed with the ultrametric

$$d\big((x_n),(y_n)\big) = \max_{n\in\mathbb{N}} 2^n|x_n - y_n|.$$

By [12], the Cantor macro-cube contains a coarse copy of each asymptotically zero-dimensional metric space of bounded geometry. Let us recall that a metric space X has *bounded geometry* if there is $\varepsilon < \infty$ such that for every $\delta < \infty$ there is an integer number $N \in \mathbb{N}$ such that each δ-ball in X can be covered by $\le N$ balls of radius ε.

The Cantor macro-cube $2^{<\mathbb{N}}$ is an asymptotic counterpart of the Cantor cube 2^ω. According to the classical Brouwer characterization [14, 7.4], a topological space X is homeomorphic to the Cantor cube 2^ω if and only if X is a zero-dimensional compact metrizable space without isolated points. A similar characterization holds also for the Cantor macro-cube [1]: *a metric space X is coarsely equivalent to the Cantor macro-cube $2^{<\mathbb{N}}$ if and only if X is an asymptotically zero-dimensional space of bounded geometry without asymptotically isolated balls.*

This characterization, combined with Theorem 1.2, implies the following "corona" characterization of $2^{<\mathbb{N}}$, which will be proved in Section 6.

Theorem 1.3. *Under $\mathfrak{u} < \mathfrak{d}$ for a metric space X of bounded geometry the following conditions are equivalent:*

(1) *X is coarsely equivalent to $2^{<\mathbb{N}}$;*
(2) *the corona $\check{X}$ of X is homeomorphic to the corona of $2^{<\mathbb{N}}$;*
(3) $\dim \check{X} = 0$ *and* $\mathsf{m}\chi(\check{X}) = \mathfrak{d}$.

Another universal metric space is the *Baire macro-space*

$$\omega^{<\mathbb{N}} = \{(x_i)_{i=1}^\infty \in \omega^{\mathbb{N}} : \exists n \in \mathbb{N} \ \ \forall m \ge n \ \ x_m = 0\}$$

endowed with the ultrametric

$$d\big((x_n),(y_n)\big) = \max(\{0\} \cup \{2^n : x_n \ne y_n\}).$$

The Baire macro-space contains a coarse copy of each separable metric space of asymptotic dimension zero. Metric spaces that are coarsely equivalent to the Baire macro-space $\omega^{<\mathbb{N}}$ have been characterized in [2]. By [18], under CH the coronas of the metric spaces $2^{<\mathbb{N}}$ and $\omega^{<\mathbb{N}}$ are homeomorphic to ω^*.

Problem 1.4. *Can the coronas of the metric spaces $2^{<\mathbb{N}}$ and $\omega^{<\mathbb{N}}$ be homeomorphic under the negation of the Continuum Hypothesis?*

2. Preliminaries

In this section we collect some information that will be used in the next sections.

By a *partial preorder* on a set P we understand any reflexive transitive binary relation $\le$ on P. A subset $A \subset P$ of a partially preordered space $(P,\le)$ is called

- *cofinal* in $(P,\le)$ if for each $x \in X$ there is $y \in A$ with $x \le y$;
- *coinitial* in $(P,\le)$ if for each $x \in X$ there is $y \in A$ with $y \le x$.

The smallest cardinality of a cofinal (resp. coinitial) subset of $(P,\le)$ is denoted by $\mathrm{cof}(P)$ (resp. $\mathrm{coin}(P)$) and called the *cofinality* (resp. *coinitiality*) of $(P,\le)$.

For example, the character $\chi(x,X)$ of a topological space X is equal to the coinitiality of the set $\mathcal{N}_x$ of all neighborhoods of X, partially ordered by the inclusion relation $\subset$.

We shall be interested in the cofinality and coinitiality of some function spaces on metric spaces.

A function $f : X \to Y$ between metric spaces is defined to be *bounded-to-bounded* if a subset $B \subset X$ is bounded in X if and only if its image $f(B)$ is bounded in Y. We shall be especially interested in bounded-to-bounded functions with values in the space ω of non-negative integers, endowed with the standard Euclidean metric. Observe that a subset $B \subset \omega$ is bounded if and only if it is finite. So, a function $\phi : \omega \to \omega$ is bounded-to-bounded if and only if it is *finite-to-one* in the sense that for each $n \in \omega$ the preimage $\phi^{-1}(n)$ is finite.

The family of all bounded-to-bounded functions $f : X \to \omega$ on a metric space X will be denoted by $\omega^{\uparrow X}$. The set $\omega^{\uparrow X}$ carries a natural partial order $\le$ in which $f \le g$ iff $f(x) \le g(x)$ for all $x \in X$.

Lemma 2.1. *For an unbounded metric space X the partially ordered set $(\omega^{\uparrow X}, \le)$ has coinitiality*

$$\mathrm{coin}(\omega^{\uparrow X}) \le \mathfrak{d}.$$

PROOF: Choose any bounded-to-bounded function $\phi : X \to \omega$. By definition of the cardinal $\mathfrak{d} = \mathrm{cof}(\omega^{\uparrow\omega})$, there exits a cofinal set $\mathcal{F} \subset \omega^{\uparrow\omega}$ of cardinality $|\mathcal{F}| = \mathfrak{d}$.

For each function $f \in \mathcal{F}$, consider the function $\bar{f} \in \omega^{\uparrow\omega}$ defined by

$$\bar{f}(n) = \max\big(\{0\} \cup \{k \in \omega : f(k) \le n\}\big).$$

We claim that the family $\mathcal{E} = \{\bar{f} \circ \phi : f \in \mathcal{F}\}$ is coinitial in $\omega^{\uparrow X}$ and hence $\mathrm{coin}(\omega^{\uparrow X}) \le |\mathcal{E}| \le |\mathcal{F}| = \mathfrak{d}$.

Indeed, take any function $g \in \omega^{\uparrow X}$ and consider the function $\tilde{g} \in \omega^{\uparrow\omega}$ defined by

$$\tilde{g}(n) = \min g\big(\phi^{-1}([n,\infty))\big) \quad \text{for} \quad n \in \omega.$$

Next, consider the function $\tilde{f} \in \omega^{\uparrow\omega}$ defined by

$$\tilde{f}(k) = \min(\tilde{g}^{-1}([k+1,\infty)) \quad \text{for} \quad k \in \omega$$

and choose any function $f \in \mathcal{F}$ with $\tilde{f} \le f$.

We claim that $\bar{f} \circ \phi \le g$. Take any point $x \in X$ and consider the number $n = \phi(x)$. Then $\tilde{g}(n) \le g(x)$. Let $k = \tilde{g}(n)$ and observe that

$$n \le \max \tilde{g}^{-1}(k) < \min \tilde{g}^{-1}([k+1,\infty)) = \tilde{f}(k) \le f(k).$$

Now the definition of $\bar{f}(n)$ implies that

$$\bar{f} \circ \phi(x) = \bar{f}(n) \le k = \tilde{g}(n) \le g(x).$$

□

Now consider the space $\omega^{\uparrow\omega}$ of bounded-to-bounded (=finite-to-one) functions on ω. Besides the coinitiality of the partial order $\le$ on $\omega^{\uparrow\omega}$ we shall be interested in the coinitiality of $\omega^{\uparrow\omega}$ endowed with the linear preorder $\le_{\mathcal{U}}$ generated by an ultrafilter $\mathcal{U} \in \omega^*$. For two functions $f, g \in \omega^{\uparrow\omega}$ we write $f \leq_{\mathcal{U}} g$ if the set $\{n \in \omega : f(n) \leq g(x)\}$ belongs to the ultrafilter $\mathcal{U}$. Following [4], we denote by $\mathfrak{q}(\mathcal{U}) = \mathrm{coin}(\omega^{\uparrow\omega}, \leq_{\mathcal{U}})$ and $\mathfrak{d}(\mathcal{U}) = \mathrm{cof}(\omega^{\uparrow\omega}, \leq_{\mathcal{U}})$ the coinitiality and the cofinality of the linearly preordered space $(\omega^{\uparrow\omega}, \leq_{\mathcal{U}})$. It is clear that $\max\{\mathfrak{q}(\mathcal{U}), \mathfrak{d}(\mathcal{U})\} \leq \mathfrak{d}$. In [8] M. Canjar constructed a ZFC-example of an ultrafilter $\mathcal{U} \in \omega^*$ with $\mathfrak{q}(\mathcal{U}) = \mathfrak{d}(\mathcal{U}) = \mathrm{cf}(\mathfrak{d})$, which can be consistently smaller than $\mathfrak{d}$.

