On McCoy condition and semicommutative rings

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Abstract. Let R be a ring and σ an endomorphism of R. We give a generalization of McCoy's Theorem [*Annihilators in polynomial rings*, Amer. Math. Monthly **64** (1957), 28–29] to the setting of skew polynomial rings of the form $R[x;\sigma]$. As a consequence, we will show some results on semicommutative and σ-skew McCoy rings. Also, several relations among McCoyness, Nagata extensions and Armendariz rings and modules are studied.

Keywords: Armendariz rings; McCoy rings; Nagata extension; semicommutative rings; σ-skew McCoy

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1. Introduction

Throughout the paper, R will always denote an associative ring with identity and M_R will stand for a right R-module. Given a ring R, the polynomial ring with an indeterminate x over R is denoted by $R[x]$. According to Nielsen [20] and Rege and Chhawchharia [22], a ring R is called *right McCoy* (resp., *left McCoy*) if, for any polynomials $f(x), g(x) \in R[x] \setminus \{0\}$, $f(x)g(x) = 0$ implies $f(x)r = 0$ (resp., $sg(x) = 0$) for some $0 \neq r \in R$ (resp., $0 \neq s \in R$). A ring is called *McCoy* if it is both left and right McCoy. By McCoy [18], commutative rings are McCoy rings. Recall that a ring R is *reversible* if $ab = 0$ implies $ba = 0$ for $a, b \in R$, and R is *semicommutative* if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. It is obvious that commutative rings are reversible and reversible rings are semicommutative, but the converse does not hold, respectively. With the help of [8, Theorem 2.2], R is a McCoy ring when $R[x]$ is semicommutative. Nielsen [20, Theorem 2] showed that reversible rings are McCoy and he gave an example of a semicommutative ring which is not right McCoy. Recall that a ring is *reduced* if it has no nonzero nilpotent elements. Rege and Chhawchharia called R an *Armendariz* ring [22, Definition 1.1], if whenever any polynomials $f(x), g(x) \in R[x]$ satisfy $f(x)g(x) = 0$, then $ab = 0$ for each coefficient a of $f(x)$ and b of $g(x)$. Any reduced ring is Armendariz by [2, Lemma 1] and Armendariz rings are clearly McCoy. We have the following diagram:

$$\left.\begin{array}{c} R \text{ is reversible} \\ R[x] \text{ is semicommutative} \\ R \text{ is Armendariz} \end{array}\right\} \Rightarrow R \text{ is McCoy}$$

The Ore extension of a ring R is denoted by $R[x;\sigma,\delta]$, where σ is an endomorphism of R and δ is a σ-derivation, i.e., $\delta\colon R\to R$ is an additive map such that $\delta(ab)=\sigma(a)\delta(b)+\delta(a)b$ for all $a,b\in R$. Recall that elements of $R[x;\sigma,\delta]$ are polynomials in x with coefficients written on the left. Multiplication in $R[x;\sigma,\delta]$ is given by the multiplication in R and the condition $xa=\sigma(a)x+\delta(a)$, for all $a\in R$. For $\delta=0$, we put $R[x;\sigma,0]=R[x;\sigma]$. Başer et al. [6], introduced a concept of σ-skew McCoy for an endomorphism σ of R. A ring R is called *σ-skew McCoy*, if for any nonzero polynomials $p(x)=\sum_{i=0}^{n}a_ix^i$ and $q(x)=\sum_{j=0}^{m}b_jx^j\in R[x;\sigma]$, $p(x)q(x)=0$ implies $p(x)c=0$ for some nonzero $c\in R$, and they have proved the following:

$$\left.\begin{array}{c} R[x;\sigma] \text{ is right McCoy} \\ R[x;\sigma] \text{ is reversible}\end{array}\right\} \Rightarrow R \text{ is } \sigma\text{-skew McCoy}$$

Hong et al. [13, Theorem 1] proved that if σ is an automorphism of R and I a right ideal of $S=R[x;\sigma,\delta]$ then $r_S(I)\neq 0$ implies $r_R(I)\neq 0$, which extends McCoy's Theorem [17].

