# Congruence lattices of intransitive G-Sets and flat M-Sets

Steve Seif

*Abstract.* An M-Set is a unary algebra $\langle X, M\rangle$ whose set $M$ of operations is a monoid of transformations of $X$; $\langle X, M\rangle$ is a G-Set if $M$ is a group. A lattice $L$ is said to be represented by an M-Set $\langle X, M\rangle$ if the congruence lattice of $\langle X, M\rangle$ is isomorphic to $L$. Given an algebraic lattice $L$, an invariant $\mathbf{\Pi}(L)$ is introduced here. $\mathbf{\Pi}(L)$ provides substantial information about properties common to all representations of $L$ by intransitive G-Sets. $\mathbf{\Pi}(L)$ is a sublattice of $L$ (possibly isomorphic to the trivial lattice), a $\Pi$-product lattice. A $\Pi$-product lattice $\Pi(\{L_i : i \in I\})$ is determined by a so-called multiset of factors $\{L_i : i \in I\}$. It is proven that if $\mathbf{\Pi}(L) \cong \Pi(\{L_i : i \in I\})$, then whenever $L$ is represented by an intransitive G-Set $\mathbf{Y}$, the orbits of $\mathbf{Y}$ are in a one-to-one correspondence $\beta$ with the factors of $\mathbf{\Pi}(L)$ in such a way that if $|I| > 2$, then for all $i \in I$, $L_{\beta(i)} \cong Con(\mathbf{X}_i)$; if $|I| = 2$, the direct product of the two factors of $\mathbf{\Pi}(L)$ is isomorphic to the direct product of the congruence lattices of the two orbits of $\mathbf{Y}$. Also, if $\mathbf{\Pi}(L)$ is the trivial lattice, then $L$ has no representation by an intransitive G-Set.

A second result states that algebraic lattices that have no cover-preserving embedded copy of the six-element lattice $A(1)$ are representable by an intransitive G-Set if and only if they are isomorphic to a $\Pi$-product lattice.

All results here pertain to a class of M-Sets that properly contain the G-Sets — the so-called flat M-Sets, those M-Sets whose underlying sets are disjoint unions of transitive subalgebras.

*Keywords:* unary algebra; congruence lattice; intransitive G-Sets; M-Sets; representations of lattices

*Classification:* 08A30, 08A35, 08A60

## 1. Introduction

An M-Set $\langle X, M\rangle$ is a unary algebra whose operations $M$ are a monoid of transformations of $X$; if $M$ acts transitively on $X$, then $\langle X, M\rangle$ is itself said to be transitive; if $M$ is a group, $\langle X, M\rangle$ is said to be a G-Set. A lattice $L$ is said to be *represented* by an M-Set $\mathbf{X} = \langle X, F\rangle$ if $L \cong Con(\mathbf{X})$.

In [10], J. Tůma proved that every algebraic lattice can be represented by a transitive G-Set. But it is not the case that every algebraic lattice can be represented by an intransitive G-Set: Even the four-element lattice $\underline{2} \times \underline{2}$ is not representable by an intransitive G-Set[1] .This raises the question that motivates

[1]This is not difficult to show directly. See Example 1.4.

this paper: What can be said about lattices that are representable by intransitive G-Sets?

Each algebraic lattice $L$ will be assigned an invariant $\mathbf{\Pi}(L)$, a certain 0,1 cover-preserving sublattice of $L$ isomorphic to a so-called $\Pi$-product lattice (see Definition 1.3). A non-trivial[2] $\Pi$-product lattice $\Pi(\{L_i : i \in I\})$ is subdirectly embedded in $(\prod_{i\in I} L_i) \times \Pi(I)$, where $\Pi(I)$ is the lattice of partitions of the index set $I$; the multiset $\{L_i : i \in I\}$ is called the set of *factors* of $\Pi(\{L_i : i \in I\})$. $\mathbf{\Pi}(L)$ provides substantial information about properties common to all representations of $L$ by intransitive flat M-Sets. Given that $L$ is an algebraic lattice with $\mathbf{\Pi}(L) = \Pi(\{L_i : i \in I\})$, it is proven that every intransitive G-Set representation of $L$ $\langle \sqcup_{j\in J} X_j, M\rangle$ with orbits indexed by $J$ satisfies the following:

- if $\mathbf{\Pi}(L)$ is the trivial lattice, $L$ has no representation by an intransitive flat M-Set;
- if $\mathbf{\Pi}(L) = \Pi(\{L_i : i \in I\})$ is non-trivial, then $|I| = |J|$; thus the number of orbits of any intransitive flat M-Set representation of $L$ is invariant;
- if $|I| = 2$, then the direct product of the congruence lattice of the orbits of $\mathbf{Y}$ is isomorphic to $L_1 \times L_2$, the direct product of the two factors of $\mathbf{\Pi}(L)$; and
- if $|I| > 2$, then there is a bijection $\beta$ from $I$ to $J$ such that for all $i \in I$, $L_i \cong Con(\mathbf{X}_{\beta(\mathbf{i})})$.

**Definition 1.1.** For $n \in \mathbb{N}$, $A(n)$ is the lattice with $n$ atoms that are also co-atoms, as well as two other atoms that join to a co-atom. (So $|A(n)| = n + 5$.)

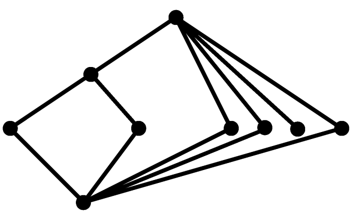

Figure 1.1. The lattice A(4)

A second main result, given in Corollary 2.5, is of a different flavor and states that if an algebraic lattice $L$ does not contain a cover-preserving embedded sublattice isomorphic to $A(1)$, then $L$ has a representation by an intransitive G-Set if and only if $L$ is isomorphic to a $\Pi$-product lattice; thus lattices in certain well-studied classes (for example, graded lattices and join-semidistributive lattices) having representations by intransitive G-Sets are given a fairly nontechnical characterization.

$\langle X, M\rangle$ is said to be a *flat M-Set* if $X$ is the disjoint union of transitive subalgebras; note that every G-Set is a flat M-Set. *The results stated above hold with*

[2]The trivial lattice is defined to be a $\Pi$-product lattice.

*"flat intransitive M-Sets" in place of "intransitive G-Sets", and were stated for G-Sets to provide a less cluttered introduction.*

Before leaving finite lattices (all results in this paper pertain to arbitrary algebraic lattices), it should also be mentioned that in stark contrast with the transitive G-Set representation result of Tůma [10] described above, using some of the ideas presented here, in [9] it is shown there exists $c > 0$ such that for high enough $n$, an isomorphism type of lattice randomly chosen from among $n$-lattices having more than one co-atom has likelihood less than $\frac{1}{2^{cn^{3/2}}}$ of being representable by an intransitive flat M-Set (whether finite or infinite).

**1.1 Background and motivation.** A *congruence* $\theta$ of an M-Set $\mathbf{X} = \langle X, M\rangle$ is an equivalence relation on $X$ such that for all $(a, b) \in \theta$ and all $m \in M$, the pair $(m(a), m(b)) \in \theta$; the congruence $\theta$ determines a quotient algebra $\mathbf{X}/\theta$. The congruences of an M-Set $\mathbf{X}$ form a lattice, denoted $Con(\mathbf{X})$. The diagonal relation on $X$, the least congruence of $\mathbf{X}$, is denoted $\Delta$; the universal congruence, the greatest congruence of $\mathbf{X}$, is denoted $\nabla$. A non-trivial M-Set $\langle X, M\rangle$ is said to be *simple* (or, *primitive*) if its only congruences are $\Delta$ and $\nabla$. If $\beta \geq \alpha$ in $Con(\mathbf{X})$, then the congruence on $\mathbf{X}/\alpha$ induced by $\beta$ will be denoted $\beta/\alpha$; also for $u \in X$, $u/\alpha$ or $\overline{u}$ will be used to denote the image of $u$ under the canonical homomorphism associated with $\alpha$. For $u, v \in X$, the congruence generated by $(u, v)$ is denoted $Cg(u, v)$. For subsets $U, V \subseteq X$, let $Cg(U \times V)$ denote the congruence on $\mathbf{X}$ generated by $U \times V$.

As mentioned, a lattice $L$ is said to be *represented* by an M-Set $\mathbf{X} = \langle X, F\rangle$ if $L \cong Con(\mathbf{X})$. G. Grätzer and T. Schmidt [2] characterized those lattices that are representable by an M-Set, the so-called *algebraic lattices*, as those complete lattices whose set of compact elements[3] generates $L$ via joins; since a finite lattice $L$ is algebraic, by [2], $L$ is represented by an M-Set. A longstanding open problem, the *Finite Lattice Representation Problem* (*FLRP*), asks whether every finite lattice can be represented by a finite M-Set.

The central role of transitive finite G-Sets in the *FLRP* was established by the theorem of P.P. Pálfy and P. Pudlák [4]. For a group $G$, let $Sub(G)$ be its lattice of subgroups. They proved that the following statements are equivalent.

- Every finite lattice is the congruence lattice of some finite M-Set.
- Every finite lattice is isomorphic to an upper interval in $Sub(G)$, where $G$ is some finite group.
- Every finite lattice is the congruence lattice of a finite <u>transitive</u> G-Set.

It is thought that the *FLRP* has a negative answer, and if that is the case, the first two of the following three questions are of interest, and the third question, while motivated in part by the role of transitivity in the *FLRP*, is a viable question regardless of the answer to the *FLRP*.

[3]An element $a$ in a complete lattice $L$ is compact if whenever $a$ is bounded above by a join of a set $S \subset L$, then there exists a finite subset $S_0 \subseteq S$ such that $a \leq \vee S_0$.

**Problem 1.** (1) *Do there exist finite lattices that have finite representations but are not representable by a finite transitive G-Set?*

(2) *Do there exist finite lattices that have finite representations but are not representable by a finite G-Set?*

(3) *What can be said about finite lattices having intransitive finite representations?*

Though not motivated by the *FLRP*, deep results on congruence lattices of intransitive G-Sets (not necessarily finite) are contained in B. Vernikov's paper [11]. Certain results from [11] are generalized here, from G-Sets to flat M-Sets, but a main purpose of this paper is to show that the $\mathbf{\Pi}$ invariant, and $\Pi$-product lattices in general, are essential in developing an understanding of the lattices that represent intransitive flat M-Sets.

**1.2 Flat M-Sets and $\Pi$-product lattices.** Flat M-Sets are exactly those M-Sets $\langle X, M\rangle$ for which the underlying set $X$ is a disjoint union of transitive subalgebras (i.e. "orbits"). It will be helpful to have another definition of flat M-Sets, one that involves the algebraic quasiorder on $X$ determined by $M$.

**Definition 1.2.** The monoid $M$ of an M-Set $\langle X, M\rangle$ determines a quasiorder $\geq_M$ on $X$: For $a, b \in X$, $a \geq_M b$ if there exists $g \in M$ such that $g(a) = b$. If $\langle X, M\rangle$ satisfies $a \geq_M b$ if and only if $b \geq_M a$, then $\langle X, M\rangle$ is said to be a **flat M-Set**.

Every transitive M-Set is flat, and, as mentioned, so are all G-Sets. An intransitive flat M-Set $\mathbf{Y}$ having $\{X_i : i \in I\}$ as its set of orbits (where $I$ is an index set) will be denoted $\mathbf{Y} = \langle \sqcup_{i\in I} X_i, M\rangle$. For $i \in I$, $X_i$ is the underlying set of a subalgebra denoted $\mathbf{X}_i$. For $\alpha \in Con(\mathbf{X})$, and for $i \in I$, let $\alpha_i = Cg(\alpha \cap (X_i \times X_i))$; abusing the notation slightly, $\alpha_i$ will also be interpreted as a congruence of $\mathbf{X}_i$. Observe that $Con(\mathbf{X}_i)$ is isomorphic to the ideal of $Con(\mathbf{Y})$ given by $\{\alpha_i : \alpha \in Con(\mathbf{X})\}$. It is convenient to have multiple ways to describe both an orbit and "restrictions" of congruences to orbits: For $u \in Y$, $X_u$ will denote the orbit containing $u$, and for $\alpha \in Con(\mathbf{Y})$, $\alpha_u$ is $Cg(\alpha \cap (X_u \times X_u))$. If $u, v$ are in different orbits, let $\alpha_{u,v} = Cg(\alpha \cap (X_u \times X_v))$, the restriction of $\alpha$ to the two orbits $X_u$ and $X_v$.

For a lattice $L$, $a$ *covers* $b$, denoted $a \succ b$, if $a > b$ and for all $c \in L$, if $a > c \geq b$, then $c = b$. Elements of $L$ that cover 0 are called *atoms*; elements of $L$ covered by 1 are called *co-atoms*. For a bounded lattice $L$, when the context is clear, 0 denotes the bottom element of $L$ and 1 the top element of $L$; exceptions are made for congruence lattices (and other sublattices of partition lattices), where $\Delta$ and $\nabla$ are used to denote the bottom and top elements, respectively. If $a, b \in L$ and $b \geq a$, let $I[a, b]$ denote the *interval* $\{c \in L : b \geq c \geq a\}$. For a bounded lattice $L$, a *0,1* sublattice $S$ of $L$ is a sublattice containing 0 and 1 of $L$. A sublattice $S$ of a lattice $L$ is said to be *cover-preserving embedded* in $L$ if $a \succ b$ in $S$ implies that $a \succ b$ in $L$ as well. $M_n$ is the lattice having $n$ atoms that are also co-atoms, a lattice with $n + 2$ elements. For an algebraic lattice $A$, $A \oplus 1$ is the lattice $A$ topped by a new greatest element. For $n \in \mathbb{N}$, let $\underline{n}$ be the $n$-element chain. (So $\underline{1}$ is the trivial lattice.) For background on algebras and lattices, consult [1] and [3].