The following lemma can be proved by analogy with Theorem 16 of [6], see also Theorem 9.4.6 of [4] or [3, pp. 82, 85]. In this lemma $\chi(\mathcal{U})$ denotes the character of an ultrafilter $\mathcal{U} \in \omega^*$ in the Stone-Čech compactification $\beta(\omega)$ of ω.

Lemma 2.2. *Any ultrafilter $\mathcal{U} \in \omega^*$ with character $\chi(\mathcal{U}) < \mathfrak{d}$ has $\mathfrak{q}(\mathcal{U}) = \mathfrak{d}(\mathcal{U}) = \mathfrak{d}$. Consequently,*

$$\max\{\chi(\mathcal{U}), \mathfrak{q}(\mathcal{U})\} = \max\{\chi(\mathcal{U}), \mathfrak{d}(\mathcal{U})\} = \max\{\chi(\mathcal{U}), \mathfrak{d}\} \geq \max\{\mathfrak{u}, \mathfrak{d}\}$$

for any ultrafilter $\mathcal{U} \in \omega^*$.

We shall need to generalize the definition of a ball $B_\varepsilon(x)$ to allow the radius to take a function value. Namely, for a function $f : X \to [0, \infty)$ defined on a metric space X, a point $x \in X$ and a subset $A \subset X$, let $B(x, f) = \{y \in X : d(y, x) \leq f(x)\} = B_{f(x)}(x)$ and

$$B(A, f) = \bigcup_{a \in A} B(a, f).$$

The set $B(A, f)$ is called the *f-neighborhood* of A in X. Sometimes for a real number $\varepsilon \geq 0$ we shall use the notation $B(x, \varepsilon)$ instead of $B_\varepsilon(x)$ identifying ε with the constant function $\varepsilon : X \to \{\varepsilon\} \subset [0, \infty)$.

For a set $A \subset X$ and a function $f : X \to [0, \infty)$, the f-neighborhood $B(A, f) \subset X$ determines the closed-and-open set $\bar{B}(A, f) = \{p \in X^\sharp : B(A, f) \in p\}$ in the compact Hausdorff space $X^\sharp \subset \beta X$ and the closed subset $\check{B}(A, f) = \{\check{p} : p \in \bar{B}(A, f)\}$ in the corona $\check{X}$ of X.

We shall use the following description of the topology $\check{X}$, mentioned in [18].

Lemma 2.3. *For each ultrafilter $p \in X^\sharp$ the family*

$$\{\check{B}(P, f) : P \in p,\ f \in \omega^{\uparrow X}\}$$

is a base of closed neighborhoods of $\check{p}$ in $\check{X}$.

This lemma implies an easy criterion for recognizing ultrafilters $p, q \in X^\sharp$ with different images $\check{p}$, $\check{q}$. We say that two subsets P, Q of a metric space (X, d) are *asymptotically disjoint* if for each real number $\varepsilon > 0$ the intersection $B(P, \varepsilon) \cap B(Q, \varepsilon)$ is bounded in X. This is equivalent to the existence of a bounded-to-bounded function $f \in \omega^{\uparrow X}$ such that the intersection $B(P, f) \cap B(Q, f)$ is bounded.

The following fact was proved by I.V.Protasov in Lemma 4.2 of [16].

Lemma 2.4. *For an unbounded metric space X two ultrafilters $p,q \in X^\sharp$ have distinct images $\check p \neq \check q$ in the corona $\check X$ if and only if there are two asymptotically disjoint sets $P,Q \subset X$ such that $P \in p$ and $Q \in q$.*

Proof: If $\check p \neq \check q$, then we can choose two disjoint neighborhoods $O(\check p)$ and $O(\check q)$ of the points $\check p$, $\check q$ in the corona $\check X$. By Lemma 2.3, we can assume that these neighborhoods are of the form $O(\check p) = \check B(P,f)$, $O(\check q) = \check B(Q,f)$ for some sets $P \in p$, $Q \in q$ and some bounded-to-bounded function $f \in \omega^{\uparrow X}$. To see that the sets P,Q are asymptotically disjoint, it suffices to check that the intersection $B(P,f)\cap B(Q,f)$ is bounded. Assuming the opposite, we could find an ultrafilter $r \in X^\sharp$ containing $B(P,f)\cap B(Q,f)$. Then $\check r \in \check B(P,f)\cap \check B(Q,f) = O(\check p)\cap O(\check q)$, which is not possible as the sets $O(\check p)$ and $O(\check q)$ are disjoint. This proves the "only if" part of the lemma.

To prove the "if" part, assume that two ultrafilters $p,q \in X^\sharp$ contain asymptotically disjoint sets $P \in p$, $Q \in q$. Choose a bounded-to-bounded function $f \in \omega^{\uparrow X}$ such that $B(P,f)\cap B(Q,f)$ is bounded. Then $\check B(P,f)$ and $\check B(Q,f)$ are two disjoint neighborhoods of the points $\check p$ and $\check q$, which implies that $\check p \neq \check q$. □

A subset A of a metric space X is called *asymptotically isolated* if A is asymptotically disjoint from its complement $X\setminus A$. This happens if and only if $B(A,f) = A$ for some bounded-to-bounded function $f \in \omega^{\uparrow X}$. For a subset $A \subset X$ let $\check A = \{\check p : A \in p \in X^\sharp\}$.

Lemma 2.5. *A subset $\mathcal U \subset \check X$ is closed-and-open in the corona $\check X$ if and only if $\mathcal U = \check U$ for some asymptotically isolated subset $U \subset X$.*

Proof: Assume that $\mathcal U = \check U$ for some asymptotically isolated subset $U \subset X$. Then $B(U,f) = U$ for some bounded-to-bounded function $f \in \omega^{\uparrow X}$. It follows from Lemma 2.3 that for each ultrafilter $p \in X^\sharp$ with $\check p \in \check U$ the set $\check B(U,f) = \check U$ is a neighborhood of $\check p$, which means that $\check U = \mathcal U$ is open in $\check X$. The set $\check U = \mathcal U$ is closed being a continuous image of the compact subset $\bar U = \{p \in X^\sharp : U \in p\}$.

Now assume that a subset $\mathcal U \subset \check X$ is closed-and-open in $\check X$. Fix any point x_0 in the metric space X. Since the set $\mathcal U$ is open in $\check X$, for each ultrafilter $p \in X^\sharp$ with $\check p \in \mathcal U$, there is a set $P_p \in p$ and a bounded-to-bounded function $f_p \in \omega^{\uparrow X}$ such that $\check B(P_p, 3f_p) \subset \mathcal U$. Replacing f_p by a smaller function, if necessary, we can assume that $B(B(x,f_p),f_p) \subset B(x,3f_p)$ and $f_p(x) \le \frac12 d(x,x_0)$ for each point $x \in X$.

By the compactness of $\mathcal U$, the cover $\{\check B(P_p,f_p) : p \in X^\sharp,\quad \check p \in \mathcal U\}$ has a finite subcover $\{\check B(P_p,f_p) : p \in F\}$ where $F \subset X^\sharp$ is a finite set. Now consider the set $U = \bigcup_{p\in F} B(P_p,f_p)$ and observe that $\check U = \bigcup_{p\in F}\check B(P_p,f_p) = \mathcal U$. Let $f = \min\{f_p : p \in F\}$ and observe that

$$\check B(U,f) = \bigcup_{p\in F}\bigcup_{x\in P_p} B(B(x,f_p),f) \subset \bigcup_{p\in F}\bigcup_{x\in P_p} B(x,3f_p) = \bigcup_{p\in F} B(P_p,3f_p)$$

and hence

$$\mathcal{U} = \check{U} \subset \check{B}(U, f) \subset \bigcup_{p \in F} \check{B}(P_p, 3f_p) \subset \mathcal{U}.$$

The equality $\check{U} = \check{B}(U, f)$ implies that the set $B(U, f) \setminus U$ is bounded. It follows from $f(x) \le \frac{1}{2} d(x, x_0)$, $x \in X$, that the set $D = \{x \in X : B(x, f) \cap (B(U, f) \setminus U) \neq \emptyset\}$ is bounded in X. Now define a bounded-to-bounded function $f_0 \in \omega^{\uparrow X}$ letting $f_0 | D \equiv 0$ and $f_0 | X \setminus D = f | X \setminus D$.

We claim that $B(U, f_0) = U$. Assuming the opposite, find a point $x \in B(U, f_0) \setminus U$ and a point $u \in U$ with $x \in B(u, f_0)$. The definition of the set D guarantees that $u \in D$ and hence $f_0(u) = 0$ and $x = u \in U$, which is a contradiction. The equality $U = B(U, f_0)$ witnesses that the set U with $\check{U} = \mathcal{U}$ is asymptotically isolated. □

Balls $B(x, f)$ with function radius $f \in \omega^{\uparrow X}$ can be used to prove the following characterization of coarse maps in spirit of uniform continuity.