In this paper, we give another generalization of McCoy's Theorem, by showing that for any right ideal I of $S=R[x;\sigma]$, we have $r_S(I)\neq 0$ implies $r_R(I)\neq 0$ when R is σ-compatible or $r_S(I)$ is σ-ideal. As a consequence, if $R[x;\sigma]$ is semicommutative then R is σ-skew McCoy. Furthermore, we show some results on Nagata extensions. For a commutative ring R, we have

1) If R is a domain, then

  (a) M_R is Armendariz if and only if $R\oplus_\sigma M_R$ is Armendariz;
  (b) the ring $R\oplus_\sigma M_R$ is semicommutative and right McCoy.

A module M_R is called *Armendariz* if whenever polynomials $m=\sum_{i=0}^{n}m_ix^i\in M[x]$ and $f=\sum_{j=0}^{m}a_jx^j\in R[x]$ satisfy $mf=0$, then $m_ia_j=0$ for each i,j.

2) If R and M_R are Armendariz such that M_R satisfies the condition $(\mathcal{C}^2_\sigma)$ (see Definition 2.7), then $R\oplus_\sigma M_R$ is Armendariz.

2. A generalization of McCoy's Theorem

McCoy [17] proved that for any right ideal I of $S=R[x_1,x_2,\ldots,x_n]$ over a ring R, if $r_S(I)\neq 0$ then $r_R(I)\neq 0$. This result was extended by Hong et al. [13] to the Ore extensions of several types, the skew monoid rings and the skew power series rings over noncommutative rings, where σ is an automorphism of R. Herein, we will extend McCoy's Theorem to skew polynomial rings of the form $R[x;\sigma]$ with σ an endomorphism of R. According to Annin [3], a ring R is *σ-compatible*, if for any $a,b\in R$, $ab=0$ if and only if $a\sigma(b)=0$. Let σ be an endomorphism of R and I an ideal of R, we say that the ideal I is *σ-ideal*, if $\sigma(I)\subseteq I$. Let σ be an endomorphism of a ring R, then for any $f(x)=\sum_{i=0}^{n}a_ix^i\in R[x;\sigma]$, we denote by $\sigma(f(x))$ the polynomial $\sum_{i=0}^{n}\sigma(a_i)x^i\in R[x;\sigma]$.

Theorem 2.1. *Let R be a ring, σ an endomorphism of R and I a right ideal in $S = R[x;\sigma]$. Suppose that R is σ-compatible or $r_S(I)$ is σ-ideal. If $r_S(I) \neq 0$ then $r_R(I) \neq 0$.*

Proof: Suppose that $r_S(I) \neq 0$. If $I = 0$, then it's trivial. Assume that $I \neq 0$. Let $g(x) = \sum_{j=0}^{m} b_j x^j \in r_S(I)$ with $b_m \neq 0$. If $m = 0$, then we are done, so we can suppose that $m \geq 1$. In this situation, if $Ib_m = 0$, then we are done. Otherwise, there exists $0 \neq f(x) = \sum_{i=0}^{n} a_i x^i \in I$ such that $f(x)b_m \neq 0$ $(*)$.

If R is σ-compatible, then $(*)$ implies $a_i\sigma^i(b_m) \neq 0$ for some $i \in \{0, 1, \ldots, n\}$, so $a_i b_m \neq 0$ because R is σ-compatible, therefore $a_i g(x) \neq 0$ for some $i \in \{0, 1, \ldots, n\}$. Take $p = \max\{i | a_i g(x) \neq 0\}$, so $a_p g(x) \neq 0$ and $a_{p+1} g(x) = \cdots = a_n g(x) = 0$. On the other hand, we get $a_p b_m = 0$ from $f(x)g(x) = 0$. So that the degree of $a_p g(x)$ is less than m such that $a_p g(x) \neq 0$. But $I(a_p g(x)) = (Ia_p)g(x) = 0$ since I is a right ideal of S, so $0 \neq a_p g(x) \in r_S(I)$. We can write $a_p g(x) = \sum_{k=0}^{\ell} a_p b_k x^k$ with $a_p b_\ell \neq 0$ and $\ell < m$. We have the two possibilities: If $\ell = 0$ then $a_p g(x)$ is a nonzero element in $r_R(I)$. Otherwise, $\ell \geq 1$. Then we will consider $a_p g(x)$ in place of $g(x)$. We have two cases $I(a_p b_\ell) = 0$ or $I(a_p b_\ell) \neq 0$. The first implies $0 \neq a_p b_\ell \in r_R(I)$, for the second, there exists $0 \neq h(x) = \sum_{k=0}^{s} c_k x^k \in I$ such that $h(x)a_p b_\ell \neq 0$. Here, we can find q as the largest integer such that $c_q a_p g(x) \neq 0$ and then $0 \neq c_q a_p g(x) \in r_S(I)$ such that the degree of $c_q a_p g(x)$ is smaller than one of $a_p g(x)$.