**1.2.1 Π-product lattices: definition, examples, and elementary properties.** For a set $I$, let $\Pi(I)$ denote the lattice of partitions[4] of $I$; for $\alpha \in \Pi(I)$ and $a, b \in I$, if $a$ and $b$ are in the same class of $\alpha$, write $a\alpha b$ or $(a, b) \in \alpha$. Let $a/\alpha = \{b \in I : a\alpha b\}$, the $\alpha$ class of $a$. If $i \in I$, and $i/\alpha$ is a singleton set, then $i$ will be said to be *$\alpha$-isolated*.

Π-product lattices are now defined. Let $\{L_i : i \in I\}$ be a multiset of lattices indexed by a set $I$. (For $i \in I$, $0_i, 1_i$ will often be used to denote the bottom and top elements of $L_i$, respectively.) The Π-product lattice $\Pi(\{L_i : i \in I\})$ is subdirectly embedded into $(\prod_{i\in I} L_i) \times \Pi(I)$, and a generic element $a$ of $(\prod_{i\in I} L_i) \times \Pi(I)$ is denoted by $(\ldots, a_k, \ldots, \sigma)$, where $k \in I$, $a_k \in L_k$, and $\sigma \in \Pi(I)$; the component $\sigma$ will be referred to as the "right-most component" or as "$R(a)$".

**Definition 1.3.** Let $I$ be an index set, with $|I| > 1$, and let $\{L_i : i \in I\}$ be a multiset of algebraic lattices. The Π-product lattice $\Pi(\{L_i : i \in I\})$ is the sublattice of $(\prod_{i\in I} L_i) \times \Pi(I)$ consisting of elements of the form $(\ldots, a_k, \ldots, \alpha)$ where for $k \in I$, if $k$ is **not** $\alpha$-isolated, then $a_k = 1_k$.

For $i \in I$, the lattice $L_i$ is said to be a **factor** of $\Pi(\{L_i : i \in I\})$. Those Π-product lattices having a finite set of factors, say $L_1, \ldots, L_n$, are denoted $\Pi(L_1, \ldots, L_n)$.

Lastly, the one-element lattice is considered a Π-product lattice.

Let $a = (\ldots, a_i, \ldots, R(a)) \in \Pi(\{L_i : i \in I\})$. For $k \in I$, the "$k$th component of $a$" will be referred to as $a(k)$ or as $a_k$. Π-product lattices are complete lattices, as is easily verified. For a set $I$, with $|I| > 1$, note that the partition lattice $\Pi(I)$ is isomorphic to the Π-product lattice having $|I|$ trivial factors. Requiring in the definition that each factor of a Π-product lattice is algebraic will promote Π-product lattices to algebraic lattices, as will be shown in Lemma 1.6 below. Lemma 1.4(2) shows that a Π-product lattice $\Pi(\{L_i : i \in I\})$ enjoys a property that the full direct product $(\prod_{i\in I} L_i) \times \Pi(I)$ does not.

**Lemma 1.4.** *Let $L = \Pi(\{L_i : i \in I\})$ be a Π-product lattice.*

(1) *An element of $L$ is maximal if and only if it is of the form $(\ldots, a_k, \ldots, \alpha)$, where for all $k \in I$, $a_k = 1_k$ and $\alpha$ is maximal in $\Pi(I)$.*
(2) *With $j$ the meet of the maximal elements of $L$, the interval $I[j, 1]$ is isomorphic to $\Pi(I)$.*

Proof: Elements of the form $(\ldots, 1_k, \ldots, \alpha)$, where $a_k = 1_k$ for all $k \in I$, and $\alpha$ is maximal in $\Pi(I)$, are clearly maximal; moreover, if $a$ is not $1_L$ and $a = (\ldots, a_i, \ldots, R(a))$, then $a$ is bounded above by the maximal element $(\ldots, 1_i, \ldots, \mu)$, where $\mu$ is maximal in $\Pi(I)$, $\mu \geq R(a)$, and for all $k \in I$, $a_i = 1_i$, completing the proof of (1). For (2), that $\Pi(\{L_i : i \in I\})$ is complete means that the meet of the maximal elements of $\Pi(\{L_i : i \in I\})$ exists; denote it by $j$. The map

[4] The lattice of partitions of $I$ is isomorphic to the lattice of equivalence relations of $I$, and no distinction will be made between these two lattices, or between partitions and equivalence relations.

$R : I[j, 1_L] \to \Pi(I)$ given by $l \to R(l)$ (for all $l \in I[j, 1_L]$) is a lattice isomorphism. $\square$

**Remark 1.5.** For a $\Pi$-product lattice $\Pi(\{L_i : i \in I\})$, let $j$ be the meet of its maximal elements. For each $i \in I$, the element $(\ldots, \ldots, 1_i, \ldots, \Delta)$, where for $k \in I$ with $k \neq i$, $a_k = 0_k$, is denoted $j_i$. Of course $I[0, j_i] \cong L_i$, $\vee_{i \in I}\, j_i = j$, and $I[0, j] \cong \prod_{i \in I} L_i$.

**Example 1.1.** Consider the following two $\Pi$-product lattices $\Pi(\underline{2}, \underline{2}, \underline{2})$ and $\Pi((\underline{2})^3, \underline{1}, \underline{1})$. Let $\alpha_1 = 1|2,3$, $\alpha_2 = 1,3|2$, $\alpha_3 = 1,2|3$. Observe that in $\Pi(\underline{2}, \underline{2}, \underline{2})$, for each of $i = 1, 2, 3$, there are two elements having right-most component $\alpha_i$. Also, there are eight elements having right-most component $\Delta$, and there is one element having right-most component $\nabla$, for a total of 15 elements. Now consider $\Pi((\underline{2})^3, \underline{1}, \underline{1})$. There are eight elements having right-most component $\alpha_1$, eight elements having right-most component $\Delta$, and for each of $i = 2, 3$, there is one element having right-most component $\alpha_i$, from which it follows that this lattice has 19 elements.

**Example 1.2.** Suppose $A$ is an algebraic lattice with a unique co-atom. Thus $A \cong (B \oplus 1)$, where $B$ is an algebraic lattice. Suppose that $B \cong C \times D$. Then, as is not difficult to show, $A \cong \Pi(C, D)$ with the unique co-atom of $\Pi(C, D)$ given by $(1_C, 1_D, \Delta)$. The last sentence also indicates that any two-factor $\Pi$-product lattice has a unique co-atom. The following has been shown: **An algebraic lattice $L$ is isomorphic to a two-factor $\Pi$-product lattice if and only if $L$ has a unique co-atom.**

Note also that for algebraic lattices $C$ and $D$, $\Pi(C, D) \cong \Pi(\underline{1}, C \times D)$, so quite different pairs of factors might determine the same isomorphism class of $\Pi$-product lattices. However, if $L$ and $N$ are $\Pi$-product lattices and $L$ has more than two factors, then $L$ and $N$ are isomorphic if and only if there is a matching between the multisets of factors of $L$ and of $M$ such that matched pairs are isomorphic lattices. See Lemma 3.2(5).

**Example 1.3.** Consider first $\langle\{0, 1, 2\}, C_3\rangle$, where $C_3$ is the three-element group of transformations of $\{0, 1, 2\}$ generated by the 3-cycle $(0, 1, 2)$. Now let $\mathbf{X}$ be the "doubling" of $\langle\{0, 1, 2\}, C_3\rangle$; that is, $\mathbf{X} = \langle\{0, 1, 2\} \sqcup \{0, 1, 2\}, C_3\rangle$, where the action of $C_3$ is the same on both copies of $\{0, 1, 2\}$.

Let $\mathcal{J}$ be the congruence on $\mathbf{X}$ that collapses each orbit to a singleton. (So with $\mathbf{X}$ above, $\mathbf{X}/\mathcal{J}$ is isomorphic to the two-element disconnected unary algebra.) For $i = 1, 2$, let $\mathcal{J}_i$ be the congruence that collapses one orbit (so for $i = 1, 2$, $\mathcal{J}_i = \{(x, y) : x = y \text{ or } \{x, y\} \subseteq X_i\}$); that each orbit in this example is simple implies that $\mathcal{J}_i$ is a minimal congruence.

Let $\{\dot{0}, \dot{1}, \dot{2}\}$ and $\{\ddot{0}, \ddot{1}, \ddot{2}\}$ denote the two copies of $X$. $\mathbf{X}$ has just three more congruences, $\alpha_1 = Cg(\dot{0}, \ddot{0})$, $\alpha_2 = Cg(\dot{0}, \ddot{1})$, and $\alpha_3 = Cg(\dot{0}, \ddot{2})$, each a minimal congruence that is also a maximal congruence; thus, $Con(\mathbf{X})$ is isomorphic to $A(3)$. Figure 2.1 provides the Hasse diagram of $Con(\mathbf{X})$ along with its so-called transitivity labeling; the transitivity labeling is described in Section 2.1.

From Lemma 1.4(2) it follows that $Con(\mathbf{X})$ is not isomorphic to a partition lattice, but it does contain numerous 0,1 cover-preserving embedded $\Pi$-product sublattices. Here is the list of them: Let $A$ be the five-element lattice with underlying set $\{\Delta, \mathcal{J}, \mathcal{J}_1, \mathcal{J}_2, \nabla\}$, a lattice isomorphic to $\Pi(\underline{2}, \underline{2})$; $B$, with underlying set $\{\Delta, \alpha_1, \alpha_2, \alpha_3, \nabla\}$, is isomorphic to $\Pi(\underline{1}, \underline{1}, \underline{1}) \cong M_3$; $C_1, C_2, C_3$, each isomorphic to $\Pi(\underline{1}, \underline{2}) \cong \underline{3}$; $D_1, D_2$, each isomorphic to $\Pi(\underline{1}, \underline{3}) \cong \underline{4}$.

Of these only $A \cong \Pi(\underline{2}, \underline{2})$ might indicate that $\mathbf{X}$ has exactly two orbits, each orbit having congruence lattice $\underline{2}$, and so, not surprisingly, $\mathbf{\Pi}(Con(\mathbf{X})) = \mathbf{\Pi}(A(3)) = A$, which can be verified from the definition of $\mathbf{\Pi}$, given by Definition 2.6 and Definition 2.7.

**Example 1.4.** It is shown that, as stated in the second paragraph of the paper, $\underline{2} \times \underline{2}$ has no intransitive G-Set representation: Let $\mathbf{X}$ be a flat M-Set, and once again let $\mathcal{J}$ be the congruence that identifies the elements within each orbit. So $\mathbf{X}/\mathcal{J}$ has singleton orbits, and $Con(\mathbf{X}/\mathcal{J})$ is isomorphic to a partition class $\Pi(I)$, where $I$ indexes the orbits of $\mathbf{X}$. Thus if $\mathbf{X}$ has more than two orbits, $Con(\mathbf{X})$ has a copy of $M(3)$, a five-element lattice. If $\mathbf{X}$ has two orbits, one of which is a singleton orbit, then, as is easy to verify, $\mathcal{J}$ is its unique maximal congruence. Lastly, if $\mathbf{X}$ has two non-singleton orbits $X_1$ and $X_2$, then it is not difficult to see that $I[\Delta, \mathcal{J}] \cong Con(\mathbf{X}_1) \times Con(\mathbf{X}_2)$, a direct product of two non-trivial lattices, and since $\mathcal{J} < \nabla$, $Con(\mathbf{X})$ has more than four elements. It now follows that $\underline{2} \times \underline{2}$ has no intransitive flat M-Set representation; thus, it has no intransitive G-Set representation.

As mentioned above, $\Pi$-product lattices are algebraic; indeed, every non-trivial $\Pi$-product lattice is isomorphic to the congruence lattice of some intransitive flat M-Set, as is shown in Lemma 1.6 below.

**Lemma 1.6.** *Let $I$ be an index set with $|I| > 1$, and suppose $\{\langle X_i, V_i\rangle : i \in I\} = \{\mathbf{X}_i : i \in I\}$ is a multiset of transitive M-Sets. Then $\Pi(\{Con(\mathbf{X}_i) : i \in I\})$ is the congruence lattice of the intransitive flat M-Set $\langle \sqcup_{i\in I} X_i, \prod_{i\in I} V_i\rangle$, where for $\mathbf{m} \in \prod_{i\in I} V_i$, and for $i \in I$ with $x_i \in X_i$, $\mathbf{m}(x_i) = m_i(x_i)$, the latter determined by $\langle X_i, V_i\rangle$. Moreover, for all $i \in I$, $Con(\langle X_i, \prod_{i\in I} V_i\rangle) \cong Con(\langle X_i, V_i\rangle)$.*

*Thus each non-trivial $\Pi$-product lattice $\Pi(\{L_i : i \in I\})$ is representable by an intransitive flat M-Set $\langle \sqcup_{i\in I} X_i, M\rangle$ where, for all $i \in I$, $L_i \cong Con(\mathbf{X}_i)$.*

Proof of Lemma 1.6: For $i \in I$, let $\mathbf{X}_i = \langle X_i, V_i\rangle$ be a transitive M-Set. Form the flat M-Set $\mathbf{X} = \langle \sqcup_{i\in I} X_i, \prod_{i\in I} V_i\rangle$ where the action of $\prod_{i\in I} V_i$ is as described in the statement of the lemma.

$\prod_{i\in I} V_i$ acts independently on the orbits, and this implies that if $i \neq j$ in $I$, and $x_i \in X_i$ and $x_j \in X_j$, then $Cg(x_i, x_j)$ contains $X_i \times X_j$.