Lemma 2.6. *A bounded-to-bounded function $f : X \to Y$ between metric spaces is coarse if and only if*
$\forall \varepsilon \in \omega^{\uparrow Y}\ \exists \delta \in \omega^{\uparrow X}\ \forall x \in X \quad f(B(x, \delta)) \subset B(f(x), \varepsilon).$

Proof: To prove the "only if" part, assume that the bounded-to-bounded function $f : X \to Y$ is coarse. In this case there is an increasing function $\xi : \omega \to \omega$ such that for any $n \in \omega$ and points $x, x' \in X$ with $d_X(x, x') \le n$ we get $d_Y(f(x), f(x')) \le \xi(n)$. Consider the bounded-to-bounded function $\zeta : \omega \to \omega$, $\zeta : m \mapsto \max\{n \in \omega : \xi(n) \le m\}$ and observe that $\xi \circ \zeta(m) \le m$ for each $m \in \omega$.

Given any bounded-to-bounded function $\varepsilon \in \omega^{\uparrow Y}$, consider the bounded-to-bounded function $\delta : X \to \omega$, $\delta(x) = \zeta \circ \varepsilon \circ f(x)$, and observe that it has the required property: $f(B(x, \delta) \subset B(f(x), \varepsilon)$ for all $x \in X$.

To prove the "if" part, choose any bounded-to-bounded function $\varepsilon \in {\uparrow} X$ and assume that there exists $\delta \in \omega^{\uparrow X}$ such that $f(B(x, \delta)) \subset B(f(x), \varepsilon)$ for all $x \in X$. To show that f is coarse, for each real number r we need to find a real number R such that $f(B_r(x)) \subset B(f(x), R)$. Since the function $\delta : X \to \omega$ is bounded-to-bounded, the set $\Delta = \delta^{-1}([0, r))$ is bounded in X and so is its r-neighborhood $B_r(\Delta) = \bigcup_{x \in \Delta} B(x, r)$. Since the functions f and ε are bounded-to-bounded, the set $f(B_r(\Delta))$ is bounded in Y and $\varepsilon \circ f(B_r(\Delta))$ is bounded in ω. It can be shown that the number

$$R = \max\left\{\varepsilon(r), \operatorname{diam}\left(\varepsilon \circ f(B_r(\Delta))\right)\right\}$$

has the required property: $f(B_r(x)) \subset B_R(f(x))$ for each $x \in X$. □

A function $\phi : X \to Y$ between two metric spaces is called *boundedly oscillating* if there is a real number D such that for any real number ε there is a bounded set $B \subset X$ such that for each point $x \in X \setminus B$ the set $\phi(B_\varepsilon(x))$ has diameter $\operatorname{diam} \phi(B_\varepsilon(x)) \le D$. It is clear that each slowly oscillating function is boundedly oscillating.

The following characterization of boundedly oscillating functions easily follows from the definition.

Lemma 2.7. *A function $\phi : X \to Y$ between metric spaces is boundedly oscillating if and only if there is a bounded-to-bounded function $\varepsilon \in \omega^{\uparrow X}$ such that* $\sup_{x\in X} \operatorname{diam} \phi(B(x,\varepsilon)) < \infty$.

Using Lemma 2.7 it is quite easy to construct boundedly oscillating functions $f : X \to \omega$ with values in ω.

Lemma 2.8. *For each metric space X there is a boundedly oscillating bounded-to-bounded function $\phi : X \to \omega$.*

Proof: Fix any point $x_0 \in X$ and choose an increasing sequence of real numbers $(r_n)_{n\in\omega}$ such that $r_0 < 0$ and $\lim_{n\to\infty} r_{n+1} - r_n = \infty$. Then the function $\phi : X \to \omega$ defined by $\phi^{-1}(n) = B_{r_{n+1}}(x_0) \setminus B_{r_n}(x_0)$ for $n \in \omega$ is boundedly oscillating and bounded-to-bounded. □

Lemma 2.9. *For any boundedly oscillating bounded-to-bounded function $\phi : X \to \omega$ on an unbounded metric space there is a bounded-to-bounded function $\tilde{\varepsilon} \in \omega^{\uparrow\omega}$ such that* $\sup_{x\in X} \operatorname{diam} \phi(B(x, \tilde{\varepsilon}\circ\phi)) < \infty$.

Proof: By Lemma 2.7, there is a bounded-to-bounded function $\varepsilon \in \omega^{\uparrow X}$ such that

$$D = \sup_{x\in X} \operatorname{diam} \phi(B(x,\varepsilon)) < \infty.$$

Since the map $\phi : X \to \omega$ is bounded-to-bounded, there is a bounded-to-bounded function $\tilde{\varepsilon} \in \omega^{\uparrow\omega}$ such that $\tilde{\varepsilon}\circ\phi \le \varepsilon$. Such function $\tilde{\varepsilon}$ can be defined by the formula

$$\tilde{\varepsilon}(n) = \min \varepsilon(\phi^{-1}([n,\infty)) \quad \text{for } n \in \omega.$$

The inequality $\tilde{\varepsilon}\circ\phi \le \varepsilon$ implies

$$\sup_{x\in X} \operatorname{diam} \phi(B(x,\tilde{\varepsilon}\circ\phi)) \le \sup_{x\in X} \operatorname{diam} \phi(B(x,\varepsilon)) < \infty.$$

□

Observe that for a bounded-to-bounded function $\phi : X \to \omega$ defined on an unbounded metric space X and an ultrafilter $p \in X^\sharp$ its image $\beta\phi(p) = \{A \subset \omega : \phi^{-1}(A) \in p\}$ lies in the set $\omega^\sharp = \omega^* \subset \beta\omega$. To shorten notations, we shall denote the image $\beta\phi(p)$ of the ultrafilter p by $\phi(p)$.

3. Dimension of the corona

By [10], for each proper metric space X of finite asymptotic dimension $\operatorname{asdim}(X)$ the corona $\check{X}$ has topological dimension $\dim(\check{X}) = \operatorname{asdim}(X)$. However it is not known if the asymptotic dimension $\operatorname{asdim}(X)$ is finite provided that the topological dimension $\dim(\check{X})$ of the corona $\check{X}$ is finite (cf. [5, §5]). In this section we give an affirmative answer to this problem for metric spaces X with zero-dimensional corona. We shall apply a characterization of asymptotic dimension zero in terms of ε-chains.

Let $\varepsilon \geq 0$ be a real number. By an *ε-chain* in a metric space (X,d) we understand any sequence of points $x_0, \dots, x_n$ of X such that $d(x_{i-1}, x_i) \leq \varepsilon$ for all positive $i \leq n$. For a point $x \in X$ its *ε-component* $C_\varepsilon(x)$ is the set of all points $y \in X$, which can be linked with x by an ε-chain $x = x_0, x_1, \dots, x_n = y$.

Theorem 3.1. *For an unbounded metric space X the following conditions are equivalent:*

(1) *X has asymptotic dimension zero;*
(2) $\sup_{x\in X} \operatorname{diam} C_\varepsilon(x) < \infty$ *for each* $\varepsilon < \infty$;
(3) *the corona $\check{X}$ has topological dimension zero.*

Proof: (1) $\Rightarrow$ (2). Assume that X has asymptotic dimension zero. Then for each $\varepsilon < \infty$ there is a cover $\mathcal{U}$ of X such that $\sup_{U\in\mathcal{U}} \operatorname{diam}(U) < \infty$ and each ε-ball $B_\varepsilon(x)$, $x \in X$, meets a unique set $U \in \mathcal{U}$. Then for each point $x \in X$ its ε-component $C_\varepsilon(x)$ lies in a unique set $U \in \mathcal{U}$, which implies that

$$\sup_{x\in X} \operatorname{diam} C_\varepsilon(x) \leq \sup_{U\in\mathcal{U}} \operatorname{diam}(U) < \infty.$$

The implication (2) $\Rightarrow$ (1) trivially follows from the fact that for each $\varepsilon < \infty$, $\mathcal{U} = \{C_\varepsilon(x) : x \in X\}$ is a disjoint cover of X such that each ε-ball $B_\varepsilon(x)$, $x \in X$, meets a unique set $U \in \mathcal{U}$ (which is equal to $C_\varepsilon(x)$).

(2) $\Rightarrow$ (3) Assume that for each $\varepsilon \geq 0$ the number $\gamma(\varepsilon) = \sup_{x\in X} \operatorname{diam} C_\varepsilon(x)$ is finite. Since the space X is unbounded, the function $\gamma : [0,\infty) \to [0,\infty)$ is bounded-to-bounded.