If $r_S(I)$ is σ-ideal, then $(*)$ implies $a_i x^i b_m \neq 0$ for some $i \in \{0, 1, \ldots, n\}$, therefore $a_i x^i g(x) \neq 0$. Take $p = \max\{i | a_i x^i g(x) \neq 0\}$, then $a_p\sigma^p(g(x)) \neq 0$ and $a_i x^i g(x) = 0$ for $i \geq p+1$. We obtain $a_p\sigma^p(b_m) = 0$ from $f(x)g(x) = 0$. Also, we have $I(a_p\sigma^p(g(x))) = (Ia_p)\sigma^p(g(x)) = 0$ because I is a right ideal of S and $\sigma^p(g(x)) \in r_S(I)$. So $0 \neq a_p\sigma^p(g(x)) \in r_S(I)$. We can write $a_p\sigma^p(g(x)) = a_p\sigma^p(b_0) + a_p\sigma^p(b_1)x + \cdots + a_p\sigma^p(b_\ell)x^\ell$, where $a_p\sigma^p(b_\ell) \neq 0$ and $\ell < m$. If $\ell = 0$ then $Ia_p\sigma^p(b_\ell) = 0$, so $0 \neq a_p\sigma^p(b_\ell) \in r_R(I)$. Otherwise, $\ell \geq 1$, then we will consider $a_p\sigma^p(g(x))$ in place of $g(x)$ and $0 \neq h(x) = \sum_{k=0}^{s} c_k x^k \in I$ such that $h(x)a_p\sigma^p(b_\ell) \neq 0$. We can find q as the largest integer such that $c_q\sigma^q(a_p\sigma^p(g(x))) \neq 0$ and then $0 \neq c_q\sigma^q(a_p\sigma^p(g(x))) \in r_S(I)$ such that the degree of $c_q\sigma^q(a_p\sigma^p(g(x)))$ is smaller than one of $a_p\sigma^p(g(x))$.

Continuing with the same manner (in the two cases), we can produce elements of the forms $0 \neq a_{t_1}a_{t_2}\ldots a_{t_s}\sigma^{t_1+t_2+\cdots+t_s}g(x)$ (resp., $0 \neq a_{t_1}a_{t_2}\ldots a_{t_s}g(x)$) in $r_S(I)$, with $s \leq m$ and the degree of these polynomials is zero. Thus $a_{t_1}a_{t_2}\cdots a_{t_s}\sigma^{t_1+t_2+\cdots+t_s}g(x) \in r_R(I)$ (resp., $0 \neq a_{t_1}a_{t_2}\ldots a_{t_s}g(x) \in r_R(I)$). Therefore $r_R(I) \neq 0$. $\square$

Corollary 2.2 ([8, Theorem 2.2])**.** *Let $f(x) \in R[x]$. If $r_{R[x]}(f(x)R[x]) \neq 0$ then $r_{R[x]}(f(x)R[x]) \cap R \neq 0$.*

Proof: Consider the right ideal $I = f(x)R[x]$. $\square$

Corollary 2.3. *Let R be a ring, σ an endomorphism of R and I a right ideal of $S = R[x;\sigma]$. If S is semicommutative, then $r_S(I) \neq 0$ implies $r_R(I) \neq 0$.*

Proof: Let I be a right ideal of $S = R[x;\sigma]$, $f(x) \in r_S(I)$ and $g(x) \in I$. Then $g(x)f(x) = 0$. Since S is semicommutative we have $g(x)Sf(x) = 0$, in particular, $g(x)xf(x) = g(x)\sigma(f)(x) = 0$, so $\sigma(f)(x) \in r_S(I)$. Thus $r_S(I)$ is σ-ideal and we have the result by Theorem 2.1. □