A function $F : Con(\mathbf{X}) \to \Pi(\{Con(\mathbf{X}_i) : i \in I\})$ is defined: For $\alpha \in Con(\mathbf{X})$, let $F(\alpha) = (\ldots, \alpha_k, \ldots, R(\alpha))$, where for $k \in I$, $\alpha_k$ is the restriction of $\alpha$ to $X_k$, and for $j, k \in I$, $(j, k) \in R(\alpha)$ if there exist $x_j \in X_j$ and $x_k \in X_k$ such that $(x_j, x_k) \in \alpha$. By the previous paragraph, it follows both that $F$ maps into $\Pi(\{Con(\mathbf{X}_i) : i \in I\})$ and that $F$ is injective. It is routine to verify that $F$ is surjective and that $F$ and $F^{-1}$ are order-preserving. Since the domain and co-domain of $F$ are both complete lattices, it follows that $F$ is a lattice isomorphism, completing the proof of the first part.

By [2], given an algebraic lattice $A$, there exists an M-Set $\langle U, M\rangle$ that represents $A$. Adding all constant operations to $U$ results in a new M-Set $\langle U, M_C\rangle$ that still represents $A$. Now given a multiset of algebraic lattices $\{L_i : i \in I\}$, using the above construction with the "constant-augmented" M-Sets, a flat intransitive M-Set can be formed that represents $\Pi(\{L_i : i \in I\})$. □

**Corollary 1.7.** (1) *Each non-trivial* $\Pi$*-product lattice is representable by an intransitive G-Set.*
(2) *The class of finite lattices that are representable by finite M-Sets is closed under* $\Pi$*-product.*

Proof: As mentioned, every algebraic lattice is representable by a transitive G-Set [10]. When the construction of Lemma 1.6 is applied to G-Sets, the monoid that acts on the disjoint union of sets is a direct product of G-Sets, which is also a G-Set, which proves (1). That same construction of Lemma 1.6 given a finite set of finite M-Sets outputs a finite flat M-Set, which is sufficient for (2). □

**Definition 1.8.** Let $\mathbf{Y} = \langle \sqcup_{i\in I} X_i, M\rangle$ be an intransitive flat M-Set. Then $\mathbf{Y}$ satisfies Property K if whenever $x_i \in X_i$, $x_j \in X_j$ where $X_i, X_j$ are distinct orbits, then $Cg(x_i, x_j)$ contains $X_i \times X_j$.

**Remark 1.9.** Note that in the definition above, "$X_i \times X_j$" could be replaced by "$X_i \times X_i$" with the resulting definition equivalent to the given one, a fact that will be used without comment.

Observe that the flat intransitive M-Set $\mathbf{Y} = \langle \sqcup_{i\in I}, \prod_{i\in I} V_i\rangle$ of Lemma 1.6 satisfies Property K.

**1.3 How $\Pi$-product lattices are 0,1 embedded in congruence lattices of flat intransitive M-Sets.** Suppose $\mathbf{Y} = \langle \sqcup_{i\in[n]} X_i, M\rangle$ is an intransitive flat M-Set.

Let the direct product $M^I$ act on $\sqcup_{i\in I} X_i$ as follows: For $\mathbf{m} = (\ldots, m_i, \ldots, m_j, \ldots) \in M^I$ and $x_i \in X_i$ (where $i \in I$), let $\mathbf{m}(x_i) = m_i(x_i)$, this last output determined by $\langle X_i, M\rangle$. $M^I$ is defined in a manner similar to a definition given in Lemma 1.6; in Lemma 1.6, one begins with a set of transitive M-Sets, whereas here the transitive M-Sets are given as orbits of a single intransitive flat M-Set.

**Definition 1.10.** Let $\Pi(\mathbf{Y}) = \langle \sqcup_{i\in I} X_i, M^I\rangle$, where the action of $M^I$ on $\sqcup_{i\in I} X_i$ is as given above.

Let $\Delta(M^I) = \{(\ldots, m_i, \ldots, m_j, \ldots) : \forall i, j \in I,\ m_i = m_j\}$ on $Y$; its action $\sqcup_{i\in I} X_i$ is identical with that of $M$; thus, $Con(\mathbf{Y})$ contains $Con(\Pi(\mathbf{Y}))$ as a $0,1$ sublattice, one that is (as will be proven in Lemma 5.5) a cover-preserving sublattice. The important $\mathcal{J}$ congruence (seen in two examples above) is defined next, followed by Lemma 1.12 which list elementary properties of $\mathcal{J}$. Then Lemma 1.13

describes relationships involving Property K, $\Pi$-product lattices, and $\Pi(\mathbf{Y})$ flat M-Sets, and the $\mathcal{J}$ relation.

**Definition 1.11.** Let $\mathbf{Y} = \langle \sqcup_{i\in I} X_i, M\rangle$ be an intransitive flat M-Set.

(1) Let $\mathcal{J} = \{(a,b) \in Y^2 : \exists i \in I \text{ such that } \{a,b\} \subseteq X_i\}$.
(2) For $i \in I$, let $\mathcal{J}_i = \{(a,b) : a = b \text{ or } \{a,b\} \subseteq X_i\}$.
(3) For $\gamma \in Con(\mathbf{Y})$, let $\mathcal{J}_\gamma$ be the $\mathcal{J}$ relation on $\mathbf{Y}/\gamma$.

Observe that $(a,b) \in \mathcal{J}$ implies that $(a/\gamma, b/\gamma) \in \mathcal{J}_\gamma$.

**Lemma 1.12.** *Let $\mathbf{Y} = \langle \sqcup_{i\in I} X_i, M\rangle$ be a flat M-Set. Then the following hold:*

(1) *$\mathcal{J}$ is a congruence, as is, for $i \in I$, $\mathcal{J}_i$;*
(2) *$\mathbf{Y}/\mathcal{J}$ is a flat M-Set whose $|I|$ orbits are all singletons;*
(3) *$Con(\mathbf{Y}/\mathcal{J})$ is isomorphic to a partition lattice; and*
(4) *$\mathcal{J} = \vee_{i\in I}\mathcal{J}_i$, and $I[\Delta,\mathcal{J}] \cong \prod_{i\in I} Con(\mathbf{X}_i)$.*

Proof: (1) is obvious. That $\mathcal{J}$ collapses orbits but does not identify elements in different orbits implies that $\mathbf{Y}/\mathcal{J}$ is a flat M-Set having singleton orbits, those orbits in one-to-one correspondence with $I$. Thus $M$ acts like the trivial monoid on $\mathbf{Y}/\mathcal{J}$, and therefore its congruence lattice is isomorphic to $\Pi(I)$. (4) is clear. $\square$

**Lemma 1.13.** *Let $\mathbf{Y}$ be an intransitive flat M-Set. Then the following hold:*

(1) *$Con(\Pi(\mathbf{Y}))$ is a $0,1$ sublattice of $Con(\mathbf{Y})$;*
(2) *$\Pi(\mathbf{Y})$ is an intransitive flat M-Set having the same orbits as $\mathbf{Y}$;*
(3) *$\Pi(\mathbf{Y})$ satisfies Property K;*
(4) *$Con(\Pi(\mathbf{Y}))$ is isomorphic to a $\Pi$-product lattice, namely $\Pi(\{Con(\mathbf{X}_i) : i \in I\})$;*
(5) *$\alpha \in Con(\Pi(\mathbf{Y}))$ if and only if $(c,d) \in \alpha - \mathcal{J}$ implies $(X_c \times X_d) \subset \alpha$;*
(6) *all elements in $Con(\mathbf{Y})$ that are comparable to $\mathcal{J}$ are in $Con(\Pi(\mathbf{Y}))$; that is, $I[\Delta,\mathcal{J}] \cup I[\mathcal{J},\nabla]$ is contained in $Con(\Pi(\mathbf{Y}))$;*
(7) *the meet of the maximal congruences of $\Pi(\mathbf{Y})$ is $\mathcal{J}$, and the map $x \to x \wedge \mathcal{J}$ is a surjective lattice homomorphism from $Con(\mathbf{Y})$ to $I[\Delta,\mathcal{J}]$; and*
(8) *$\mathbf{Y}$ satisfies Property K if and only if $Con(\Pi(\mathbf{Y})) = Con(\mathbf{Y})$. Thus, if $\mathbf{Y} = \langle \sqcup_{i\in I} X_i, M\rangle$ satisfies Property K, then $Con(\mathbf{Y})$ is isomorphic to a $\Pi$-product lattice, namely $\Pi(\{Con(\mathbf{X}_i) : i \in I\})$.*

Proof: The explanation for (1) was given right after Definition 1.10. The proof of Lemma 1.6 can be slightly modified to produce proofs for (2), (3), and (4) of this lemma. For (5), if $\alpha \in Con(\Pi(\mathbf{Y}))$, that $M^I$ acts independently on orbits and $M^I$ is transitive on each orbit ensure that if $(c,d) \in \alpha - \mathcal{J}$, then $(X_c \times X_d) \subset \alpha$. Conversely, using that $\mathcal{J}$ is invariant under $M^I$, if $\alpha \in Con(\mathbf{X})$ has the property that $(c,d) \in \alpha - \mathcal{J}$ implies $(X_c \times X_d) \subset \alpha$, then $\alpha$ is also invariant under $M^I$; therefore, $\alpha \in Con(\Pi(\mathbf{Y}))$. For (6), a congruence $\alpha \in I[\mathcal{J},\nabla]$ (an interval of $Con(\mathbf{Y})$) contains $\mathcal{J}$ so it certainly satisfies $(c,d) \in \alpha - \mathcal{J}$ implies $X_c \times X_d \subseteq \alpha$, and congruences in $I[\Delta,\mathcal{J}]$ satisfy (in a vacuous way) that same sufficient condition for containment in $Con(\Pi(\mathbf{Y}))$.

For (7), by Lemma 1.12(3), $I[\mathcal{J}, \nabla]$ is isomorphic to a non-trivial partition lattice (a lattice whose maximal elements meet to their bottom element); thus, $\mathcal{J}$ is a meet of the maximal congruences contained in $\mathbf{Y}$. Suppose for contradiction that there exists $\phi$ maximal in $Con(\Pi(\mathbf{Y}))$ but $\phi \not\geq \mathcal{J}$. Thus there exists $i \in I$ such that $\phi_i$ does not contain $\mathcal{J}_i = Cg(X_i \times X_i)$. Since $\Pi(\mathbf{Y})$ satisfies Property K, it follows that if $j \in I$ and $i \neq j$, then $X_i \times X_j \cap \phi = \emptyset$. Let $\rho = \mathcal{J}_i \vee Cg(Y - \{X_i\})$, a congruence of $\Pi(\mathbf{Y})$ that is not maximal (since $\phi_i < Cg(X_i \times X_i)$) and contains $\phi$, a contradiction. It now follows that the meet of the maximal congruences of $\Pi(\mathbf{Y})$ is $\mathcal{J}$. The last part of (7) now follows from the first part of (7) and the fact that in any $\Pi$-product lattice whose meet of maximal elements is $j$, the map $x \to (x \wedge j)$ is a surjective lattice homomorphism onto $I[0, j]$.

For the "if and only if" part of (8), suppose first that $\mathbf{Y}$ satisfies Property K. Property K implies that for all congruences $\alpha \in Con(\mathbf{Y})$, if $(c, d) \in \alpha - \mathcal{J}$, then $X_c \times X_d \subset \alpha$, which implies that $\alpha \in Con(\Pi(\mathbf{Y}))$. Conversely, if $Con(\mathbf{Y}) = Con(\Pi(\mathbf{Y}))$, then since $\Pi(\mathbf{Y})$ satisfies Property K (as shown in (3) of this lemma) and Property K is a property defined in terms of principal congruences and orbits, and these agree with those of $\Pi(\mathbf{Y})$, it follows readily that $\mathbf{Y}$ satisfies Property K. For the last part of (8), assume $\mathbf{Y}$ satisfies Property K. As the first part of (8) showed, $Con(\mathbf{Y}) = Con(\Pi(\mathbf{Y}))$; now from (4) of this lemma, $Con(\Pi(\mathbf{Y})) \cong \Pi(\{Con(\mathbf{X}_i) : i \in I\})$, and the last part of (8) now follows. $\square$

**Remark 1.14.** (1) For the flat M-set $\mathbf{X}$ of Example 1.3, $Con(\Pi(\mathbf{X}))$ has underlying set $\{\Delta, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}, \nabla\}$ (a sublattice denoted $A$): Adding the non-diagonal operations of $C_3 \times C_3$ kills off the congruences $\alpha_1, \alpha_2, \alpha_3$. Notice that $Con(\Pi(\mathbf{X})) = \mathbf{\Pi}(Con(\mathbf{X}))$, an equality that Theorem 2.8 states holds for an arbitrary flat intransitive M-Set $\mathbf{Z}$.

(2) By Lemma 1.13(8) a flat intransitive M-Set satisfying Property K has a congruence lattice isomorphic to a $\Pi$-product lattice. The converse is also true, and is part of the statement of Theorem 2.4. (See (4) implies (2)).

## 2. Statement of the two main theorems

Theorem 2.4 provides necessary and sufficient conditions for a flat intransitive M-Set to satisfy Property K, and Corollary 2.5 states that a lattice having no $A(1)$ cover-preserving sublattice can be represented by an intransitive flat M-Set if and only if it is isomorphic to a $\Pi$-product lattice. Several definitions are needed before Theorem 2.4 can be stated.

**2.1 Transitivity labeling of congruence lattices of M-Sets.** Let $\mathbf{X} = \langle X, M \rangle$ be an *arbitrary M-Set*. The monoid of transformations $M$ determines a quasiordering $\geq_M$ on $X$, called the *algebraic quasiorder*, given by $a \geq_M b$ if there exists $t \in M$ such that $t(a) = b$. Let $\mathcal{J}$ be the equivalence relation given by $(a, b) \in \mathcal{J}$ if $a \geq_M b$ and $b \geq_M a$. Observe that $\mathbf{X}$ is transitive if and only if $\mathcal{J}$ is the universal relation, and that the $\mathcal{J}$ relation given in Definition 1.11 for flat M-Sets coincides on flat M-Sets with the $\mathcal{J}$ relation just defined.