To show that the corona $\check{X}$ of X has topological dimension zero, fix any ultrafilter $p \in X^\sharp$ and a neighborhood $U \subset \check{X}$ of its equivalence class $\check{p}$. By Lemma 2.3, we can assume that U is of the form $U = \check{B}(P,f)$ where $P \in p$ and $f : X \to \omega$ is a bounded-to-bounded function.

Fix any point $x_0 \in X$ and put $\|x\| = d(x, x_0)$ for any point $x \in X$. Replacing f by a smaller function, if necessary, we can assume that $f(x) \leq \frac{1}{2}\|x\|$. This condition guarantees that for any point $x \in X$ and $y \in B(x,f)$ we get

$$\|y\| = d(y,x_0) \leq d(y,x) + d(x,x_0) \leq f(x) + d(x,x_0) \leq \frac{1}{2}\|x\| + \|x\| = \frac{3}{2}\|x\|$$

and

$$\|x\| = d(x,x_0) \leq d(x,y) + d(y,x_0) \leq f(x) + \|y\| \leq \frac{1}{2}\|x\| + \|y\|,$$

which implies $\frac{1}{2}\|x\| \leq \|y\|$. Consequently,

$$\frac{2}{3}\|y\| \leq \|x\| \leq 2\|y\| \quad \text{for any points } \; x \in X \; \text{ and } \; y \in B(x,f). \tag{1}$$

Consider the bounded-to-bounded function $\varepsilon : X \to [0,\infty)$ defined by

$$\varepsilon(x) = \frac{1}{2}\sup\{\varepsilon \geq 0 : \gamma(\varepsilon) \leq f(x)\} \text{ for } x \in X,$$

and observe that $C_{\varepsilon(x)}(x) \subset B(x, f(x))$ for all $x \in X$. Using the inequalities (1), one can check that the function

$$\delta : X \to [0,\infty), \quad \delta : x \mapsto \inf\{\varepsilon(y) : x \in C_{\varepsilon(y)}(y)\},$$

is bounded-to-bounded. So, we can choose a bounded-to-bounded function $\tilde{f} : X \to \omega$ such that $\tilde{f}(x) \leq \delta(x)$ for all $x \in X$.

The choice of the function ε guarantees that the set $\tilde{P} = \bigcup_{x\in P} C_{\varepsilon(x)}(x)$ belongs to the ultrafilter p and lies in the f-neighborhood $B(P, f)$ of the set P. Moreover, $B(\tilde{P}, \tilde{f}) = \tilde{P}$. Indeed, for each point $x \in \tilde{P}$ we can find a point $y \in P$ with $x \in C_{\varepsilon(y)}(y)$. Then definition of the function δ guarantees that $\tilde{f}(x) \leq \delta(x) \leq \varepsilon(y)$, which implies that $B(x, \tilde{f}) \subset C_{\varepsilon(y)}(y) \subset \tilde{P}$. So, $B(\tilde{P}, \tilde{f}) = \tilde{P}$, which implies that $\check{B}(\tilde{P}, \tilde{f}) \subset \check{B}(P, f)$ is a closed-and-open neighborhood of $\check{p}$ in $\check{X}$.

(3) $\Rightarrow$ (2) To derive a contradiction, assume that $\dim(\check{X}) = 0$ but there is $\varepsilon < \infty$ such that $\sup_{x\in X} \operatorname{diam} C_\varepsilon(x) = \infty$. For two subsets $A, B \subset X$ put $\operatorname{dist}(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}$. Fix any point $\theta \in X$.

Claim 3.2. *There is a sequence $(C_n)_{n\in\omega}$ of bounded ε-connected subsets of X such that* $\operatorname{diam} C_n > n$ *and* $\operatorname{dist}(C_n, C_{<n}) \geq n$ *where* $C_{<n} = B_n(\theta) \cup \bigcup_{k<n} C_k$.

Proof: The sets C_n, $n \in \omega$, will be constructed by induction. Assume that for some number $n \in \omega$ bounded ε-connected sets $C_0, \dots, C_{n-1}$ have been constructed. Consider the bounded set $C_{<n} = B_n(\theta) \cup \bigcup_{k<n} C_k$ and its n-neighborhood $B = B_n(C_{<n}) = \bigcup_{c\in C_{<n}} B_n(c)$.

Now we consider two cases.

(i) $D = \sup_{x\in B} \operatorname{diam} C_\varepsilon(x) < \infty$. Since $\sup_{x\in X} C_\varepsilon(x) = \infty$, we can choose a point $x \in X$ such that $\operatorname{diam} C_\varepsilon(x) > 2\max\{n, D\}$. It follows that $x \notin B$ and moreover, $C_\varepsilon(x) \cap B = \emptyset$ (in the opposite case, for a point $y \in B \cap C_\varepsilon(x)$, its ε-connected component $C_\varepsilon(y) = C_\varepsilon(x)$ has diameter $\operatorname{diam} C_\varepsilon(y) > 2D \geq D$, which contradicts the definition of D). So, $C_\varepsilon(x) \cap B = \emptyset$.

Since $\operatorname{diam} C_\varepsilon(x) > 2n$, we can choose a point $y \in C_\varepsilon(x)$ such that $d(y, x) > n$. By the definition of the set $C_\varepsilon(x)$, the points $x, y \in C_\varepsilon(x)$ can be linked by an ε-chain $x = x_0, \dots, x_m = y$. Then $C_n = \{x_0, \dots, x_m\}$ is a required bounded ε-connected subset of X that has diameter $\operatorname{diam} C_n \geq d(x, y) > n$ and

$$\operatorname{dist}(C_n, C_{<n}) \geq \operatorname{dist}(C_\varepsilon(x), C_{<n}) \geq \operatorname{dist}(X \setminus B, C_{<n}) \geq n.$$

(ii) The second case happens when $\sup_{x\in B} \operatorname{diam} C_\varepsilon(x) = \infty$. In this case we can choose a point $y \in B$ such that $\operatorname{diam} C_\varepsilon(y) > 2(\operatorname{diam}(B) + n + \varepsilon)$. Then there is a point $x \in C_\varepsilon(y)$ with $d(x, y) > \operatorname{diam}(B) + n + \varepsilon$, which can be linked with y by an ε-chain $x = x_0, \dots, x_m = y$. Since $d(x_0, x_m) = d(x, y) > \operatorname{diam}(B) + n + \varepsilon$, we can choose the smallest number $k \le m$ such that $d(x_0, x_k) > n$. Then $d(x_0, x_i) \le n$ for every $i < k$ and hence

$$\begin{aligned} d(x_i, B) &\ge d(x_i, y) - \operatorname{diam}(B) \\ &\ge d(x_0, y) - d(x_0, x_i) - \operatorname{diam}(B) \\ &> \operatorname{diam}(B) + n + \varepsilon - n - \operatorname{diam}(B) = \varepsilon. \end{aligned}$$

Also $d(x_k, B) \ge d(x_{k-1}, B) - d(x_{k-1}, x_k) > \varepsilon - \varepsilon = 0$. Consequently, the bounded ε-connected set $C_n = \{x_0, \dots, x_k\}$ has diameter $\operatorname{diam}(C_n) \ge d(x_0, x_k) > n$ and is disjoint with the set $B = B_n(C_{<n})$, which implies that $\operatorname{dist}(C_n, C_{<n}) \ge n$. This completes the inductive construction. □

Claim 3.2 yields a sequence $(C_n)_{n\in\omega}$ of ε-connected sets such that $\operatorname{diam}(C_n) > n$ and $\operatorname{dist}(C_n, C_{<n}) \ge n$ for each $n \in \omega$. For every $n \in \omega$ choose two points $x_n, y_n \in C_n$ on distance $d(x_n, y_n) > n$. The choice of the sets $C_n \subset X \setminus B_n(\theta)$, $n > 0$, implies that the sequences $\vec{x} = (x_n)_{n\in\omega}$ and $\vec{y} = (y_n)_{n\in\omega}$ tend to infinity and the sets $P = \{x_n\}_{n\in\omega}$ and $Q = \{y_n\}_{n\in\omega}$ are unbounded and asymptotically disjoint.

The sequences $\vec{x}$ and $\vec{y}$ can be thought as functions $\vec{x} : \omega \to X$ and $\vec{y} : \omega \to Y$ and so have the Stone-Čech extensions $\beta\vec{x} : \beta\omega \to \beta X_d$ and $\beta\vec{y} : \beta\omega \to \beta X_d$. Since the sequences $\vec{x}$ and $\vec{y}$ tend to infinity, $\beta\vec{x}(\omega^*) \cup \beta\vec{y}(\omega^*) \subset X^\sharp$. Take any free ultrafilter $\mathcal{F} \in \omega^*$ and consider its images $p = \beta\vec{x}(\mathcal{F}) \in X^\sharp$ and $q = \beta\vec{y}(\mathcal{F}) \in X^\sharp$. Since the sets $\vec{x}(\omega) \in p$ and $\vec{y}(\omega) \in q$ are asymptotically disjoint, $\check{p} \ne \check{q}$ according to Lemma 2.4.