Corollary 2.4. *Let σ be an endomorphism of a ring R. If $R[x;\sigma]$ is a semicommutative ring then R is σ-skew McCoy.*

Proof: It follows directly from Corollary 2.3, by letting $I = f(x)R[x;\sigma]$. □

From Corollary 2.4, we obtain immediately [6, Corollary 6] and [8, Corollary 2.3]. According to Clark [7], a ring R is said to be *quasi-Baer* if the right annihilator of each right ideal of R is generated (as a right ideal) by an idempotent. Following Başer et al. [4] and Zhang and Chen [24], a ring R is said to be *σ-semicommutative* if, for any $a, b \in R$, $ab = 0$ implies $aR\sigma(b) = 0$. A ring R is called *right* (*left*) *σ-reversible* [5, Definition 2.1] if whenever $ab = 0$ for $a, b \in R$, $b\sigma(a) = 0$ ($\sigma(b)a = 0$). A ring R is called *σ-reversible* if it is both right and left σ-reversible. Hong et al. [9], proved that, if R is σ-rigid then R is quasi-Baer if and only if $R[x;\sigma]$ is quasi-Baer. Hong et al. [12] have proved the same result when R is semi-prime and all ideals of R are σ-ideals.

Proposition 2.5. *Let R be a σ-semicommutative ring. If $R[x;\sigma]$ is quasi-Baer then R is so.*

Proof: Let I be a right ideal of R. We have $r_{R[x;\sigma]}(IR[x;\sigma]) = eR[x;\sigma]$ for some idempotent $e = e_0 + e_1x + \cdots + e_mx^m \in R[x;\sigma]$. By [4, Proposition 3.9], $r_R(IR[x;\sigma]) = e_0R$. Clearly, $r_R(IR[x;\sigma]) \subseteq r_R(I)$. Conversely, let $b \in r_R(I)$ then $Ib = 0$. Since R is σ-semicommutative, we have $IR[x;\sigma]b = 0$, so $b \in r_R(IR[x;\sigma])$. Therefore $r_R(I) = e_0R$. □

Example 2.6. Let $\mathbb{Z}$ be the ring of integers and consider the ring

$$R = \{(a, b) \in \mathbb{Z} \oplus \mathbb{Z} \mid \ a \equiv b \pmod 2\}$$

and $\sigma\colon R \to R$ defined by $\sigma(a, b) = (b, a)$.

1) $R[x;\sigma]$ is quasi-Baer and R is not quasi-Baer, by [9, Example 9].

2) R is not σ-semicommutative. Let $a = (2, 0)$, $b = (0, 2)$. We have $ab = 0$, but $a\sigma(b) = (2, 0)(2, 0) = (4, 0) \neq 0$. Thus R is not σ-semicommutative. Therefore the condition "R is σ-semicommutative" is not a superfluous condition in Proposition 2.5.

Definition 2.7. Let R be a ring, M_R an R-module and σ an endomorphism of R. For $m \in M_R$ and $a \in R$, we say that M_R satisfies the condition ($\mathcal{C}^1_\sigma$) (resp., ($\mathcal{C}^2_\sigma$)) if $ma = 0$ (resp., $m\sigma(a)a = 0$) implies $m\sigma(a) = 0$.

Proposition 2.8. *Let σ be an endomorphism of a ring R.*

(1) *If R is semicommutative and satisfies the condition $(\mathcal{C}_\sigma^2)$ then it is σ-skew McCoy.*