For $u \in X$, let $J_u$ denote the equivalence class of $\mathscr{J}$ containing $u$. Let $\mathbf{X}/\mathscr{J}$ denote the set of $\mathscr{J}$ classes of $\mathbf{X}$, and observe that there is a partial order on $\mathbf{X}/\mathscr{J}$ given by: $J_u \geq_M J_v$ if $u, v \in X$ and $u \geq_M v$. An M-Set $\langle X, M\rangle$ is flat if and only if $\mathbf{X}/\mathscr{J}$ is an anti-chain.

The transitivity labeling for the Hasse diagram of an arbitrary M-Set is defined.

**Definition 2.1.** Let $\mathbf{X}$ be a flat M-Set. If $\beta \succ \alpha$ in $Con(\mathbf{X})$, then the edge $\langle \alpha, \beta\rangle$ of the Hasse diagram of $Con(\mathbf{X})$ is labeled as follows:

(1) $+$ (written as $Tr\langle \alpha, \beta\rangle = +$) if for all $(a, b) \in \beta - \alpha$, there exists $s \in M$ such that $(s(a), b) \in \alpha$;

(2) otherwise, $\langle \alpha, \beta\rangle$ is labeled $-$ (written as $Tr\langle \alpha, \beta\rangle = -$).

**Remark 2.2.** Let $\mathbf{X}$ be any M-Set.

(1) Note that if $\gamma \in Con(\mathbf{X})$ and $\phi \succ \mu \geq \gamma$ in $Con(\mathbf{Y})$, then the definition of the transitivity labeling guarantees that $Tr\langle \mu, \phi\rangle = Tr\langle \mu/\gamma, \phi/\gamma\rangle$.

(2) If $\beta \succ \Delta$, observe that $Tr\langle \Delta, \beta\rangle = +$ if and only if $\beta \leq \mathscr{J}$.

Using basic properties of algebraic lattices and a Zorn's Lemma argument, it is not difficult to show (but it will not be used in this paper) that for any M-Set $\mathbf{X} = \langle X, M\rangle$, the M-Set $\mathbf{X}$ is transitive if and only if all transitivity labels of the Hasse diagram of $Con(\mathbf{X})$ are $+$ . An interval $I[\mu, \gamma]$ is transitive if for all $(a/\mu, b/\mu) \in \mathbf{X}/\mu$, there exists $f \in M$ such that $f(a/\mu) = b/\mu$. It is not difficult to prove that $I[\mu, \gamma]$ is transitive if and only if all transitivity labels of the Hasse diagram of the interval $I[\mu, \gamma]$ are $+$ . The transitivity labeling of the eight-element congruence lattice of the six-element M-Set $\mathbf{X}$ of Example 1.3 is given below.

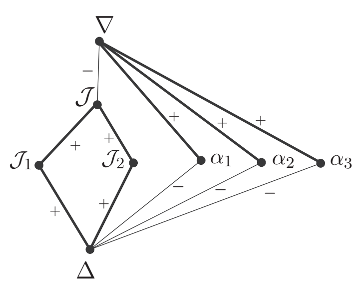


Figure 2.1. $Con(\mathbf{X}) \cong A(3)$, transitivity-labeled

**Remark 2.3.** The remarks that follow involve M-Sets that are not necessarily flat, and they will not be used in this paper: For arbitrary M-Sets, $\mathscr{J}$ is not in general a congruence. See [6] (in which the transitivity labeling was called the signed labeling), where it was proven that there exists a non-obvious semimodularity[5] constraint on congruence lattices of arbitrary algebras, and [7], where the transitivity labeling is used to show that there exist *transitivity-forcing* algebraic

[5]A lattice $L$ is semimodular, if $a, b, c \in L$, $b \succ a$ and $c \wedge b = a$, then $a \vee c \succ c$.

lattices, lattices whose every representation (finite or infinite) is transitive, and there also exist *strongly-transitivity-forcing lattices* $L$ satisfying the following: If $\mathbf{X}$ is an M-Set, $\gamma > \mu$ in $Con(\mathbf{X})$, and $L \cong I[\mu, \gamma]$, then $I[\mu, \gamma]$ is a transitive interval. For example, if $L$ is a non-semimodular algebraic lattice all of whose proper subintervals are semimodular, then $L$ is strongly-transitivity-forcing, as is proven in [7].

**2.1.1 The statement of the first main theorem.** The first main theorem incorporates a slight reformulation of a non-trivial result ((1) implies (2) of Theorem 2.4 below) of B. Vernikov in [11]. In fact, the proof of (1) implies (2) is that of Vernikov (except for small details), and is given here for completeness and to frame the proof of the equivalence of (2) through (5) in Theorem 2.4.

**Theorem 2.4.** *Let $I$ be an index set with $|I| > 1$, and let $\mathbf{X} = \langle \sqcup_{i \in I} X_i, L \rangle$ be an intransitive flat M-Set. Consider the five statements below.*

(1) $Con(\mathbf{X})$ *contains no cover-preserving sublattice isomorphic to* $A(1)$.
(2) $\mathbf{X}$ *satisfies Property K.*
(3) $Con(\mathbf{X})$ *contains no non-uniformly labeled cover-preserving sublattice isomorphic to* $A(1)$.
(4) $Con(\mathbf{X})$ *is isomorphic to a* $\Pi$*-product lattice.*
(5) $Con(\mathbf{X}) \cong \Pi(\{Con(\mathbf{X_i}) : i \in I\})$.

*Then*

(a) (*due to B. Vernikov*) (1) *implies* (2).
(b) (2), (3), (4), *and* (5) *are equivalent.*

The next corollary completely characterizes, for a broad class containing several important classes of lattices, those algebraic lattices that have a representation by an intransitive flat M-Set.

**Corollary 2.5.** *An algebraic lattice $L$ having no cover-preserving sublattice isomorphic to $A(1)$ has an intransitive M-Set representation if and only if $L$ is isomorphic to a* $\Pi$*-product lattice.*

**2.2 The statement of the second main theorem, Theorem 2.8.** For an algebraic lattice $L$, a certain subset of the $\Pi$-product sublattices of $L$, its $\Pi$-possible sublattices, are defined in Definition 2.6.

**Definition 2.6.** Let $L$ be a lattice. A sublattice $A$ of $L$ is $\Pi$**-possible** if the following hold:

(1) $A$ is a $0, 1$ cover-preserving embedded sublattice of $L$;
(2) $A$ is isomorphic to a $\Pi$-product lattice; and
(3) let $k$ be the meet of the maximal elements of $A$. Then the following must hold:
    (a) the intervals of $L$ given by $I[0, k]$ and $I[k, 1]$ are both contained in $A$, and
    (b) if $b \leq k$ and $I[b, k]$ is isomorphic to a non-trivial partition lattice, then $n$ is maximal in $L$ and $n \geq b$ jointly imply that $n \in A$.

Note that the only $\Pi$-possible sublattice of $A(3) \cong Con(\mathbf{X})$, the two-orbit flat M-Set of Example 1.3, is the lattice $A$, having underlying set $\{\Delta, \mathcal{J}_1, \mathcal{J}_2, \mathcal{J}, \nabla\}$; the 3-chains $D_1, D_2, D_3$ all fail 3(b), while $B$ and the 4-chains $D_1$ and $D_2$ fail 3(a).

**Definition 2.7.** Let $L$ be a non-trivial algebraic lattice. If $L$ has a $\Pi$-possible sublattice $A$ such that every $\Pi$-possible sublattice of $L$ is contained in $A$, then $\mathbf{\Pi}(L) = A$; otherwise, $\mathbf{\Pi}(L) = \underline{1}$.

Since the only $\Pi$-possible sublattice of $A(3)$ is $A$, it follows that $\mathbf{\Pi}(A(3)) = A$. The notation"$\Pi$" is used in quite a few ways, and the tight interplay of these usages is the crux of the first part of Theorem 2.8. As it turns out, Theorem 2.8(1), (2), and (3) all follow from that first part of the theorem, in tandem with Lemma 3.2(5), a lemma that catalogs properties of $\Pi$-product lattices.

**Theorem 2.8.** *If $\mathbf{X}$ is an intransitive flat M-Set, then $\mathbf{\Pi}(Con(\mathbf{X})) = Con(\Pi(\mathbf{X}))$. Moreover, for an algebraic lattice $L$, the following hold when $\mathbf{X}$ is a flat M-Set.*

(1) *If $\mathbf{\Pi}(L)$ is the trivial lattice and $L \cong Con(\mathbf{X})$, then $\mathbf{X}$ is transitive.*
(2) *Suppose $\mathbf{\Pi}(L) = \Pi(L_1, L_2)$. If $L \cong Con(\mathbf{X})$, then either $\mathbf{X}$ is transitive or $\mathbf{X}$ has two orbits, $X_1$ and $X_2$, and $L_1 \times L_2 \cong Con(\mathbf{X_1}) \times Con(\mathbf{X_2})$; and*
(3) *if $\mathbf{\Pi}(L) = \Pi(\{L_i : i \in I\})$, $|I| > 2$, and $L \cong Con(\mathbf{X})$, then either $\mathbf{X}$ is transitive or $\mathbf{X} = \langle \sqcup_{i \in I} X_i, M\rangle$, and after suitably ordering the orbits, $L_i \cong Con(\mathbf{X_i})$.*

Section 3 provides more information about $\Pi$-product lattices. Section 4 consists of the proof of Theorem 2.4, and Section 5 provides the proof of Theorem 2.8. Section 6 discusses further work and present some problems.

## 3. Structure of $\Pi$-product lattices

Recall that a non-trivial lattice $L$ is *directly indecomposable* if whenever there exist lattices $M$ and $N$ such that $L \cong M \times N$, then at least one of $M, N$ is trivial; the congruence distributivity of the variety of lattices implies the following fact (one that will be used later): For a lattice $L$, if $n$ and $m$ are positive integers, $L$ has a direct factorization into $n$ directly indecomposable lattices, and $L$ is the direct product of $m$ non-trivial lattices, then $n \leq m$. As Lemma 1.4 indicates, partition lattices play an important role in this paper.

It is well known that the join of the atoms of a partition lattice is $\nabla$, and the meet of its maximal elements is $\Delta$; further properties that will used in this paper are given below.

**Observation 3.1.** *Observations concerning partition lattices follow.*

(1) *It is well known that if $I$ is a non-empty set with $|I| > 1$, then $\Pi(I)$ is simple. Thus, non-trivial partition lattices are directly indecomposable.*
(2) *Observe that if $\Pi(X)$ is a non-trivial partition lattice, and $\rho > \Delta$, then there exists $\phi \in \Pi(X)$ such that $\rho \succ \phi$ in $\Pi(X)$: To form such $\phi$, divide*

*a non-singleton class $C$ of $\rho$ into two non-empty subsets, but otherwise leave $\rho$ alone. Moreover, with $\rho > \Delta$ and $\phi$ chosen so that $\rho \succ \phi$, there exists a maximal equivalence relation $\nu$ such that $\nu \geq \phi$ but $\nu \ngeq \rho$.*

(3) *Suppose $I$ has more than two elements and $i \in I$. Let $\sigma_i$ be the partition of $I$ having two classes $\{i\}$ and $I - \{i\}$. So $\sigma_i$ is maximal in $\Pi(I)$, and $\sigma_i$ has the additional property that $I[\Delta, \sigma_i]$ is isomorphic to $\Pi(I - \{i\})$. A property that will be used several times in the paper is this: If $\mu$ is maximal in $\Pi(I)$, the interval $I[\Delta, \mu]$ is directly indecomposable if and only if $\mu = \sigma_i$, for some $i \in I$.*

In Lemma 3.2, for the $\Pi$-product lattice $L$, let $j$ be the meet of the maximal elements of $L$, and let $j_i = (\dots, 1_i, \dots, \Delta)$, where for $k \neq i$, $j_i(k) = 0_k$. Observe again that $I[0, j_i] \cong L_i$ and $\vee_{i \in I} j_i = j$. Recall that for $a \in L$ with $a = (\dots, a_i, \dots, \alpha)$, $R(a) = \alpha$. Also, let $1_L$ be denoted by 1.