Since the space $\check{X}$ has topological dimension zero, there are disjoint open-and-closed sets $\mathcal{U}, \mathcal{V} \subset \check{X}$ such that $\check{p} \in \mathcal{U}$ and $\check{q} \in \mathcal{V}$. By Lemma 2.5 there are asymptotically isolated sets $U, V \subset X$ such that $\mathcal{U} = \check{U}$ and $\mathcal{V} = \check{V}$. Since U, V are asymptotically isolated in X, there is a bounded-to-bounded function $f \in \omega^{\uparrow X}$ such that $B(U, f) = U$ and $B(V, f) = V$.

It follows from $\check{U} \cap \check{V} = \mathcal{U} \cap \mathcal{V} = \emptyset$ that the intersection $U \cap V$ is bounded. Choose $n \in \omega$ so large that

- the n-ball $B_n(\theta)$ contains the bounded set $U \cap V$, and
- $f(x) > \varepsilon$ for each $x \in X \setminus B_n(\theta)$.

It follows from $\check{p} \in \mathcal{U} = \check{U}$ and $\check{q} \in \mathcal{V} = \check{V}$ that $U \in p = \beta\vec{x}(\mathcal{F})$ and $V \in q = \vec{y}(\mathcal{F})$. Consider the (infinite) set $F = \vec{x}^{-1}(U \setminus B_n(\theta)) \cap \vec{y}^{-1}(V \setminus B_n(\theta)) \in \mathcal{F}$. Choose any number $m \in F$ with $m > n$ and consider the ε-connected set C_m. By Claim 3.2, $C_m \cap B_n(\theta) \subset C_m \cap B_m(\theta) = \emptyset$. Choose an ε-chain $x_m = z_0, \dots, z_k = y_m$ linking the points x_m and y_m of the set C_m. Observe that $z_0 = x_m \in U \setminus B_n(\theta)$ and $z_k = y_m \in V \setminus B_n(\theta) \subset X \setminus U$. So, the largest number $l \le k$ such that $z_l \in U$ is not equal to k. It follows from $z_l \in C_m \subset X \setminus B_m(\theta) \subset X \setminus B_n(\theta)$ and the choice of the number n that $f(z_l) > \varepsilon$.

Then $z_{l+1} \in B_\varepsilon(z_l) \subset B_{f(z_l)}(z_l) = B(z_l, f) \subset B(U, f) = U$, which contradicts the definition of l. □

4. Evaluating the character of a point in the corona

In this section, for an unbounded metric space (X,d) and an ultrafilter $p \in X^\sharp$ we shall evaluate the character $\chi(\check{p}, \check{X})$ of the point $\check{p}$ in the corona $\check{X}$ of X.

First we derive an upper bound on $\chi(\check{p}, \check{X})$ from Lemmas 2.1 and 2.3.

Lemma 4.1. *For each ultrafilter $p \in X^\sharp$ the point $\check{p} \in \check{X}$ has character*

$$\chi(\check{p}, \check{X}) \le \max\{\chi(p, X^\sharp), \mathfrak{d}\}.$$

Proof: Let $\kappa = \max\{\chi(p, X^\sharp), \mathfrak{d}\}$. Since $\chi(p, X^\sharp) \le \kappa$, there is a family $\mathcal{P} \subset p$ of cardinality $|\mathcal{P}| = \chi(p, X^\sharp) \le \kappa$ such that for each set $P \in p$ there is a set $Q \in \mathcal{P}$ with $\bar{Q} \subset \bar{P}$, where $\bar{Q} = \{q \in X^\sharp : Q \in q\}$. We claim that the complement $Q \setminus P$ is bounded. In the other case, there is an ultrafilter $q \in X^\sharp$ such that $Q \setminus P \in p$. Then $q \in \bar{Q} \setminus \bar{P}$, which is a contradiction.

Fix any point $\theta \in X$ and consider the enriched family $\mathcal{P}' = \{P \setminus B_n(\theta) : P \in \mathcal{P},\ n \in \omega\} \subset p$. It is clear that $|\mathcal{P}'| \le \aleph_0 \cdot |\mathcal{P}| \le \kappa$ and for each set $P \in p$ there is a set $P' \in \mathcal{P}'$ with $P' \subset P$.

By Lemma 2.1, the partially ordered set $(\omega^{\uparrow\omega}, \le)$ has coinitiality $\mathrm{coin}(\omega^{\uparrow X}) \le \mathfrak{d}$. So, we can find a coinitial set $\mathcal{F} \subset \omega^{\uparrow X}$ of cardinality $|\mathcal{F}| \le \mathfrak{d}$.

It follows that for each set $P \in p$ and a function $g \in \omega^{\uparrow X}$ there is a set $P' \in \mathcal{P}'$ and a function $f \in \mathcal{F}$ such that $P' \subset P$ and $f \le g$. Then $p \in \bar{B}(P', f) \subset \bar{B}(P, g)$ and hence $\check{p} \in \check{B}(P', f) \subset \check{B}(P, g)$, which implies that $\{\check{B}(P, f) : P \in \mathcal{P}',\ f \in \mathcal{F}\}$ is a neighborhood base at $\check{p}$ and $\chi(\check{p}, \check{X}) \le |\mathcal{P}'| \cdot |\mathcal{F}| \le \kappa$. □

Lemma 4.2. *If $\phi : X \to \omega$ is a boundedly oscillating bounded-to-bounded function, then for each ultrafilter $p \in X^\sharp$ the point $\check{p} \in \check{X}$ has character*

$$\chi(\check{p}, \check{X}) \ge \chi(\phi(p), \omega^*).$$

Proof: Assume conversely that the cardinal $\kappa = \chi(\check{p}, \check{X})$ is smaller than $\chi(\phi(p), \omega^*)$. Using Lemma 2.3, choose a transfinite sequence of pairs $(P_\alpha, f_\alpha) \in p \times \omega^{\uparrow X}$, $\alpha < \kappa$, such that for each pair $(P, f) \in p \times \omega^{\uparrow X}$ there is an ordinal $\alpha < \kappa$ with $\check{B}(P_\alpha, f_\alpha) \subset \check{B}(P, f)$.

By Lemma 2.9, there is a function $\tilde{f} \in \omega^{\uparrow\omega}$ such that

$$D = \sup_{x \in X} \operatorname{diam} \phi\big(B(x, \tilde{f} \circ \phi)\big) < \infty.$$

Let $f = \tilde{f} \circ \phi$ and choose any natural number $l > 2D$.

Since $\phi(p)$ is an ultrafilter on $\omega = \bigcup_{i=0}^{l-1} l\omega + i$, there is a non-negative integer number $i < d$ such that the set $l\omega + i = \{ln + i : n \in \omega\}$ belongs to $\phi(p)$.

For every $\alpha < \kappa$ consider the set $Q_\alpha = (l\omega + i) \cap \phi(P_\alpha) \in \phi(p)$. Since the family $\{Q_\alpha\}_{\alpha<\kappa}$ has cardinality $\le \kappa < \chi(\phi(p), \omega^*)$, there exists a set $Q \in \phi(p)$ such that $Q_\alpha \setminus Q$ is infinite for all $\alpha < \kappa$.

Let $P = \phi^{-1}(Q \cap (l\omega + i))$ and for the neighborhood $\check{B}(P, g)$ of $\check{p}$ in $\check{X}$ find an ordinal $\alpha < \kappa$ such that $\check{B}(P_\alpha, f_\alpha) \subset \check{B}(P, f)$. By the choice of the set Q, the complement $Q_\alpha \setminus Q$ is infinite. Then we can construct a sequence of points $(a_k)_{k\in\omega}$ such that $\phi(a_k) \in Q_\alpha \setminus Q$ and $\phi(a_{k+1}) > \phi(a_k)$ for every $k \in \omega$.

The set $A = \{a_k\}_{k\in\omega}$ is not bounded because it has infinite image $\phi(A) \subset \omega$ under the bounded-to-bounded function ϕ.