(2) *If R is reduced and right σ-reversible then it is σ-skew McCoy.*

Proof: (1) Immediately from [23, Proposition 3.4]. (2) Clearly from (1). □

3. Nagata extensions and McCoyness

Let R be a commutative ring, M_R be an R-module and σ an endomorphism of R. The R-module $R \oplus_\sigma M_R$ acquires a ring structure (possibly noncommutative), where the product is defined by $(a,m)(b,n) = (ab, n\sigma(a) + mb)$, for $a, b \in R$ and $m, n \in M_R$. We shall call this extension the *Nagata extension* of R by M_R and σ. If $\sigma = id_R$, then $R \oplus_{id_R} M_R$ (denoted by $R \oplus M_R$) is a commutative ring. Anderson and Camillo [1] have proved that if R is a commutative domain then M_R is Armendariz if and only if $R \oplus M_R$ is Armendariz. We will see that this result holds for $R \oplus_\sigma M_R$ as well. Kim et al. [21] have proved that, if R is a commutative domain and σ is a monomorphism of R then $R \oplus_\sigma R$ is reversible, and so it is McCoy. Recall that if σ is an endomorphism of a ring R, then the map $R[x] \to R[x]$ defined by $\sum_{i=0}^n a_i x^i \mapsto \sum_{i=0}^n \sigma(a_i) x^i$ is an endomorphism of the polynomial ring $R[x]$ and clearly this map extends σ. We shall also denote the extended map $R[x] \to R[x]$ by σ and the image of $f \in R[x]$ by $\sigma(f)$. In this section, we will discuss when the Nagata extension $R \oplus_\sigma M_R$ is McCoy.

Let R be a commutative domain. The set $T(M) = \{m \in M | r_R(m) \neq 0\}$ is called the *torsion submodule* of M_R. If $T(M) = M$ (resp., $T(M) = 0$) then M_R is *torsion* (resp., *torsion-free*).

Lemma 3.1. *If M_R is a torsion-free module then it is Armendariz.*

Proof: Let $m(x) = m_0 + m_1 x + \cdots + m_p x^p \in M[x]$ and $f(x) = a_0 + a_1 x + \cdots + a_q x^q \in R[x]$ such that $m(x)f(x) = 0$. We may assume that $a_0 \neq 0$ (if not, set $f(x) = f'(x)x^k$ with a minimal k such that $a_k \neq 0$). This implies the following system of equations:

$$m_0 a_0 = 0, \tag{0}$$

$$m_0 a_1 + m_1 a_0 = 0, \tag{1}$$

$$m_0 a_2 + m_1 a_1 + m_2 a_0 = 0, \tag{2}$$

$$\cdots$$

$$m_p a_q = 0. \tag{$p+q$}$$

Since M_R is a torsion-free module, then from these equations, we obtain $m_i = 0$ for all $i \in \{0, 1, \ldots, p\}$. Thus M_R is an Armendariz module. □

Proposition 3.2. *Let R be a commutative domain and M_R an R-module. Then $R \oplus_\sigma M_R$ is Armendariz if and only if M_R is Armendariz. In particular, if M_R is torsion-free then $R \oplus_\sigma M_R$ is Armendariz.*

Proof: Let $R' = R \oplus_\sigma M_R$, then we have $R'[x] = R[x] \oplus_\sigma M[x]$. Suppose that R' is Armendariz. Let $m = \sum_{i=0}^{p} m_i x^i \in M[x]$ and $f = \sum_{j=0}^{q} a_j x^j \in R[x]$ with $mf = 0$. We have $(0, m) = \sum_{i=0}^{p}(0, m_i)x^i \in R'[x]$ and $(f, 0) = \sum_{j=0}^{q}(a_j, 0)x^j \in R'[x]$, since R' is Armendariz then $(0, m_i)(a_j, 0) = (0, m_i a_j) = (0, 0)$ for all i, j. Thus $m_i a_j = 0$ for all i, j. Conversely, suppose that M_R is Armendariz. Let $f, g \in R[x]$ and $m, n \in M[x]$ such that $(f, m)(g, n) = (0, 0)$. Write $(f, m) = \sum(a_i, m_i)x^i \in R'[x]$ and $(g, n) = \sum(b_j, n_j)x^j \in R'[x]$. From $(f, m)(g, n) = (0, 0)$, we have $(fg, n\sigma(f) + mg) = (0, 0)$. Since $R[x]$ is a commutative domain, then $f = 0$ or $g = 0$. If $f = 0$, we get $mg = 0$. Then $m_i b_j = 0$ and $a_i = 0$ for all i, j. Thus $(a_i, m_i)(b_j, n_j) = (a_i b_j, n_j\sigma(a_i) + m_i b_j) = (0, 0)$. Otherwise, we get $n\sigma(f) = 0$. Then $b_j = 0$ and $n_j\sigma(a_i) = 0$ for all i, j. Thus $(a_i, m_i)(b_j, n_j) = (a_i b_j, n_j\sigma(a_i) + m_i b_j) = (0, 0)$. Therefore $R \oplus_\sigma M_R$ is Armendariz. In particular, if M_R is torsion-free then M_R is Armendariz by Lemma 3.1. Therefore $R \oplus_\sigma M_R$ is Armendariz. □

