Some evolution equations under the List's flow and their applications

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Abstract. In this paper, we consider some evolution equations of generalized Ricci curvature and generalized scalar curvature under the List's flow. As applications, we obtain L^2-estimates for generalized scalar curvature and the first variational formulae for non-negative eigenvalues with respect to the Laplacian.

Keywords: List's flow; eigenvalue; scalar curvature

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1. Introduction

Let $(M^n, g(t))$, $(t \in [0,T))$ be a solution to the following List's flow which was introduced by B. List [5]:

$$\begin{cases} \frac{\partial}{\partial t} g = -2\mathrm{Ric} + 2\alpha\, d\varphi \otimes d\varphi, \\ \varphi_t = \Delta\varphi, \end{cases} \tag{1.1}$$

where $\alpha > 0$ is a constant and φ is a smooth function on M^n. Δ denotes the Laplace-Beltrami operator given by $g(t)$. Throughout this paper, we always assume that M^n is compact without boundary. The motivation to study the system (1.1) stems from its connection to general relativity. In [7], Lott and Sesum obtain some long time behavior of the List's flow when $\alpha = 2$. Li [4] studied the eigenvalues and entropies under the harmonic-Ricci flow, he derived some monotonicity formulas for eigenvalues of Laplacian. Wang [10] proved some differential Harnack inequalities about the coupled Ricci flow which is closely related to List's flow. In [2], Fang also obtained some differential Harnack inequalities associated with the List's flow which generalize the results of Cao and Hamilton in [1]. For the study of List's flow and some developments, see [6], [5], [8], [3] and the references therein.

Let $h_{ij} = R_{ij} - \alpha\, \varphi_i \varphi_j$ be a symmetric two-tensor which we call the *generalized Ricci curvature*. Then (1.1) becomes

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2h_{ij}, \\ \varphi_t = \Delta\varphi. \end{cases} \tag{1.2}$$

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In this paper, we consider the following evolution equation:

$$\begin{cases} \frac{\partial}{\partial t} g_{ij} = -2\tilde{h}_{ij}, \\ \varphi_t = \Delta\varphi, \end{cases} \tag{1.3}$$

where $\tilde{h}_{ij} = h_{ij} - (ar+b)g_{ij}$. Here a, b are two constants,

$$r = \frac{\int_M (\mathrm{tr}_{\mathrm{g}} h)\, dV_g}{\int_M dV_g}$$

is the average of the trace of two-tensor h_{ij} and $\mathrm{tr}_{\mathrm{g}} h = g^{ij} h_{ij} = R - \alpha|\nabla\varphi|^2$ which is called the *generalized scalar curvature*. Note that if $a = \frac{1}{n}$ and $b = 0$, then the volume of M^n is a constant for all t. In this case, the equation (1.3) with $a = \frac{1}{n}$ and $b = 0$ is called the normalized List's flow.

The rest of this paper is organized as follows: In Section 2, we first derive some evolution equations of generalized Ricci curvature and generalized scalar curvature under the List's flow. As applications, we obtain L^2-estimates for generalized scalar curvature and the first variational formulae for non-negative eigenvalues with respect to the Laplacian.

2. Some evolution equations

We first recall the following lemma (see Lemma 1.4 in [5]):

Lemma 2.1 ([5]). *Let $g(t)$ be a solution to the evolution equation*

$$\frac{\partial}{\partial t} g_{ij} = v_{ij}$$

on M^n. Then the following evolution equations hold:

(1) $\frac{\partial}{\partial t} g^{ij} = -v^{ij}$, *where* $v^{ij} = g^{ik} g^{jl} v_{kl}$;
(2) $\frac{\partial}{\partial t} R_{ij} = -\frac{1}{2}\Delta v_{ij} - \frac{1}{2}(\mathrm{tr}_{\mathrm{g}} v)_{ij} + \frac{1}{2} g^{kl}(v_{ik,jl} + v_{jk,il})$, *where* $\mathrm{tr}_{\mathrm{g}} v = g^{ij} v_{ij}$;
(3) $\frac{\partial}{\partial t} R = -\Delta(\mathrm{tr}_{\mathrm{g}} v) + g^{ij} g^{kl}(v_{ik,lj} - R_{ik} v_{jl})$;
(4) $\frac{\partial}{\partial t} dV_g = \frac{1}{2}(\mathrm{tr}_{\mathrm{g}} v)\, dV_g$,

where $, i$ denotes the covariant derivative in the direction of i.