We claim that the sets A and $B(P, f)$ are asymptotically disjoint. This will follow as soon as we check that

$$d(a_k, B(P, f)) \ge f(a_k) = \tilde{f} \circ \phi(a_k).$$

Assume conversely that $d(a_k, x) < f(a_k)$ for some $x \in B(P, f)$ and find a point $y \in P$ such that $x \in B(y, f)$. The choice of the function $f = \tilde{f} \circ \phi$ guarantees that $|\phi(a_k) - \phi(x)| \le \operatorname{diam} \phi(B(a_k, f)) \le D$ and $|\phi(x) - \phi(y)| \le \operatorname{diam} \phi(B(y, f)) \le D$. Taking into account that $\phi(a_k) \in Q_\alpha \subset l\omega + i$ and $\phi(y) \in \phi(P) \subset l\omega + i$, we conclude that $\phi(a_k) - \phi(y) \in l\mathbb{Z}$. This fact combined with the upper bound

$$|\phi(a_k) - \phi(y)| \le |\phi(a_k) - \phi(x)| + |\phi(x) - \phi(y)| \le D + D < l$$

implies that $\phi(a_k) = \phi(y)$, which is not possible as $\phi(y) \in Q$ and $\phi(a_k) \in Q_\alpha \setminus Q$.

This contradiction shows that the sets A and $B(P, f)$ are asymptotically disjoint. Therefore, there exists $q \in A^\sharp$ such that $\check{q} \notin \check{B}(P, f)$ according to Lemma 2.4. On the other hand, $A \subset P_\alpha \subset B(P_\alpha, f_\alpha)$ implies $\check{q} \in \check{B}(P_\alpha, f_\alpha) \subset \check{B}(P, f)$. This contradiction completes the proof. $\square$

Lemma 4.3. *If the space X has no asymptotically isolated balls, then for each boundedly oscillating bounded-to-bounded function $\phi : X \to \omega$ and each ultrafilter $p \in X^\sharp$ the point $\check{p} \in \check{X}$ has character $\chi(\check{p}, \check{X}) \ge \mathfrak{q}(\phi(p))$.*

PROOF: Given any ultrafilter $p \in X^\sharp$, we need to check that $\chi(\check{p}) \ge \mathfrak{q}(\phi(p))$. To derive a contradiction, assume that the cardinal $\kappa = \chi(\check{p})$ is smaller than $\mathfrak{q}(\phi(p))$.

Using Lemma 2.3, choose a transfinite sequence of pairs $\{(P_\alpha, f_\alpha)\}_{\alpha<\kappa} \subset p \times \omega^{\uparrow X}$ such that for each $(P, f) \in p \times \omega^{\uparrow X}$ there is $\alpha < \chi(\check{p})$ such that $\check{B}(P_\alpha, f_\alpha) \subset \check{B}(P, f)$.

For every $\alpha < \kappa$ choose a bounded-to-bounded function $\tilde{f}_\alpha : \omega \to \omega$ such that $\tilde{f}_\alpha \circ \phi \le f_\alpha$. Such a function $\tilde{f}_\alpha$ can be defined by the formula $\tilde{f}_\alpha(n) = \min f_\alpha\big(\phi^{-1}([n, \infty))\big)$ for $n \in \omega$. Since $\kappa < \mathfrak{q}(\phi(p)) = \operatorname{coin}(\omega^{\uparrow\omega}, \le_{\phi(p)})$, there exists a non-decreasing function $\tilde{f} \in \omega^{\uparrow\omega}$ such that $\tilde{f} \le_{\phi(p)} \tilde{f}_\alpha$ for all $\alpha < \kappa$.

Since the function $\phi : X \to \omega$ is boundedly oscillating and bounded-to-bounded we can replace $\tilde{f}$ by a smaller function, if necessary and assume additionally that

$$D = \sup_{x\in X} \operatorname{diam} \phi(B(x, \tilde{f} \circ \phi)) < \infty,$$

see Lemma 2.9. Let $f = \tilde{f} \circ \phi \in \omega^{\uparrow X}$ and choose an integer number $l > 3D$.

Since X has no asymptotically isolated balls, there exists a non-decreasing function $\rho \in \omega^{\uparrow\omega}$ such that $\rho(n) \ge n$ and $B(x, \rho(n)) \not\subset B(x, n)$ for all $n \in \omega$ and $x \in X$. Let $n_0 \ge D$ be an integer number such that $\tilde{f}(n_0) \ge 4\rho(0)$. For every $n < n_0$ put $g(n) = 0$ and for every $n \ge n_0$ let $\tilde{g}(n)$ be the largest number $m \in \omega$

such that $\rho(6m) \le \frac{1}{4}\tilde f(n)$. In this way we define a non-decreasing bounded-to-bounded function $\tilde g : \omega \to \omega$ such that

$$6\tilde g(n) \le \rho(6\tilde g(n)) \le \tfrac{1}{4}\tilde f(n) \quad \text{for all} \quad n \ge n_0.$$

The function $\tilde g$ induces a bounded-to-bounded function $g = \tilde g \circ \phi : X \to \omega$.

For every $n \in \omega$ using Zorn's Lemma, choose a maximal subset $S_n \subset \phi^{-1}(n)$, which is $\tilde f(n)$-*separated* in the sense that $d(x,y) \ge \tilde f(n)$ for any distinct points $x, y \in S_n$.

For every $i < l$, consider the set $X_i = \phi^{-1}(l\omega + i) \subset X$ where $l\omega + i = \{ln + i : n \in \omega\}$. Divide each set X_i into two subsets

$$B_i = X_i \cap \bigcup_{n \in l\omega + i} B(S_n, 2g) \quad \text{and} \quad A_i = X_i \setminus B_i.$$

Since p is an ultrafilter, there is a set $P \in p$ such that $P = A_i$ or $P = B_i$ for some $0 \le i < l$. By Lemma 2.3, the set $\check B(P, g)$ is a neighborhood of $\check p$ in $\check X$, so we can find an ordinal $\alpha < \kappa$ such that $\check B(P_\alpha, f_\alpha) \subset \check B(P, g)$.

By the choice of the function $\tilde f$, the set $\tilde Q_\alpha = \{n \in \omega : \tilde f(n) \le \tilde f_\alpha(n)\}$ belongs to the ultrafilter $\phi(p)$. Then the set

$$Q_\alpha = P \cap P_\alpha \cap \phi^{-1}\big(\tilde Q_\alpha \cap (l\omega + i)\big)$$

belongs to the ultrafilter p and hence is unbounded. This allows us to choose a sequence of points $(a_k)_{k\in\omega}$ in Q_α such that $\phi(a_{k+1}) > \phi(a_k) + 2 > n_0 + 2$ for every $k \in \omega$.

Now we consider two cases.

1) $P = A_i$. For every $k \in \omega$ the maximality of the $\tilde f(\phi(a_k))$-separated set $S_{\phi(a_k)} \subset \phi^{-1}(\phi(a_k)) \subset X_i$ yields a point $s_k \in S_{\phi(a_k)}$ such that $d(a_k, s_k) < \tilde f(\phi(a_k)) = f(a_k)$. Since $\phi(s_k) = \phi(a_k) \to \infty$, the set $\Sigma = \{s_k\}_{k\in\omega}$ is unbounded and hence belongs to some ultrafilter $q \in X^\sharp$.

We claim that $\check q \in \check B(P_\alpha, f_\alpha) \setminus \check B(P, g)$, which will contradict the choice of α.

To see that $\check q \in \check B(P_\alpha, f_\alpha)$, observe that for every $k \in \omega$ we get $\phi(a_k) \in \tilde Q_\alpha$ and hence $\tilde f \circ \phi(a_k) \le \tilde f_\alpha \circ \phi(a_k) \le f_\alpha(a_k)$. This implies

$$s_k \in B(a_k, \tilde f \circ \tilde\phi(a_k)) \subset B(a_k, f_\alpha) \subset B(P_\alpha, f_\alpha)$$

and $\Sigma \subset B(P_\alpha, f_\alpha)$.

Lemma 2.4 will imply that $\check q \notin \check B(P, g)$ as soon as we show that the sets $\Sigma = \{s_k\}_{k\in\omega}$ and $B(P, g)$ are asymptotically disjoint. This will follow as soon as we check that $d(s_k, B(P, g)) \ge g(s_k)$ for every $k \in \omega$. Assume conversely that $d(s_k, x) < g(s_k)$ for some $x \in B(P, g)$. Since $d(s_k, x) < g(s_k) = \tilde g \circ \phi(s_k) \le \tilde f \circ \phi(s_k) = f(s_k)$, the choice of the function $\tilde f$ guarantees that $|\phi(x) - \phi(s_k)| \le \operatorname{diam} \phi\big(B(s_k, f)\big) \le D$.