Corollary 3.3. *Let R be a commutative domain and M_R an R-module satisfying the condition $(\mathcal{C}^2_{id_R})$. Then $R \oplus_\sigma M_R$ is Armendariz.*

Proof: Since M_R is semicommutative then it is Armendariz by [23, Lemma 3.3]. □

Proposition 3.4. *Let R be a commutative ring and M_R an R-module such that R satisfies $(\mathcal{C}^1_\sigma)$ and M_R satisfies $(\mathcal{C}^2_\sigma)$. Then $R \oplus_\sigma M_R$ is a semicommutative ring.*

Proof: We will use freely the conditions $(\mathcal{C}^1_\sigma)$ and $(\mathcal{C}^2_\sigma)$. Let $(r, m), (s, n) \in R \oplus_\sigma M_R$ such that

$$(r, m)(s, n) = (rs, n\sigma(r) + ms) = (0, 0). \tag{1}$$

We will show that for any $(t, u) \in R \oplus_\sigma M_R$

$$(r, m)(t, u)(s, n) = (rts, n\sigma(rt) + u\sigma(r)s + mts) = (0, 0). \tag{2}$$

It suffices to show $n\sigma(rt) + u\sigma(r)s + mts = 0$. Multiplying $n\sigma(r) + ms = 0$ of equation (1) on the right hand by r, gives $n\sigma(r)r = 0$, so we get $n\sigma(r) = 0$ and hence $ms = 0$. Thus $n\sigma(rt) = mts = 0$. Clearly $rs = 0$ implies $\sigma(r)s = 0$ and so $u\sigma(r)s = 0$. Therefore $n\sigma(rt) + u\sigma(r)s + mts = 0$. □

Proposition 3.5. *Let R be a commutative domain and M_R an R-module. Then $R \oplus_\sigma M_R$ is a semicommutative right McCoy ring.*

Proof: Consider equations (1) and (2) of Proposition 3.4. From equation (1), we get $r = 0$ or $s = 0$ since R is a domain. Say $r = 0$, then $rts = n\sigma(rt) = u\sigma(r)s = 0$, and $mts = 0$ from (1), hence we have (2). Next say $s = 0$, it follows $rts = u\sigma(r)s = mts = 0$ and $n\sigma(rt) = 0$ from (1), and so we have (2). Therefore $(r, m)(R \oplus_\sigma M)(s, n) = 0$. For McCoyness, let $(r, m), (s, n) \in R' = R \oplus_\sigma M_R$. Suppose that $(r, m)(s, n)^2 = (rs^2, n\sigma(r^2) + ns\sigma(r) + ms^2) = 0$, then $r = 0$ or $s = 0$ which implies $(r, m)(s, n) = (rs, n\sigma(r) + ms) = 0$. Thus by Proposition 2.8(1), $R \oplus_\sigma M_R$ is right McCoy. □

The next example shows that under the conditions of Proposition 3.5, $R \oplus_\sigma M_R$ cannot be reversible.

Example 3.6. Let D be a commutative domain and $R = D[x]$ be the polynomial ring over D with an indeterminate x. Consider the endomorphism $\sigma\colon R \to R$ defined by $\sigma(f(x)) = f(0)$. Since $(x,1)(0,1) = (0,0)$ and $(0,1)(x,1) = (0,x) \neq (0,0)$, then $R \oplus_\sigma R$ is not reversible. Thus $R \oplus_\sigma M_R$ cannot be reversible under the conditions of Proposition 3.5.