**Lemma 3.2.** *Let $L = \Pi(\{L_i : i \in I\})$ be a non-trivial $\Pi$-product lattice with $j$ the meet of its maximal elements.*

(1) *$\Pi(\{L_i : i \in I\})$ is isomorphic to a partition lattice if and only if for all $i \in I$ the lattice $L_i$ is trivial.*
(2) *If $m \in L$ and $I[m, 1]$ is isomorphic to a partition lattice, then $m \geq j$.*
(3) *For all $a \in L$, the interval $I[a, 1]$ is isomorphic to a $\Pi$-product lattice.*
(4) *Suppose $|I| > 2$. Let $i \in I$ and let $\sigma_i \in \Pi(I)$ consist of two classes, a singleton class $\{i\}$ and the class $I - \{i\}$. Then $\{(\dots, a_i, \dots, \sigma_i) :$ If $k \neq i$ then $a_k = 1_k\} = R^{-1}(\sigma_i)$, and $R^{-1}(\sigma_i)$ is isomorphic to both $L_i$ and to $I[0, j_i]$.*
(5) *Suppose $L = \Pi(\{L_i : i \in I\})$ and $N = \Pi(\{N_j : j \in J\})$ are $\Pi$-product lattices. Then $\Pi(\{L_i : i \in I\}) \cong \Pi(\{N_j : j \in J\})$ if and only if*
   (a) *$|I| = 2$ and $(L_1 \times L_2) \cong (N_1 \times N_2)$, or*
   (b) *$|I| > 2$ and there is a bijection $\phi : I \to J$ such that for all $i \in I$, $L_i \cong N_{\phi(i)}$.*

Proof: If $L = \Pi(\{L_i : i \in I\})$, then $L$ is subdirectly embedded into $(\prod_{i \in I} L_i) \times \Pi(I)$. If $|I| > 1$ and $L$ is isomorphic to a partition lattice, then $L$ must be non-trivial and simple. Since $\Pi(I)$ is also non-trivial, it follows readily for all $i \in I$ that $L_i$ are trivial. The other direction of (1) follows from the definition of a $\Pi$-product lattice. For (2), if $m \in L$ is such that $I[m, 1]$ is isomorphic to a partition lattice, then the maximal elements in $I[m, 1]$ meet to $m$; by Lemma 1.4, $m \geq j$. For (3), let $a = (\dots, a_i, \dots, \alpha) \in \Pi(\{L_i : i \in I\})$. If $a = 1$, then $I[a, 1]$ is the trivial lattice, a lattice that is by definition a $\Pi$-product lattice. Suppose $a < 1$. In that case, $\alpha < 1_{\Pi(I)}$. Let $I/\alpha$ denote the set of classes of $\alpha$. So $|I/\alpha| > 1$. A multiset of lattices $\{M_U : U \in I/\alpha\}$ is defined: If $U$ is a non-singleton class of $\alpha$, let $M_U = \underline{1}$; if $U = \{i\}$ is a non-singleton class of $\alpha$, let $M_U = I[a_i, 1_i]$ (where $a_i \in L_i$ is the component of $a$ contained in $L_i$). Consider $S : I[a, 1] \to \Pi(\{M_U : U \in I/\alpha\})$ where for $b \geq a$ and $b = (\dots, b_i, \dots, \beta)$, $S(b)(U) = 1_U$ if $U$ is a non-singleton class of $\alpha$ (since $M_U$ is the trivial lattice no other choice for $S(b)(U)$ is possible), while if $U = \{k\} \subset I$, then $S(b)(U) = a_k$ (where $a_k$ is the component of $a$ that is contained in $L_k$); also, $R(S(b)) = \beta/\alpha \in \Pi(I/\alpha)$. If $y = (\ldots, c_i, \ldots, \rho/\alpha) \in \Pi(\{M_U : U \in I/\alpha\})$ (so $\rho \geq \alpha$ in $\Pi(I)$), then $y$ has a unique preimage $S^{-1}(y) = d$ in $I[a,1]$: The right-most element of $d$ will be $\rho$, and for $n \in I$, $d(n) = 1_n$ if $n/\rho$ is a non-singleton class, and $d(n) = a_n$ otherwise. Both $S$ and $S^{-1}$ are order-preserving, and since $I[a,1]$ and $\Pi(\{M_U : U \in I/\alpha\})$ are both complete, $S$ is a lattice isomorphism, completing the proof of (3). (4) follows directly from definitions. (5)(a) follows from the observation of Example 1.2 that a $\Pi$-product lattice with a unique co-atom must have two factors, and that if $L = \Pi(A,B)$ is a two-factor $\Pi$-product lattice, then $L \cong (A \times B) \oplus 1$.

For (5)(b), suppose $|I| > 2$ and that $T : L = \Pi(\{L_i : i \in I\}) \to M = \Pi(\{N_j : j \in J\})$ is an isomorphism. Let $j, k$ be the meet of maximal elements of $L$ and $M$ respectively. By Lemma 1.4, $I[j,1]$ and $I[k,1]$ are isomorphic to non-trivial partition lattices, so both are simple lattices. Since $T$ is an isomorphism, $T(I[j,1])$ is both isomorphic to a non-trivial partition lattice and an upper interval in $M$. From (2) of this lemma, $T(j) \geq k$. Similarly, $T^{-1}(k) \geq j$. It follows that $T(j) = k$.

As in Observation 3.1, for $i \in I$, let $\sigma_i$ be the maximal element of $\Pi(I)$ having classes $\{i\}$ and $I - \{i\}$. Abusing the notation slightly, let $\sigma_i$ also refer to a maximal element in $L$, namely $(\ldots, 1_i, \ldots, \sigma_i)$, where for all $i \in I$, $a_i = 1_i$. It follows from Observation 3.1 that the interval in $L$ given by $I[j, \sigma_i]$ is isomorphic to $\Pi(I - \{i\})$, a simple, non-trivial lattice. Because $T$ is a lattice isomorphism, it follows that there exists a maximal element $\tau_j \in M$ such that $I[k, \tau_j]$ is simple and $T(\sigma_i) = \tau_j$. Thus $T$ induces a bijection $\phi : I \to J$. Observe that $R^{-1}(\sigma_i)$ is an interval of $L$ whose every element is bounded above by exactly one maximal element of $L$, namely $\sigma_i$. Thus, $T(R^{-1}(\sigma_i))$ is an interval whose every element is bounded by just one maximal element of $M$, and this maximal element must be $\phi(\sigma_i)$. Now using (4) of this lemma, it follows that $L_i \cong N_{\phi(i)}$.

Conversely, if there is a bijection $\phi : I \to J$ such that for all $i \in I$ there exists an isomorphism $\phi_i : L_i \to N_{\phi(i)}$, then the map $W : \Pi(\{L_i : i \in I\}) \to \Pi(\{N_j : j \in J\})$ given by $W(\ldots, a_i, \ldots, \alpha) = (\ldots, b_{\phi(i)}, \ldots, \phi(\alpha))$, where $b_{\phi(i)} = \phi_i(a_i)$ and $\phi(\alpha)$ is the partition of $J$ formed from $\alpha$ under the bijection $\phi$, is easily seen to be an isomorphism between $\Pi(\{L_i : i \in I\})$ and $\Pi(\{N_j : j \in J\})$. $\square$

## 4. Proof of Theorem 2.4

An important property of congruences of intransitive flat M-Sets that are not contained in $\mathcal{J}$ is given in Lemma 4.1; its corollary, Corollary 4.2, provides a useful characterization of Property K. Lemma 4.1 and Corollary 4.2 are both due to Vernikov.

**Lemma 4.1.** *Suppose that $\mathbf{X} = \langle \sqcup_{i \in I} X_i, M \rangle$ is an intransitive flat M-Set, $\theta \in Con(\mathbf{X})$, and $(c,d) \in \theta - \mathcal{J}$. Then $\mathbf{X_c}/\theta_c \cong (X_c \sqcup X_d)/\theta_{c,d} \cong \mathbf{X_d}/\theta_d$.*

Proof: It can be assumed that $X = X_c \sqcup X_d$. Let $F : \mathbf{X_c}/\theta_c \to \mathbf{X}/\theta$ be such that for all $u/\theta_c \in \mathbf{X_c}/\theta_c$, $F(u/\theta_c) = u/\theta$. It is clear that $F$ is a well-defined and injective map. Let $v \in X_d$, and consider $v/\theta \in (X_c \sqcup X_d)/\theta$. Because $M$ acts transitively on $X_d$, there exists $m \in M$ such that $m(d) = v$. Thus, $(m(c), m(d)) =$

$(m(c), v) \in \theta$ and $F(m(c)/\theta_c) = v/\theta$, from which it follows that $F$ is bijective. For $m \in M$, $F(m(u/\theta_c)) = F(m(u)/\theta_c) = m(u)/\theta = m(u/\theta) = m(F(u/\theta_c))$, the first and second-to-last equality because $\theta \in Con(\mathbf{X})$, completing the proof. $\square$

**Corollary 4.2.** *Let $\mathbf{X} = \langle \sqcup_{i\in I} X_i, M\rangle$ be a flat intransitive M-Set. Then $\mathbf{X}$ does not satisfy Property K if and only if there exists a congruence $\gamma \in Con(\mathbf{X})$ containing a pair $(c,d) \in \gamma - \mathcal{J}$, $\mathbf{X_c}/\gamma_c \cong \mathbf{X_d}/\gamma_d$, and $\mathbf{X_c}/\gamma_c$ is not isomorphic to the trivial M-Set.*

**Lemma 4.3.** *Let $\mathbf{X} = \langle \sqcup_{i\in I} X_i, M\rangle$ be an intransitive M-Set, and let $u, v \in X$ with $X_u \neq X_v$.*

(1) *$i : \langle X_u \sqcup X_v, M\rangle \to \mathbf{X}$, given by $i(x) = x$, is a unary algebra monomorphism.*
(2) *$\iota : Con(\langle X_u \sqcup X_v, M\rangle) \to I[\Delta, \nabla_{u,v}]$ where for all $\alpha \in Con(\langle X_u \sqcup X_v, M\rangle)$, $\iota(\alpha) = Cg(\alpha)$ is a cover-preserving lattice isomorphism that satisfies the following: If $\beta \succ \alpha$ in $Con(\langle X_u \sqcup X_v, M\rangle)$, then $Tr\,\langle \alpha, \beta\rangle = Tr\,\langle \iota(\alpha), \iota(\beta)\rangle$.*

Proof: It is routine to verify that $i$ is a unary algebra monomorphism and that $\iota$ is an order-preserving lattice injection, and these guarantee that $Tr\,\langle \alpha, \beta\rangle = Tr\,\langle \iota(\alpha), \iota(\beta)\rangle$. $\square$

**Lemma 4.4.** *Suppose $\mathbf{X}$ is a flat M-Set satisfying Property K. Then all homomorphic images of $\mathbf{X}$ satisfy Property K.*

Proof: Let $\mathbf{X}$ and $\gamma \in Con(\mathbf{X})$. For contradiction, assume $\mathbf{X}/\gamma$ does not satisfy Property K, a failure witnessed by $a, b, c \in X$, where $(\overline{a}, \overline{b}) \in \mathcal{J}_\gamma - Cg(\overline{a}, \overline{c})$, and $(\overline{a}, \overline{c}) \notin \mathcal{J}_\gamma$. Observe that $(a, c) \notin \mathcal{J}$, and $\mathbf{X}$ satisfies Property K now implies that $X_a \times X_a$ is contained in $Cg(a, c)$.

That $(a, b) \in \mathcal{J}$ would then imply that $(a, b) \in Cg(a, c)$, from which it follows that $(\overline{a}, \overline{b}) \in Cg(\overline{a}, \overline{c})$, a contradiction. On the other hand, if $(a, b) \notin \mathcal{J}$, then $(\overline{a}, \overline{b}) \in \mathcal{J}_\gamma$ implies there exists $(u, v) \in \gamma \wedge (J_a \times J_b)$. That $\mathbf{X}$ satisfies Property K implies that $(a, b) \in Cg(u, v)$. That $Cg(u, v) \leq \gamma$ means that $\overline{a} = \overline{b}$, which is not possible. $\square$

The following observation is folklore and will be used without comment.

**Lemma 4.5.** *Let $\alpha, \beta, \gamma \in Con(\mathbf{X})$, where $\mathbf{X}$ is an M-Set. If $\beta \succ \alpha$, $\gamma \wedge \beta = \alpha$, and $\gamma \vee \beta \subseteq \gamma \circ \beta \circ \gamma$, then $\beta \vee \gamma \succ \gamma$.*

Theorem 2.4 will be proven in four parts. First it will be shown in a single proof that (1) implies (2) and that (3) implies (2). Then it will be shown that (2) implies (3). (2) implies (5) has been already proven in Lemma 1.13(8). Observe that (5) implies (4) is obvious, so the proof of Theorem 2.4 will be completed with the proof of (4) implies (2).

**Proof of (1) implies (2), and proof of (3) implies (2).** It will be shown that if a flat M-Set $\mathbf{X}$ fails Property K, then $Con(\mathbf{X})$ contains a cover-preserving embedded non-uniformly labeled $A(1)$. Repeated use will be made (in this part, and in other parts of the proof of this theorem) of Remark 2.2(1) which guarantees that if $\rho$ is a congruence of an M-Set $\mathbf{X}$ and $Con(\mathbf{X}/\rho)$ has a non-uniformly labeled cover-preserving embedded $A(1)$, then so does $Con(\mathbf{X})$.

Failures of Property K occur in two-orbit subalgebras, so by the statement of Lemma 4.3 (which states, essentially, that for $c, d$ in different orbits, the transitivity labeling of the Hasse diagram of $Con(\langle X_c \sqcup X_d, M\rangle)$ coincides with the transitivity labeling of $I[\Delta, \nabla_{c,d}]$), it can be assumed that $\mathbf{X}$ has two orbits. So let $\mathbf{X} = \langle X_c \sqcup X_d, M\rangle$, a two-orbit flat M-Set that does not satisfy Property K, as witnessed by $Cg(c, d)$. By Lemma 4.1 and Corollary 4.2, it can be assumed that $\mathbf{X_c}/Cg(c, d)_c \cong \mathbf{X_d}/Cg(c, d)_d$, a non-trivial M-Set. Let $\rho = Cg(c, d)_c \vee Cg(c, d)_d$; observe $\mathbf{X}/\rho$ has two isomorphic orbits (the two orbits isomorphic to $\mathbf{X_c}/Cg(c, d)_c$ and $\mathbf{X_d}/Cg(c, d)_d$, respectively).

Now let $\mathbf{Z} = \mathbf{X}/\rho$. Since the two orbits of $\mathbf{Z}$ are isomorphic, without loss of generality $\mathbf{Z}$ can be assumed to have the following form: $\mathbf{Z} = \langle \dot{V} \sqcup \ddot{V}, M\rangle$, where $M$ acts the same on both orbits; that is, for $a, b \in V$, and for $m \in M$, $m(\dot{a}) = \dot{b}$ if and only if $m(\ddot{a}) = \ddot{b}$.