Since $x \in B(P, g)$, there is a point $y \in P$ with $d(x, y) \le g(y)$. The inequality $d(x, y) \le g(y) = \tilde g \circ \phi(y) \le \tilde f \circ \phi(y)$ implies that $|\phi(x) - \phi(y)| \le l$. It follows from

$\phi(s_k) - \phi(y) \in (l\omega + i) - (l\omega + i) = l\mathbb{Z}$ and

$$|\phi(s_k) - \phi(y)| \le |\phi(s_k) - \phi(x)| + |\phi(x) - \phi(y)| \le D + D < l$$

that $\phi(s_k) = \phi(y) = n$ for some number $n \in \omega$. Taking into account that $y \in P = A_i = X_i \setminus B_i \subset X_i \setminus B(s_k, 2\tilde{g}(n))$, we conclude that $d(y, s_k) > 2\tilde{g}(n)$ and hence

$$d(x, s_k) \ge d(y, s_k) - d(x, y) > 2\tilde{g}(n) - g(\phi(y)) = 2\tilde{g}(n) - \tilde{g}(n) = \tilde{g}(n) = g(s_k),$$

which contradicts our assumption. So, the sets Σ and $B(P, g)$ are asymptotically disjoint and $\check{q} \notin \check{B}(P, g)$.

2) Now consider the second case $P = B_i$. By the choice of the function ρ, for every $k \in \omega$ there is a point $b_k \in B(a_k, \rho(6g(a_k))) \setminus B(a_k, 6g(a_k))$. Since $d(b_k, a_k) \le \rho(6g(a_k)) = \rho(6\tilde{g} \circ \phi(a_k)) \le \tilde{f} \circ \phi(a_k)$, the choice of the number D and the function $\tilde{f}$ guarantees that $|\phi(b_k) - \phi(a_k)| \le D$. Since the sequence $(\phi(a_k))_{k\in\omega}$ tends to infinity, so does the sequence $(\phi(b_k))_{k\in\omega}$, which implies that the set $\Sigma = \{b_k\}_{k\in\omega}$ is unbounded. So we can find an ultrafilter $q \in X^\sharp$ with $\Sigma \in q$.

We claim that $\check{q} \in \check{B}(P_\alpha, f_\alpha)$. Indeed, for every $k \in \omega$ we get $\phi(a_k) \in \tilde{Q}_\alpha$ and hence

$$b_k \in B\big(a_k, \rho(6g(a_k))\big) \subset B(a_k, \tilde{f} \circ \phi(a_k)) \subset B(a_k, f_\alpha(a_k)) \subset B(P_\alpha, f_\alpha).$$

Consequently, $\Sigma \subset B(P_\alpha, f_\alpha)$ and $\check{q} \in \check{B}(P_\alpha, f_\alpha)$.

Next, we show that $\check{q} \notin \check{B}(P, g)$. By Lemma 2.4, it suffices to show that the sets Σ and $B(P, g)$ are asymptotically disjoint. Since $\tilde{g}(\phi(b_k) - D) \to \infty$, this will follow as soon as we check that

$$d(b_k, B(P, g)) \ge \tilde{g}(\phi(b_k) - D) \text{ for every } k \in \omega.$$

Assuming the converse, find a point $x \in B(P, g)$ such that $d(b_k, x) < \tilde{g}(\phi(b_k) - D)$.

Since

$$d(a_k, b_k) \le \rho(6\tilde{g}(\phi(a_k))) \le \tilde{f} \circ \phi(a_k),$$

the choice of the number D guarantees that $|\phi(a_k) - \phi(b_k)| \le D$. Taking into account that $a_k \in P = B_i$, find a point $s_k \in S_{\phi(a_k)}$ such that $a_k \in B(s_k, 2g)$ and $\phi(a_k) = \phi(s_k) \in l\omega + i$.

Since

$$d(b_k, x) < \tilde{g}(\phi(b_k) - D) \le \tilde{g}(\phi(b_k)) \le \tilde{f}(\phi(b_k)),$$

the choice of the number D guarantees that $|\phi(b_k) - \phi(x)| \le \operatorname{diam} \phi(B(b_k, f)) \le D$. Since $x \in B(P, g)$, there is a point $y \in P$ such that $x \in B(y, g) \subset B(y, f)$ and hence $|\phi(x) - \phi(y)| \le D$. Since $y \in P = B_i$, there is a point $s \in S_{\phi(y)}$ such that $y \in B(s, 2g)$ and $\phi(s) = \phi(y) \in l\omega + i$.

Taking into account that $\phi(s)-\phi(s_k)\in(l\omega+i)-(l\omega+i)=l\mathbb{Z}$ and

$$\begin{aligned}|\phi(s)-\phi(s_k)|\le{}&|\phi(s)-\phi(y)|+|\phi(y)-\phi(x)|\\&+|\phi(x)-\phi(b_k)|+|\phi(b_k)-\phi(a_k)|+|\phi(a_k)-\phi(s_k)|\\ \le{}& 0+D+D+D+0<l,\end{aligned}$$

we conclude that $\phi(s)=\phi(s_k)$. Let $n=\phi(s)=\phi(s_k)=\phi(a_k)=\phi(y)$.

If $s=s_k$, then

$$\begin{aligned}d(b_k,x)&\ge d(b_k,a_k)-d(a_k,s_k)-d(s_k,s)-d(s,y)-d(x,y)\\&\ge 6g(a_k)-2g(s_k)-0-2g(s)-g(y)\\&=6\tilde g(\phi(a_k))-2\tilde g(\phi(s_k))-2\tilde g(\phi(s))-g(\tilde{(y)})\\&=6\tilde g(n)-2\tilde g(n)-2\tilde g(n)-\tilde g(n)\\&=\tilde g(n)=\tilde g(\phi(a_k))\ge\tilde g(\phi(b_k)-D),\end{aligned}$$

which contradicts the choice of the point x.

If $s\ne s_k$, then $d(s,s_k)\ge\tilde f(n)$ by the choice of the $\tilde f(n)$-separated set S_n and then

$$\begin{aligned}d(b_k,x)&\ge d(s_k,s)-d(s_k,a_k)-d(a_k,b_k)-d(x,y)-d(y,s)e\\&\ge\tilde f(n)-2g(s_k)-\rho(6g(a_k))-g(y)-2g(s)\\&=\tilde f(n)-2\tilde g(n)-\rho(6\tilde g(n))-\tilde g(n)-2\tilde g(n)\\&=\tilde f(n)-\rho(6\tilde g(n))-6\tilde g(n)\ge\tilde f(n)-\rho(6\tilde g(n))-\rho(6\tilde g(n))\\&\ge\tilde f(n)-2\rho(6\tilde g(n))\ge\tilde f(n)-\frac12\tilde f(n)=\frac12\tilde f(n)\\&\ge\tilde g(n)=\tilde g(\phi(a_k))\ge\tilde g(\phi(b_k)-D).\end{aligned}$$

Therefore $d(b_k,B(P,g))\ge\tilde g(\phi(b_k)-D)\to\infty$, which implies that the sets $B=\{b_k\}_{k\in\omega}$ and $B(P,g)$ are asymptotically disjoint and $\check q\notin\check B(P,g)$. □

Lemma 4.4. *If an unbounded metric space X has asymptotically isolated balls, then its corona $\check X$ contains a closed-and-open subset, homeomorphic to ω^* and hence $\mathsf{m}\chi(\check X)\le\mathsf{m}\chi(\omega^*)=\mathfrak{u}$.*

Proof: Since X has asymptotically isolated balls, there is $\varepsilon>0$ such that for each finite $\delta\ge\varepsilon$ there is an ε-ball $B_\varepsilon(x)$ equal to the δ-ball $B_\delta(x)$. In particular, for the number $\delta_0=2\varepsilon$, we can find a point $x_0\in X$ such that $B_\varepsilon(x_0)=B_{\delta_0}(x_0)$. By induction we shall construct an increasing sequence of real numbers $(\delta_n)_{n=1}^\infty$ and a sequence of points $(x_n)_{n\in\omega}$ in X such that for every $n\in\mathbb{N}$ the following conditions are satisfied:

(1) $\delta_n\ge(n+2)\varepsilon$;
(2) $B_{\delta_n-\varepsilon}(x_k)\not\subset B_{2\varepsilon}(x_k)$ for all $k<n$;
(3) $B_{\delta_n}(x_n)=B_\varepsilon(x_n)$.

These conditions imply that for every $k < n$ we get $d_X(x_k, x_n) \geq \delta_n$. Assuming the opposite, we get $x_k \in B_{\delta_n}(x_n) = B_\varepsilon(x_n)$ and hence $d_X(x_k, x_n) < \varepsilon$ and

$$B_{\delta_n - \varepsilon}(x_k) \subset B_{\delta_n}(x_n) = B_\varepsilon(x_n) \subset B_{2\varepsilon}(x_k),$$

which contradicts the condition (2).