Lemma 3.7. *Let M_R be an Armendariz module, $m(x) \in M[x]$ and $f(x), g(x) \in R[x]$ such that $m(x) = \sum_{i=0}^n m_i x^i$, $f(x) = \sum_{j=0}^p a_j x^j$ and $g(x) = \sum_{k=0}^q b_k x^k$. Then*

$$m(x)f(x)g(x) = 0 \Leftrightarrow m_i a_j b_k = 0 \ \text{ for all } \ i,j,k.$$

Proof: ($\Leftarrow$) Clear. ($\Rightarrow$) If $m(x)f(x) = 0$ then $m(x)a_j = 0$ for all j. Now, if $m(x)f(x)g(x) = 0$ then $m(x)[f(x)b_k] = 0$ for all k. Since M_R is Armendariz we have $m_i(a_j b_k) = 0$ for all i,j. Thus $m_i a_j b_k = 0$ for all i,j,k. □

Lemma 3.8. *If M_R is an Armendariz module satisfying the condition $(\mathcal{C}_\sigma^2)$. Then $M[x]_{R[x]}$ satisfies the condition $(\mathcal{C}_\sigma^2)$.*

Proof: Let $m(x) = \sum_{i=0}^n m_i x^i \in M[x]$ and $f(x) = \sum_{j=0}^p a_j x^j \in R[x]$. Suppose that $m(x)\sigma(f(x))f(x) = 0$. By Lemma 3.7, $m_i\sigma(a_j)a_k = 0$ for all i,j,k. In particular, $m_i\sigma(a_j)a_j = 0$ for all i,j. Then $m_i\sigma(a_j) = 0$ for all i,j. Therefore $m(x)\sigma(f(x)) = 0$. □

Theorem 3.9. *Let R be a commutative Armendariz ring, σ an endomorphism of R and M_R a module satisfying the condition $(\mathcal{C}_\sigma^2)$. Then M_R is Armendariz if and only if $R \oplus_\sigma M_R$ is Armendariz.*

Proof: Let $f, g \in R[x]$ and $m, n \in M[x]$ such that $(f,m)(g,n) = (0,0)$. Write $(f,m) = \sum (a_i, m_i)x^i \in R'[x]$ and $(g,n) = \sum (b_j, n_j)x^j \in R'[x]$. From $(f,m)(g,n) = (0,0)$, we have $(fg, n\sigma(f) + mg) = (0,0)$. Since R is Armendariz, then $a_i b_j = 0$ for all i,j. Multiplying $n\sigma(f) + mg = 0$ on the right by f. By Lemma 3.8, we have $n\sigma(f)f = 0$, then $n\sigma(f) = 0$ and so $mg = 0$. Since M_R is Armendariz we have $m_i b_j = 0$ and $n_i\sigma(a_j) = 0$ for all i,j. Thus $(a_i, m_i)(b_j, n_j) = (a_i b_j, n_j\sigma(a_i) + m_i b_j) = (0,0)$. Therefore R' is Armendariz. The converse is clear. □

Corollary 3.10. *If R is a commutative reduced ring which satisfies the condition $(\mathcal{C}_\sigma^1)$ then $R \oplus_\sigma R$ is semicommutative and Armendariz.*

Proof: Immediately by Proposition 3.4 and Theorem 3.9. □

Example 3.11. Consider the ring $R = \mathbb{Z}_2 \oplus \mathbb{Z}_2$ with the usual addition and multiplication. Let $\sigma\colon R \to R$ be defined by $\sigma(a,b) = (b,a)$. Clearly R is a commutative reduced ring but not a domain. Let $A = ((0,1),(0,1))$, $B = ((1,0),(0,1))$ and $C = ((1,0),(1,0))$. We have

$$AB = ((0,1),(0,1))((1,0),(0,1)) = ((0,0),((0,1)\sigma(0,1) + (0,1)(1,0))) = 0.$$

But

$$ACB = ((0,1),(0,1))((1,0),(1,0))((1,0),(0,1)) = ((0,0),(1,0))((1,0),(0,1)) = ((0,0),(1,0)) \neq 0.$$

Hence $R \oplus_\sigma R$ is not semicommutative. On other hand, we have $(1,0)(0,1) = 0$, but $(1,0)\sigma((0,1)) = (1,0)(1,0) = (1,0) \neq 0$, so R does not satisfy the condition $(\mathcal{C}_\sigma^1)$. Thus the condition $(\mathcal{C}_\sigma^1)$ in Corollary 3.10 is not superfluous.

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