That $Con(\dot{\mathbf{V}})$ is algebraic implies there exist $\dot{\beta} \succ \dot{\alpha}$, a pair of covering congruences in $Con(\dot{\mathbf{V}})$. Since $\dot{\mathbf{V}}$ and $\ddot{\mathbf{V}}$ are isomorphic, there exists a corresponding covering pair $\ddot{\beta} \succ \ddot{\alpha}$ in $Con(\ddot{\mathbf{V}})$. Observe that $\dot{\beta} \vee (\dot{\alpha} \vee \ddot{\alpha}) = \dot{\beta} \vee \ddot{\alpha} = \dot{\beta} \cup \ddot{\alpha} \subseteq (\dot{\alpha} \vee \ddot{\alpha}) \circ \dot{\beta} \circ (\dot{\alpha} \vee \ddot{\alpha})$. From Lemma 4.5 and since the inclusion of the last sentence hold with $\ddot{\beta}$ in place of $\dot{\beta}$, the following holds:

(1) $\dot{\beta} \vee (\ddot{\alpha} \vee \dot{\alpha}) \succ \ddot{\alpha} \vee \dot{\alpha}$
(2) $\ddot{\beta} \vee (\ddot{\alpha} \vee \dot{\alpha}) \succ \ddot{\alpha} \vee \dot{\alpha}$

Now replace $\mathbf{Z} = \langle \dot{V} \sqcup \ddot{V}, M\rangle$ by $\mathbf{Z}/(\dot{\alpha} \vee \ddot{\alpha}) = \mathbf{W}$, and observe that the two orbits of $\mathbf{W}$ are isomorphic to $\dot{\mathbf{V}}/\dot{\alpha}$ and $\ddot{\mathbf{V}}/\ddot{\alpha}$, respectively, and by the manner in which $\dot{\alpha}$ and $\ddot{\alpha}$ were selected (they were "counterpart" congruences in a pair of identical orbits), it can be assumed that $\mathbf{W}$ has identical orbits. But now in addition, $\mathbf{W}$ has a pair of minimal congruences, both contained in the $\mathcal{J}$ congruence of $\mathbf{W}$.

Some properties of $\mathbf{W}$ are given.

- The flat M-Set $\mathbf{W}$ has two orbits; the orbits are identical. Abusing the notation slightly, denote the two orbits by $\dot{V}$ and $\ddot{V}$.
- $\mathbf{W}$ does not satisfy Property K. Indeed, with $c \in V$, the congruence $Cg(\dot{c}, \ddot{c}) = \Delta \cup \{(\dot{k}, \ddot{k}) : k \in V\}$ is minimal and meets $\mathcal{J}$ trivially.
- $\mathbf{W}$ has a minimal congruence contained in $\mathcal{J}_1$, and, once again abusing the notation slightly, let this congruence be denoted $\dot{\beta}$. That $\mathbf{W}$ is a "doubling" of the transitive M-Set $\dot{\mathbf{V}}$ implies there exists a counterpart congruence $\ddot{\beta}$, one that is minimal and contained in $\mathcal{J}_2$ and satisfies the following: For all $a, b \in V$, if $(\dot{a}, \dot{b}) \in \dot{\beta}$ if and only if $(\ddot{a}, \ddot{b}) \in \ddot{\beta}$.
- $(\dot{\beta} \vee \ddot{\beta}) \wedge Cg(\dot{c}, \ddot{c}) = \Delta$.

**Claim 4.6.** The following hold:

(1) $Cg(\dot{c}, \ddot{c}) \vee \dot{\beta} = Cg(\dot{c}, \ddot{c}) \vee \ddot{\beta}$;
(2) $Cg(\dot{c}, \ddot{c}) \vee \dot{\beta} \succ Cg(\dot{c}, \ddot{c})$; and

(3) $Cg(\dot{c},\ddot{c}) \vee \dot{\beta} \succ (\dot{\beta} \vee \ddot{\beta})$.

Proof: As remarked above, $(\dot{a},\dot{b}) \in \dot{\beta}$ if and only if $(\ddot{a},\ddot{b}) \in \ddot{\beta}$. From $(\dot{a},\ddot{a})$, $(\dot{b},\ddot{b})$ are both contained in $Cg(\dot{c},\ddot{c})$, it follows both that $\ddot{\beta} \leq Cg(\dot{c},\ddot{c}) \vee \dot{\beta}$, and that $\dot{\beta} \leq Cg(\dot{c},\ddot{c}) \vee \ddot{\beta}$, from which (1) follows.

Observe that $Cg(\dot{c},\ddot{c}) \vee \dot{\beta} \subseteq Cg(\dot{c},\ddot{c}) \circ \dot{\beta} \circ Cg(\dot{c},\ddot{c})$. By Lemma 4.5, $Cg(\dot{c},\ddot{c}) \vee \dot{\beta} \succ Cg(\dot{c},\ddot{c})$, completing the proof of (2).

For (3), using (1) and the proof of (2), it follows that $Cg(\dot{c},\ddot{c}) \vee (\dot{\beta} \vee \ddot{\beta}) = Cg(\dot{c},\ddot{c}) \vee \dot{\beta} \subseteq Cg(\dot{c},\ddot{c}) \circ \dot{\beta} \circ Cg(\dot{c},\ddot{c})$. Now observe that $Cg(\dot{c},\ddot{c}) \circ \dot{\beta} \circ Cg(\dot{c},\ddot{c}) \subseteq \ddot{\beta} \cup \dot{\beta} \cup (Cg(\dot{c},\ddot{c}) \circ \dot{\beta}) \cup (\dot{\beta} \circ Cg(\dot{c},\ddot{c}))$. Moreover, $Cg(\dot{c},\ddot{c}) \circ \dot{\beta} = \ddot{\beta} \circ Cg(\dot{c},\ddot{c})$, from which it now follows that the union given in the previous sentence is contained in $(\dot{\beta} \vee \ddot{\beta}) \circ Cg(\dot{c},\ddot{c}) \circ (\dot{\beta} \vee \ddot{\beta})$. Applying Lemma 4.5 yields (3), and completes the proof of the claim.

Let $\phi = Cg(\dot{c},\ddot{c}) \vee \dot{\beta}$. From Claim 4.6, it now follows that

(1) $\phi \succ (\dot{\beta} \vee \ddot{\beta}) \succ \dot{\beta} \succ \Delta$;
(2) $(\dot{\beta} \vee \ddot{\beta}) \succ \ddot{\beta}$; and
(3) $\phi \succ Cg(\dot{c},\ddot{c}) \succ \Delta$.

Thus, the six congruences named above ($\{\Delta, \dot{\beta}, \ddot{\beta}, \dot{\beta} \vee \ddot{\beta}, Cg(\dot{c},\ddot{c}), \phi\}$) form a cover-preserving embedded $A(1)$ in $Con(\mathbf{W})$. Since $\dot{\alpha}, \ddot{\alpha}$ are both minimal congruences and both are contained in $\mathcal{J}$, it follows from Remark 2.2(2) that $Tr\langle \Delta, \dot{\beta}\rangle = + = Tr\langle \Delta, \ddot{\beta}\rangle$. That $Cg(\dot{c},\ddot{c}) \cap \mathcal{J} = \Delta$, from Remark 2.2(2) again, the minimality of $Cg(\dot{c},\ddot{c})$ implies that $Tr\langle \Delta, Cg(\dot{c},\ddot{c})\rangle = -$. That is, the above cover-preserving embedded $A(1)$ is non-uniformly labeled.

As remarked above, the transitivity-labeled congruence lattice of $\mathbf{W}$ is identical to that of an interval of $Con(\mathbf{X})$ (where $\mathbf{X}$ was the original flat M-Set $\mathbf{X}$ that failed Property K); thus $Con(\mathbf{X})$ contains a cover-preserving, non-uniformly labeled $A(1)$, completing the proofs of both (1) implies (2) and (3) implies (2).

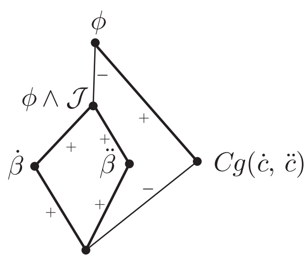


Figure 4.1. $\mathbf{X}$ fails Property K leads to above labeled cover-preserving $A(1)$

Note that $\phi \subseteq Cg(\dot{c},\ddot{c}) \circ \dot{\beta} \circ Cg(\dot{c},\ddot{c})$ implies that $\phi \wedge \mathcal{J} = \dot{\beta} \vee \ddot{\beta}$, as indicated in Figure 4.1 above.

**Proof of (2) implies (3).** By Lemma 4.4, for flat M-Sets, Property K is preserved under homomorphism. As remarked, the transitivity labeling of the

Hasse diagram of $I[\theta, \nabla]$ coincides with the transitivity labeling of the Hasse diagram of $Con(\mathbf{X}/\theta)$. Thus, to prove (2) implies (3) it suffices to show that if a flat M-Set $\mathbf{X}$ has a congruence lattice that contains a cover-preserving embedded non-uniformly labeled copy of $A(1)$, the minimal element of which is $\Delta$, then $\mathbf{X}$ fails Property K, so for contradiction, assume that $\mathbf{X}$ satisfies Property K.

**Claim 4.7.** If an intransitive flat M-Set $\mathbf{Z}$ satisfies Property K, $\phi \succ \Delta$ in $Con(\mathbf{Z})$, and $Tr\langle\Delta, \phi\rangle = -$, then there exist $u, v \in Z$ such that $\phi - \Delta = \{(u,v),(v,u)\}$. Moreover, if $\rho \succ \Delta$ in $Con(\mathbf{X})$ and $\rho \neq \phi$, then $(\rho \vee \phi) \succ \rho$.

Proof: That $Tr\langle\Delta, \phi\rangle = -$ implies by Remark 2.2(2) that $\phi \wedge \mathcal{J} = \Delta$; therefore, $(u,v) \in \phi - \Delta$ implies that $X_u$ and $X_v$ are distinct orbits. That $\mathbf{X}$ satisfies Property K and $\phi \wedge \mathcal{J} = \Delta$ implies that $\mathcal{J}_u = \mathcal{J} \wedge Cg(X_u \times X_u) = \Delta$. So $X_u$ and $X_v$ are singleton orbits, from which it follows that $\phi - \Delta = \{(u,v),(v,u)\}$. With $\rho \succ \Delta$, observe that $\rho \vee \phi \subseteq \rho \circ \phi \circ \rho$, and this implies that $\rho \vee \phi \succ \rho$, completing the proof of the claim.

Because $\gamma \vee \alpha$ does not cover $\alpha$, it follows from Claim 4.7 that $Tr\langle\Delta, \gamma\rangle = +$. Because the labeling is not uniform, it follows from Remark 2.2(2) that $\alpha \vee \gamma = \beta \vee \gamma \not\leq \mathcal{J}$ (otherwise, all labels are $+$). Thus $Tr\langle\Delta, \alpha\rangle = - = Tr\langle\Delta, \alpha\rangle$. It will be shown that there is no $\{r,s,t\} \subseteq X$ consisting of three distinct elements that satisfy "$r \ \alpha \ s \ \gamma \ t$": If such a trio existed, then $(s,t) \in \mathcal{J} - \Delta$ and $(r,s) \in \alpha - \mathcal{J}$. That $\mathbf{X}$ satisfies Property K implies that $(s,t) \in Cg(r,s) = \alpha$, which is not possible. Now it follows readily that $(\alpha \circ \gamma) \cup (\gamma \circ \alpha) \subseteq (\alpha \cup \gamma)$, from which it follows that $(\alpha \vee \gamma) \subseteq (\gamma \circ \alpha) \cup \alpha \cup \gamma$. By Lemma 4.5, $\gamma \vee \alpha \succ \alpha$, and this is not the case. Thus $\mathbf{X}$ satisfies Property K implies there exists no non-uniformly labeled cover-preserving embedded $A(1)$ in $Con(\mathbf{X})$. □

**Proof of (2) implies (5).** As mentioned, this part was proved with Lemma 1.13(8).

**Proof of (4) implies (2).** Assume for contradiction that $\mathbf{X}$ fails Property K but $Con(\mathbf{X})$ is isomorphic to a Π-product lattice.

**Claim 4.8.** If $I$ is a set with more than one element, $L \cong \Pi(I)$, and $L$ represents an intransitive flat M-Set $\mathbf{Y}$, then $\mathbf{Y}$ has $|I|$ orbits, each of which is a singleton set.

Proof of claim: It is known that $\Pi(I)$ is semimodular; therefore, it has no cover-preserving embedded copy of $A(1)$. Using the already proven logical equivalence of (2) and (3), $\mathbf{Y}$ satisfies Property K, and using (2) implies (5) (also proven) it follows that $Con(\mathbf{Y})$ is isomorphic to the non-trivial Π-product lattice $\Pi(\{Con(\mathbf{X}_i) : i \in I\})$, where the factors are the congruence lattices of the orbits of $\mathbf{Y}$. That $\mathbf{Y}$ satisfies Property K implies that $Con(\mathbf{Y}) = Con(\Pi(\mathbf{Y}))$, and Lemma 1.13(7) implies that $\mathcal{J}$ is the meet of the maximal congruences of $\mathbf{Y}$. That $Con(\mathbf{Y})$ is a Π-product lattice whose maximal elements meet to $\mathcal{J}$ implies that the map $x \to (x \wedge \mathcal{J})$ (for all $x \in Con(\mathbf{Y})$) is a lattice homomorphism. That

$L \cong Con(\mathbf{Y})$ and $L$ is simple implies that the lattice homomorphism is an isomorphism or is trivial. It is not an isomorphism because then $F(\nabla) = F(\mathcal{J})$ would imply that $\nabla = \mathcal{J}$, which is not possible since $\mathbf{X}$ is an intransitive flat M-Set. Thus, $\mathcal{J}$ must be $\Delta$, and since $I[\Delta, \mathcal{J}] \cong \prod_{i \in I} Con(\mathbf{X}_i)$, the direct product of the congruence lattices of the orbits of $\mathbf{X}$, this implies that all orbits of $\mathbf{X}$ are trivial, which establishes the claim.