Consider the subspace $D = \{x_n\}_{n\in\omega} \subset X$ and its ε-neighborhood

$$D_\varepsilon = \bigcup_{n\in\omega} B_\varepsilon(x_n) = \bigcup_{n\in\omega} B_{\delta_n}(x_n).$$

It follows that the characteristic function $f : X \to \{0,1\}$ of the set D_ε is slowly oscillating. It induces a continuous map $\check{f} : \check{X} \to \{0,1\}$ such that the preimage $\check{f}^{-1}(1)$ is a clopen subset of $\check{X}$ that coincides with the corona $\check{D}_\varepsilon$ of the set D_ε.

It is easy to check that the identity embedding $e : D \to D_\varepsilon$ is a coarse equivalence, which induces a homeomorphism $\check{e} : \check{D} \to \check{D}_\varepsilon$. Since each function on D is slowly oscillating, the corona $\check{D}$ of D coincides with the Stone-Čech remainder $D^\sharp = \beta D \setminus D$ of the discrete space D. Consequently, the corona $\check{X}$ contains a clopen subset $\check{D}_\varepsilon$, which is homeomorphic to $\omega^* = \beta\omega \setminus \omega$ and hence $\mathsf{m}\chi(\check{X}) \leq \mathsf{m}\chi(\check{D}) = \mathsf{m}\chi(\omega^*) = \mathfrak{u}$. □

Lemmas 4.1, 4.2, 4.3 and 2.2 imply the following theorem, which is the main result of this section.

Theorem 4.5. *Let X be an unbounded metric space and $\phi : X \to \omega$ be a boundedly oscillating bounded-to-bounded function. For each ultrafilter $p \in X^\sharp$ the point $\check{p} \in \check{X}$ has character*

(1) $\chi(\check{p}, \check{X}) \leq \max\{\chi(p, X^\sharp), \mathfrak{d}\}$;
(2) $\chi(\check{p}, \check{X}) \geq \chi(\phi(p), \omega^*) \geq \mathfrak{u}$;
(3) $\chi(\check{p}, \check{X}) \geq \max\{\chi(\phi(p), \omega^*), \mathfrak{q}(\phi(p))\} \geq \max\{\mathfrak{u}, \mathfrak{d}\}$ *if the space X has no asymptotically isolated balls.*

5. Proof of Theorem 1.2

We need to prove that for an unbounded metric space X its corona $\check{X}$ has minimal character

- $\mathsf{m}\chi(\check{X}) = \mathfrak{u}$ if X has asymptotically isolated balls and
- $\mathsf{m}\chi(\check{X}) = \max\{\mathfrak{u}, \mathfrak{d}\}$, otherwise.

If X has asymptotically isolated balls, then the corona $\check{X}$ has minimal character $\mathsf{m}\chi(\check{X}) \leq \mathfrak{u}$ by Lemma 4.4. The inequality $\mathsf{m}\chi(\check{X}) \geq \mathfrak{u}$ follows from Theorem 4.5(2).

If X does not have asymptotically isolated balls, then $\mathsf{m}\chi(\check{X}) \geq \max\{\mathfrak{u}, \mathfrak{d}\}$ by Theorem 4.5(3). To prove the reverse inequality, take any injective function $f : \omega \to X$ such that $\lim_{n\to\infty} d(f(n), f(0)) = \infty$. Choose any ultrafilter $\mathcal{U} \in \omega^*$ with $\chi(\mathcal{U}, \omega^*) = \mathfrak{u}$ and consider its image $p = \beta f(\mathcal{U}) \in \beta X$. The choice of the function f guarantees that $p \in X^\sharp$. It follows that $\chi(p, X^\sharp) = \chi(\mathcal{U}, \omega^*) = \mathfrak{u}$ and then

$$\mathsf{m}\chi(\check{X}) \le \chi(\check{p}, \check{X}) \le \max\{\chi(p, X^\sharp), \mathfrak{d}\} = \max\{\mathfrak{u}, \mathfrak{d}\}$$

according to Theorem 4.5(1).

6. Proof of Theorem 1.3

It is easy to see that the Cantor macro-cube $C = 2^{<\mathbb{N}}$ has no asymptotically isolated balls. Consequently, $\mathsf{m}\chi(\check{C}) = \max\{\mathfrak{u}, \mathfrak{d}\} = \mathfrak{d}$ by Theorem 1.2. By [10], $\dim(\check{C}) = \mathrm{asdim}(C) = 0$. Now we are ready to prove the implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (1) of Theorem 1.3. Let (X, d_X) be a metric space of bounded geometry.

(1) $\Rightarrow$ (2). If X is coarsely homeomorphic to the Cantor macro-cube $C = 2^{<\mathbb{N}}$, then the coronas of X and C are homeomorphic according to [19, 2.42].

(2) $\Rightarrow$ (3) If the coronas $\check{X}$ and $\check{C}$ are homeomorphic, then $\dim(\check{X}) = \dim(\check{C}) = \mathrm{asdim}(C) = 0$ and $\mathsf{m}\chi(\check{X}) = \mathsf{m}\chi(\check{C}) = \mathfrak{d}$.

(3) $\Rightarrow$ (1) Assume that $\dim(\check{X}) = 0$ and $\mathsf{m}\chi(\check{X}) = \mathfrak{d} > \mathfrak{u}$. By Proposition 3.1 and Theorem 1.2(1), the metric space X has asymptotic dimension zero and has no asymptotically isolated balls. Since X has bounded geometry, the characterization theorem [1] implies that the metric space X is coarsely equivalent to the Cantor macro-cube $2^{<\mathbb{N}}$.

References

[1] Banakh T., Zarichnyi I., *Characterizing the Cantor bi-cube in asymptotic categories*, Groups Geom. Dyn. **5** (2011), no. 4, 691–728.

[2] Banakh T., Zarichnyi I., *A coarse characterization of the Baire macro-space*, Proc. of Intern. Geometry Center **3** (2010), no. 4, 6–14 (available at http://arxiv.org/abs/1103.5118).

[3] Banakh T., Zdomskyy L., *The coherence of semifilters: a survey. Selection principles and covering properties in topology*, 53–105, Quad. Mat., 18, Dept. Math., Seconda Univ. Napoli, Caserta, 2006.

[4] Banakh T., Zdomskyy L., *Coherence of Semifilters*, book in progress, `http://www.franko.lviv.ua/faculty/mechmat/Departments/Topology/booksite.html`.

[5] Bell G., Dranishnikov A., *Asymptotic dimension*, Topology Appl. **155** (2008), no. 12, 1265–1296.

[6] Blass A., *Near coherence of filters, I. Cofinal equivalence of models of arithmetic*, Notre Dame J. Formal Logic **27** (1986), 579–591.

[7] Blass A., *Combinatorial cardinal characteristics of the continuum*, in: Handbook of Set Theory, Chapter 6, pp. 395–489, Springer, Dordrecht, 2010.

[8] Canjar M., *Cofinalities of countable ultraproducts: the existence theorem*, Notre Dame J. Formal Logic **30** (1989), no. 4, 539–542.

[9] van Douwen E., *The integers and topology*, in: Handbook of Set-theoretic Topology, 111–167, North-Holland, Amsterdam, 1984.

[10] Dranishnikov A., *Asymptotic topology*, Uspekhi Mat. Nauk **55** (2000), no. 6, 71–116.

[11] Dranishnikov A.N., Keesling J., Uspenskij V.V., *On the Higson corona of uniformly contractible spaces*, Topology **37** (1998), no. 4, 791–803.

[12] Dranishnikov A., Zarichnyi M., *Universal spaces for asymptotic dimension*, Topology Appl. **140** (2004), no. 2–3, 203–225.

[13] Fremlin D., *Consequences of Martin's Axiom*, Cambridge Tracts in Mathematics, 84, Cambridge University Press, London, 1984.
[14] Kechris A., *Classical Descriptive Set Theory*, Springer, New York, 1995.
[15] Laflamme C., Zhu J.-P., *The Rudin-Blass ordering of ultrafilters*, J. Symbolic Logic **63** (1998), 584–592.
[16] Protasov I.V., *Normal ball structures*, Mat. Stud. **20** (2003), 3–16.
[17] Protasov I.V., *Coronas of balleans*, Topology Appl. **149** (2005), no. 1–3, 149–160.
[18] Protasov I.V., *Coronas of ultrametric spaces*, Comment. Math. Univ. Carolin. **52** (2011), 303–307.
[19] Roe J., *Lectures on Coarse Geometry*, American Mathematical Society, Providence, RI, 2003.
[20] Vaughan J., *Small uncountable cardinals and topology*, in: Open Problems in Topology, 195–218, North-Holland, Amsterdam, 1990.

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