Returning to the proof, because $\mathbf{X}$ does not satisfy Property K, there exist $c, d \in X$ such that $X_c \neq X_d$ and $Cg(c,d)_c < \mathcal{J}_c$ and $Cg(c,d)_d < \mathcal{J}_d$. Let $\gamma \in Con(\mathbf{X})$ be such that $\gamma \leq \mathcal{J}$ and $\gamma$ is the congruence that identifies all orbits with the exception of $X_c$ and $X_d$; that is, $\gamma = Cg((X - (X_c \cup X_d))^2)$. By Lemma 3.2(3), $Con(\mathbf{X}/\gamma)$ is also isomorphic to a $\Pi$-product lattice. By choice of $\gamma$, $\mathbf{X}/\gamma$ does not satisfy Property K. As is the case for any flat intransitive M-Set, $\mathcal{J}_\gamma$ is a meet of some subset of the maximal congruences of $\mathbf{X}/\gamma$. Observe that $(\mathbf{X}/\gamma)/\mathcal{J}_\gamma \cong \Pi(m)$, where $m$ is the number of orbits of both $\mathbf{X}/\gamma$ and of $(\mathbf{X}/\gamma)/\mathcal{J}_\gamma$. Note that $m \leq 3$.

Let $\mu \in Con(\mathbf{X})$ be such that $\mu \geq \gamma$ and $\mu/\gamma$ is the meet of all maximal congruences of $\mathbf{X}/\gamma$. Using that $\mathbf{X}/\gamma$ is a $\Pi$-product lattice, since $\mu/\gamma$ is the meet of maximal elements of a $\Pi$-product lattice, Lemma 1.4(2) guarantees that $I[\mu/\gamma, \nabla]$ is isomorphic to a partition lattice. By Claim 4.8, $\mathbf{X}/\mu$ has singleton orbits. Since $\mathcal{J}_\gamma \geq \mu/\gamma \geq \gamma/\gamma$, it follows that $\mathbf{X}/\gamma$, $(\mathbf{X}/\gamma)/\mathcal{J}_\gamma$, and $(\mathbf{X}/\gamma)/(\mu/\gamma)$ all have the same number of orbits, namely $m$.

Then $I[\mu/\gamma, \nabla] \cong \Pi(m) \cong I[\mathcal{J}_\gamma, \nabla]$, and the finiteness of $m$ implies that $\mu/\gamma = \mathcal{J}_\gamma$. Thus $\mathcal{J}_\gamma$ is the meet of the maximal congruences of the $\Pi$-product lattice $Con(\mathbf{X}/\gamma)$, from which it follows that the map $x \to (x \wedge \mathcal{J}_\gamma)$ is a lattice homomorphism of $Con(\mathbf{X}/\mathcal{J})$.

For $x \in X$ let $\overline{x} = x/\gamma$. Observe that $[Cg(\overline{c}, \overline{d}) \vee (Cg(X_{\overline{c}} \times X_{\overline{c}}))] \wedge \mathcal{J}_\gamma$ contains $Cg(X_{\overline{d}} \times X_{\overline{d}})$, while $(Cg(\overline{c}, \overline{d}) \wedge \mathcal{J}_\gamma) \vee (Cg(X_{\overline{c}} \times X_{\overline{c}}) \wedge \mathcal{J}_\gamma)$ does not contain $Cg(X_{\overline{d}} \times X_{\overline{d}})$. (In fact, the specific failure of Property K under discussion is determined by (and equivalent to) the non-containment of $X_{\overline{d}} \times X_{\overline{d}}$ in $Cg(\overline{c}, \overline{d})$.) Thus $x \to (x \wedge \mathcal{J}_\gamma)$ is not a homomorphism, from which it follows that $Con(\mathbf{X})$ is not isomorphic to a $\Pi$-product lattice, a contradiction that completes the proof of (4) implies (2), and the proof of Theorem 2.4. $\square$

## 5. Proof of Theorem 2.8

A series of lemmas and a corollary lead up to Lemma 5.5, where it is proved that if $\mathbf{X}$ is a flat intransitive M-Set, then $Con(\Pi(\mathbf{X}))$ is a $\Pi$-possible sublattice of $Con(\mathbf{X})$.

**Lemma 5.1.** *Let $\mathbf{X} = \langle \sqcup_{i \in I} X_i, M \rangle$ be a flat M-Set with $\gamma \in Con(\mathbf{X})$ and $\gamma \leq \mathcal{J}$. Then $\Pi(\mathbf{X}/\gamma) \cong \Pi(\mathbf{X})/\gamma$.*

Proof: By Lemma 1.13(2), $\gamma \leq \mathcal{J}$ implies that $\gamma$ is a congruence relation on $\Pi(\mathbf{X})$. So "$\Pi(\mathbf{X})/\gamma$" makes sense. Also $\gamma \leq \mathcal{J}$ implies that $\mathbf{X}/\gamma = \langle \sqcup_{i \in I} X_i/\gamma_i, M \rangle$. Thus $\Pi(\mathbf{X}/\gamma)$ is $\langle \sqcup_{i \in I} X_i/\gamma_i, M^I \rangle$ where $M^I$ acts in the usual way on $\sqcup_{i \in I} X_i$. But

$\Pi(\mathbf{X})/\gamma$ is also $\langle \sqcup_{i\in I} X_i/\gamma_i, M^I\rangle$, with $M^I$ acting the same as it does in $\Pi(\mathbf{X}/\gamma)$, and the two flat M-Sets are isomorphic under an identity function involving the underlying sets. $\square$

**Lemma 5.2.** *Let $L$ be an algebraic lattice with a $\Pi$-possible sublattice $A$. Let the meet of the maximal elements of $A$ be $k$, and let $l \in L$ be such that $k \geq l$. (So by definition of a $\Pi$-possible sublattice, $l \in A$). Then $A \cap I[l,1]$ is a $\Pi$-possible sublattice of the lattice $I[l,1]$, and the meet of the maximal elements of $A\cap I[l,1]$ is $k$.*

Proof: That $A$ is cover-preserving embedded in $L$ implies that $A\cap I[l,1]$ is cover-preserving embedded in $I[l,1]$. By Lemma 3.2(3), $A \cap I[l,1]$ is isomorphic to a $\Pi$-product lattice. Thus, Definition 2.6(1) and (2) (of a $\Pi$-possible sublattice) are satisfied.

That $k \geq l$ implies that the maximal elements of $I[l,1]$ meet to $k$. Because $A$ is a $\Pi$-possible sublattice of $L$, by (3)(a) of Definition 2.6, $I[l,k]$ and $I[k,1]$, as intervals of $L$, are contained in $A \cap I[l,1]$, so $A \cap I[l,1]$ also satisfies (3)(a). That $A$ is $\Pi$-possible and satisfies Property (3)(b) of that definition, and that the meet of maximals of $A \cap I[l,1]$ is $k$, together imply that $A \cap I[l,1]$ also satisfies Property (3)(b). Thus $A \cap I[l,1]$ is a $\Pi$-possible sublattice of $I[l,1]$. $\square$

Let $\mathbf{X} = \langle \sqcup_{i\in I} X_i, M\rangle$ be an intransitive flat M-Set, and $\alpha \in Con(\mathbf{X})$. Recall that $R(\alpha)$ partitions $I$ as follows: $(i,j) \in R(\alpha)$ if $\alpha \cap (X_i \times X_j) \neq \emptyset$.

**Lemma 5.3.** *Let $\mathbf{X} = \langle \sqcup_{i\in I} X_i, M\rangle$ be an intransitive flat M-Set having a maximal congruence $\gamma$. If $\gamma \not\geq \mathcal{J}$, then $R(\gamma)$ is the universal relation on $I$.*

Proof: Suppose $\gamma \not\geq \mathcal{J}$ and is maximal. Let $\Gamma \geq \mathcal{J}$ be the congruence that identifies all orbits identified by $R(\gamma)$. (So $\Gamma = \vee\{Cg(X_i \times X_j) : (i,j) \in \gamma\}$.) If $\gamma$ is not universal on $I$, then $\Gamma$ is a proper congruence that contains $\gamma$, so the maximality of $\gamma$ implies that $\Gamma = \gamma$, but then $\gamma \geq \mathcal{J}$, a contradiction. $\square$

**Corollary 5.4.** *Let $\mathbf{X}$ be a flat M-Set having a singleton orbit. Then the meet of the maximal congruences of $\mathbf{X}$ is $\mathcal{J}$.*

Proof: Let $\gamma \in Con(\mathbf{X})$ be such that $R(\gamma)$ is the universal relation on $I$. That there exists a singleton orbit and that each orbit is a transitive M-Set together imply that $\gamma = \nabla$. The lemma now follows from Lemma 5.3. $\square$

**Lemma 5.5.** *Let $\mathbf{X} = \langle \sqcup_{i\in I} X_i, M\rangle$ be an intransitive flat M-Set. Then $Con(\Pi(\mathbf{X}))$ is a $\Pi$-possible sublattice of $Con(\mathbf{X})$.*

Proof: By Lemma 1.13(4) and (7), $\Pi(\mathbf{X})$ is known to be a $\Pi$-product lattice whose maximal elements meet to $\mathcal{J}$, and which satisfies $I[\Delta,\mathcal{J}] \cup I[\mathcal{J},\nabla] \subseteq Con(\Pi(\mathbf{Y}))$. Thus (2) and (3)(a) of the $\Pi$-possible definition (Definition 2.6) are satisfied.

It is proven next that (1) of the $\Pi$-possible definition is satisfied; that is, that $Con(\Pi(\mathbf{X}))$ is a cover-preserving embedded sublattice of $Con(\mathbf{X})$. So suppose $\gamma \succ \mu$ in $Con(\Pi(\mathbf{X}))$. Let $(\dots,\mu_i,\dots,R(\mu))$ be $\Pi$-product coordinates of $\mu$, and let $(\ldots, \gamma_i, \ldots, R(\gamma))$ be $\Pi$-product coordinates of $\gamma$. Suppose first that $R(\gamma) = R(\mu)$, from which it follows readily that $\gamma \succ \mu$ in $Con(\Pi(\mathbf{X}))$ implies that there exists $i \in I$ such that $\gamma_i \succ \mu_i$ (in $Con(\Pi(\mathbf{X})))$ and for all $j \in I - \{i\}$, $\gamma_j = \mu_j$. That $\gamma_i, \mu_i \leq \mathcal{J}$ implies that $\gamma_i$ and $\mu_i$ are congruences of $\mathbf{X}$ itself. That $I[\Delta, \mathcal{J}]$ is an interval in $Con(\Pi(\mathbf{X}))$ implies that for any $(x, y) \in \gamma - \mu$, $(x, y) \in \gamma_i - \mu_i$, and $\mu_i \vee Cg(x, y) = \gamma_i$ in $Con(\mathbf{X})$, from which it follows that $\gamma \succ \mu$ in $Con(\mathbf{X})$.

For the second and last case, suppose that $\gamma \succ \mu$ in $Con(\Pi(\mathbf{X}))$ and $R(\gamma) \neq R(\mu)$. It follows readily that $R(\gamma) \succ R(\mu)$ and that $\mu_i = \gamma_i$ for all $i \in I$. That $R(\gamma) \succ R(\mu)$ now implies for all $(s, t) \in X^2$ that $(s, t) \in \gamma - \mu$ if and only if for some $i, j \in I$, $s \in X_i$, $t \in X_j$, $(i, j) \in R(\gamma) - R(\mu)$. That $\gamma \succ \mu$ in $Con(\Pi(\mathbf{X}))$ and $\mu \vee Cg(s, t) = \gamma$ implies that $(X_i \times X_i) \cup (X_j \times X_j) \subset \mu$. It follows that $\gamma \succ \mu$ in $Con(\mathbf{X})$.

To complete the proof of the lemma, it must be proven that Property (3)(b) is satisfied by $Con(\Pi(\mathbf{X}))$. Suppose $\gamma \in Con(\Pi(\mathbf{X}))$ is such that $\gamma \leq \mathcal{J}$ and that $I[\gamma, \mathcal{J}]$ is isomorphic to a non-trivial partition lattice $B$.

By Lemma 3.2(3), $Con(\Pi(\mathbf{Y})/\gamma)$ is isomorphic to a $\Pi$-product lattice, and by Lemma 1.13(4), $Con(\Pi(\mathbf{Y})/\gamma) \cong \Pi(\{Con(\mathbf{X}_i/\gamma_i) : i \in I\})$. Moreover, the meet of the maximal congruences of $\Pi(\mathbf{Y})/\gamma$ is $\mathcal{J}_\gamma$.

Thus $I[\gamma, \mathcal{J}]$ is isomorphic to $\prod_{i \in I} Con(\mathbf{X}_i/\gamma_i) \cong B$, and the lattice $B$ is directly indecomposable. That $\gamma \leq \mathcal{J}$ means that $\mathbf{X}/\gamma$ is still intransitive; therefore, for some $i \in I$, $Con(\mathbf{X}_i/\gamma_i)$ is trivial, and this implies that $\mathbf{X_i}/\gamma_i$ is a singleton orbit. By Corollary 5.4, the maximal congruences of $\mathbf{X}/\gamma$ meet to $\mathcal{J}_\gamma$, and it follows that every maximal congruence bounded below by $\gamma$ is also bounded below by $\mathcal{J}$, completing the proof that Property (3)(b) is satisfied, and completing the proof of the lemma. $\square$

**Lemma 5.6.** *Suppose $A$ and $B$ are both $\Pi$-possible sublattices of an algebraic lattice $L$, with $a$ the meet of the maximal elements of $A$, and with $b$ the meet of the maximal elements of $B$. If $a$ and $b$ are comparable, then $a = b$.*

Proof: Suppose for contradiction that $a > b$. The hypotheses imply that $I[a, 1]$ and $I[b, 1]$ are both isomorphic to partition lattices. Observation 3.1(2) guarantees that there exists $a_0 \in L$ such that $a \succ a_0 \geq b$ and a lattice element $n$ that is maximal in $I[b, 1]$, satisfying $n \geq a_0$ but $n \not\geq a$; thus, $A$ fails Property (3)(b), a contradiction. $\square$

**Lemma 5.7.** *Let $\mathbf{X} = \langle \sqcup_{i \in I} X_i, M \rangle$ be a flat intransitive M-Set. Suppose $\mu \in Con(\mathbf{X})$ is such that $\mu \wedge \mathcal{J} = \Delta$. Then $I[\Delta, \mu] \cong I[\Delta, R(\mu)]$, the latter an interval in $\Pi(I)$.*

Proof: Suppose $(u, v) \in \mu - \mathcal{J}$. Since $\mu \wedge \mathcal{J} = \Delta$, it follows that if $w \in X_v$ and $(u, w) \in \mu$, then $v = w$. That $X_u, X_v$ are orbits and $M$ acts transitively on them implies that $\mu_{u,v} \cap (X_u \times X_v)$ is a matching of the elements in $X_u$ and $X_v$; moreover, if $u_0 \in X_u, v_0 \in X_v$, and $(u_0, v_0) \in \mu$, then $Cg(u, v) = Cg(u_0, v_0)$.

Let $\mu \leq \beta$. Of course $R(\mu) \geq R(\beta)$, and, from the previous paragraph, $\beta$ is completely determined by $R(\beta)$. Thus the map $R : I[\Delta, \mu] \to I[\Delta, R(\mu)]$ is an order-preserving injection, and if $\Gamma \leq R(\mu)$, the structure of the congruences in $I[\Delta,\mu]$ (as described above) determines $\gamma \in I[\Delta,\mu]$ such that $R(\Gamma) = \gamma$. Note that $R$ and its inverse are clearly both order-preserving; thus, $R$ is a lattice isomorphism. $\square$

**Lemma 5.8.** *Let* $\mathbf{X}$ *be an intransitive flat M-Set, let* $A$ *be a* $\Pi$*-possible sublattice of* $Con(\mathbf{X})$*, and let* $\mu$ *be the meet of the maximal elements of* $A$*. Then* $\mu = \mathcal{J}$*.*

Proof: Suppose not. By Lemma 5.5, $Con(\Pi(\mathbf{X}))$ is a $\Pi$-possible sublattice of $Con(\mathbf{X})$, and the meet of the maximals of $Con(\Pi(\mathbf{X}))$ is $\mathcal{J}$. So by Lemma 5.6(2), it can be assumed that $\mu$ and $\mathcal{J}$ are incomparable.

**Case 1.** $\mu \wedge \mathcal{J} = \Delta$.

By Lemma 5.7, $I[\Delta,\mu] \cong I[\Delta, R(\mu)]$, the latter an interval of $\Pi(I)$ and therefore isomorphic to a direct product of non-trivial partition lattices. It follows from Observation 3.1(2) that there exists $\gamma \in I[\Delta,\mu]$ such that $\mu \succ \gamma$. In view of the description of the congruences in $I[\Delta,\mu]$ given in Lemma 5.7 and its proof, such a $\gamma$ can be formed by meeting $\mu$ with a maximal congruence $\nu$ such that $\nu \geq \mathcal{J}$. But this implies that $A$ fails Property (3)(b) (with $\nu \geq \gamma$ but $\nu \ngeq \mu$).

**Case 2.** $\mathcal{J} > \mu \wedge \mathcal{J} > \Delta$.

Let $\rho = \mu \wedge \mathcal{J}$. The meet of the maximal congruences of $\Pi(\mathbf{Y})/\rho$ is $\mathcal{J}_\rho$, and by Lemma 5.2, $A \cap I[\rho,\nabla]$ is a $\Pi$-possible sublattice of $Con(\mathbf{X}/\rho)$. It is clear that the maximal elements of $A \cap [\rho,\nabla]$ meet to $\mu/\rho$. But $\mathcal{J}_\rho \wedge \mu/\rho = \Delta$, and from Case 1, it follows that $A \cap I[\rho,\nabla]$ is not a $\Pi$-possible sublattice of $\mathbf{X}/\rho$, a contradiction. $\square$

The first part of Theorem 2.8 is proven next.

**Lemma 5.9.** *If* $\mathbf{Y}$ *is an intransitive flat M-Set, then* $Con(\Pi(\mathbf{Y})) = \mathbf{\Pi}(Con(\mathbf{Y}))$*.*

Proof: To prove the lemma it suffices to show that if $A$ is a $\Pi$-possible sublattice of $Con(\mathbf{X})$, then $A$ is contained in $Con(\Pi(\mathbf{X}))$. Since $A$ is $\Pi$-possible and thus cover-preserving embedded in $Con(\mathbf{X})$, it follows that maximal elements of $A$ are also maximal in $Con(\mathbf{X})$. By Lemma 5.8, the meet of the maximal elements of $A$ is $\mathcal{J}$.

Suppose for contradiction that $A$ contains a congruence $\alpha$ not in $Con(\Pi(\mathbf{X}))$. By Lemma 1.13, then there exists $(c,d) \in \alpha$ with $X_c \neq X_d$, but $X_c \times X_d$ is not contained in $\alpha$. But by arguments used earlier, then the map $x \to x \wedge \mathcal{J}$ would not be a lattice homomorphism of $A$ into $I[\Delta,\mathcal{J}]$, contradicting that $A$ is isomorphic to a $\Pi$-product lattice having maximal element $\mathcal{J}$. So $A \subseteq Con(\Pi(\mathbf{X}))$, completing the proof of the lemma. $\square$

Completing the proof of Theorem 2.8: Suppose $L$ is an algebraic lattice and $\mathbf{Y}$ is a flat M-Set that represents $L$. If $\mathbf{\Pi}(L)$ is trivial but $\mathbf{Y}$ is intransitive, then $Con(\Pi(\mathbf{Y}))$ is non-trivial. But by Lemma 5.8, $Con(\Pi(\mathbf{Y})) = \mathbf{\Pi}(Con(\mathbf{Y})) = \mathbf{\Pi}(L)$, which is trivial, a contradiction. Thus $\mathbf{\Pi}(L)$ is trivial implies $L$ has no representation by an intransitive flat M-Set.

Now given an algebraic lattice $L$ such that $\mathbf{\Pi}(L) = \Pi(\{L_j : j \in J\})$, a non-trivial $\Pi$-product lattice, by Lemma 5.9 any representation of $L$ by a flat intransitive M-Set $\mathbf{Y} = \langle \sqcup_{i\in I} X_i, M\rangle$ satisfies $Con(\Pi(\mathbf{Y})) \cong \Pi(\{L_j : i \in J\})$. By Lemma 1.13(4), $Con(\Pi(\mathbf{Y})) \cong \Pi(\{Con(\mathbf{X_j}) : j \in J\})$. Thus, $\Pi(\{Con(\mathbf{X}_j) : j \in J\}) \cong \Pi(\{L_i : i \in I\})$. By Lemma 3.2(5), if $|I| > 2$, the factors match up isomorphically with the congruence lattices of the orbits of $\mathbf{Y}$, and if $|I| = 2$, all other flat intransitive M-Sets that represent $L$ have two orbits and the product of the congruence lattices of the two orbits must be isomorphic to the direct product of the factors of $\Pi(\{L_i : i \in I\})$. $\square$

## 6. Conclusion

Theorem 2.4 and Theorem 2.8 provide, via $\Pi$-product lattices and $A(n)$, fundamental information about the congruence theory of flat intransitive M-Sets.

**6.0.1 Two-orbit flat M-Sets.** Two-orbit flat M-Sets are the obvious building blocks of the flat intransitive M-Sets, but given a two-orbit M-Set $\mathbf{X} = \langle X_1 \sqcup X_2, M\rangle$, that $\mathbf{\Pi}(Con(\mathbf{X})) = (Con(\mathbf{X}_1) \times Con(\mathbf{X}_2)) \oplus 1$ reveals nothing that was not already known. In [5], S. Radeleczki shows that if $\mathbf{T}$ is transitive, then for each congruence $\alpha$ of $\mathbf{T}$, the subgroup lattice of the automorphism group of $\mathbf{T}/\alpha$ occurs isomorphically as an ideal of the interval $I[\alpha, \nabla]$ of $Con(\mathbf{T})$. In [8], in some sense generalizing the above result in [5], it is shown that the more interesting aspects of the structure of congruence lattices of a two-orbit flat M-Set $\mathbf{X}$ are largely determined by coset lattices[6] of automorphism groups of certain homomorphic images of $\mathbf{X}$.

**6.1 Problems.** In Corollary 1.7 it is shown that finite lattices that are representable by a finite M-Set are closed under $\Pi$-product. Peter Mayr suggested the following question.

**Problem 2.** *If $L_1, \ldots, L_k$ are each representable by a finite transitive G-Set, is $\Pi(L_1, \ldots, L_k)$ representable by a finite transitive G-Set?*

As mentioned, using some of the methods developed here, it is shown that, with high probability, a finite lattice having more than one co-atom has no representation by an intransitive flat M-Set [9]; however, little seems to be known about the intransitive finite representations of a finite lattice, or about the likelihood that a finite $n$-lattice has intransitive finite representations. An M-Set $\mathbf{X} = \langle X, M\rangle$ is *cyclic* if there exists $a \in X$ such that $X = \{m(a) : m \in M\}$; note that $\mathbf{X}$ is cyclic if and only if it contains a unique maximal $\mathcal{J}$ class. Let $i(n)$ be the number of isomorphism classes of $n$-lattices having a finite intransitive M-Set representation, let $l(n)$ be the number of isomorphism classes of $n$-lattices, and let $b(n)$ be the number of isomorphism classes of $n$-classes having a finite non-cyclic M-Set representation.

[6] The empty set along with the cosets of a group $G$ forms a lattice, ordered under set inclusion, denoted $\mathfrak{R}(G)$. $\mathfrak{R}(G)$ contains multiple cover-preserving embedded copies of $Sub(G)$.

**Problem 3.** *Find bounds for any of the following (perhaps after restricting to a class $\mathcal{C}$ of finite lattices):*

(1) $\limsup_{n\to\infty} \frac{i(n)}{l(n)}$;
(2) $\limsup_{n\to\infty} \frac{b(n)}{l(n)}$; *or*
(3) $\limsup_{n\to\infty} \frac{b(n)}{i(n)}$.

**6.1.1 0-Flat M-Sets.** While flat M-Sets are important if for no other reason than that they generalize G-Sets, few M-Sets have an intransitive flat M-Set homomorphic image. It would be interesting to generalize the methods and results here to a class of M-Sets broader than the flat M-Sets, a class onto which more M-Sets map homomorphically. The *0-flat M-Sets* are such a class.

Let $\langle X, M\rangle$ be an M-Set. Recall that an element $0 \in X$ (the "zero" of $\mathbf{X}$) satisfies the following: For all $m \in M$, $m(0) = 0$; moreover, for all $a \in X$, there exists $n \in M$ such that $n(a) = 0$.

**Definition 6.1.** (1) $\langle X, M\rangle$ is said to be 0-transitive if $X = J \sqcup \{0\}$, where $J$ is a $\mathcal{J}$ class.
(2) $\langle X, M\rangle$ is said to be 0-flat if $\mathbf{X}$ has a 0, and for all $a, b \in X$ with $b \neq 0$, if $a \geq_M b$, then $b \geq_M a$.

Note that 0-transitive M-Sets are 0-flat, and that a 0-flat M-Set $\mathbf{Y}$ can be viewed as a "fusion at 0" of a set of 0-transitive algebras. Suppose $\mathbf{X}$ is not flat. Let $J_M$ be the set of the maximal $\mathcal{J}$ classes of $\mathbf{X}$, and let $\theta_{J_M} = \Delta \cup \{(x, y) \in X^2 : \exists u, v \in J_M$ such that $u > x$ and $v > y\}$. Observe that $\theta_{J_M}$ is a congruence of $\mathbf{X}$ and that $\mathbf{X}/\theta_{J_M}$ is 0-flat. Thus each non-flat M-Set has an upper interval in its congruence lattice that is isomorphic to the congruence lattice of a 0-flat M-Set. (Unfortunately, however, the direct product of 0-flat M-Sets need not be 0-flat. Thus there exist non-flat M-Sets having no minimal 0-flat congruence.)

Suppose $\mathbf{X}$ is 0-flat. It is not true in general that $\mathcal{J}$ is a congruence. Let $\mathcal{K}^* = \{(a, b) \in X^2 : \forall f \in M\ (f(a), a) \in \mathcal{J}$ if and only if $(f(b), b) \in \mathcal{J}\}$, and let $\mathcal{K} = \mathcal{J} \wedge \mathcal{K}^*$. It is not difficult to verify that $\mathcal{K}^*$ and $\mathcal{K}$ are congruences, and that $\mathbf{X}$ is 0-transitive implies that $\mathcal{K}$ is its unique maximal congruence. The following generalizes Property K from flat M-Sets to 0-flat M-Sets: $\mathbf{X}$ has Property K if for all $(a, b) \in \mathcal{K}$, if $(a, c) \notin \mathcal{J}$, then $(a, b) \in Cg(a, c)$. $\mathcal{J}$, $\mathcal{K}^*$, $\mathcal{K}$, and Property K may be of some help in finding generalizations of Theorem 2.4 and Theorem 2.8 to 0-flat M-Sets, which in turn might be of help in Problem 3.

**Problem 4.** *Generalize Theorem* 2.4 *and Theorem* 2.8 *to 0-flat M-Sets.*

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Department of Mathematics, University of Louisville, Louisville, KY 40292, USA

*E-mail:* swseif01@louisville.edu

(*Received* January 24, 2011, *revised* May 23, 2013)