Using Lemma 2.1 above, we can prove the following results:

Lemma 2.2. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on M^n. Then the following evolution equations hold:*

(1) $(\frac{\partial}{\partial t} - \Delta) h_{ij} = -g^{kl} R_{ik} h_{jl} - g^{kl} R_{jk} h_{il} - 2R_{lijk} h^{lk} + 2\alpha(\Delta\varphi)\varphi_{ij}$;
(2) $(\frac{\partial}{\partial t} - \Delta)(\mathrm{tr}_{\mathrm{g}} h) = 2|h_{ij}|^2 + 2\alpha(\Delta\varphi)^2 - 2(ar+b)\mathrm{tr}_{\mathrm{g}} h$.

Proof: Using Lemma 2.1, we have

$$\frac{\partial}{\partial t} R_{ij} = -\frac{1}{2}\Delta v_{ij} - \frac{1}{2}(\mathrm{tr}_{\mathrm{g}} v)_{ij} + \frac{1}{2} g^{kl}(v_{ik,jl} + v_{jk,il}). \tag{2.1}$$

Taking $v_{ij} = -2\tilde{h}_{ij} = -2[h_{ij} - (ar+b)g_{ij}]$, we have

$$\begin{aligned}-\frac{1}{2}\Delta v_{ij} &= \Delta h_{ij} = \Delta R_{ij} - \alpha\,\Delta(\varphi_i\varphi_j)\\ &= \Delta R_{ij} - \alpha\,(\Delta\varphi_i)\varphi_j - \alpha\,\varphi_i(\Delta\varphi_j) - 2\alpha\varphi_{ik}\varphi_{jk}.\end{aligned}$$

Here and hereafter we use moving frames in all calculations. The second term of the right hand side in (2.1) gives

$$-\frac{1}{2}(\mathrm{tr_g} v)_{ij} = [R - \alpha|\nabla\varphi|^2 - n(ar+b)]_{ij} = R_{,ij} - 2\alpha\varphi_{ki}\varphi_{kj} - 2\alpha\varphi_k\varphi_{kji}.$$

Note that

$$\begin{aligned}\frac{1}{2}v_{ik,jk} &= -h_{ik,jk}\\ &= -R_{ik,jk} + \alpha(\varphi_i\varphi_k)_{,jk}\\ &= -(R_{ik,kj} + R_{lk}R_{lijk} + R_{il}R_{lkjk})\\ &\quad + \alpha[\varphi_{ijk}\varphi_k + (\Delta\varphi)\varphi_{ij} + \varphi_{ik}\varphi_{jk} + \varphi_i\varphi_{jkk}]\\ &= -\frac{1}{2}R_{,ij} - R_{lk}R_{lijk} - R_{ik}R_{jk}\\ &\quad + \alpha[\varphi_{ikj}\varphi_k + \varphi_k\varphi_l R_{lijk} + (\Delta\varphi)\varphi_{ij} + \varphi_{ik}\varphi_{jk} + \varphi_i\varphi_{jkk}].\end{aligned}$$

Hence, we obtain from (2.1) that

$$\begin{aligned}\frac{\partial}{\partial t}R_{ij} &= \Delta R_{ij} - 2R_{ik}R_{jk} - 2R_{lk}R_{lijk} + 2\alpha\varphi_k\varphi_l R_{lijk}\\ &\quad + 2\alpha(\Delta\varphi)\varphi_{ij} - 2\alpha\varphi_{ki}\varphi_{kj}.\end{aligned} \tag{2.2}$$

By a direct calculation, we have

$$\begin{aligned}\frac{\partial}{\partial t}(\varphi_i\varphi_j) &= (\varphi_t)_i\varphi_j + \varphi_i(\varphi_t)_j\\ &= (\Delta\varphi)_i\varphi_j + \varphi_i(\Delta\varphi)_j\\ &= \Delta(\varphi_i)\varphi_j - R_{ik}\varphi_k\varphi_j + \varphi_i\Delta(\varphi_j) - R_{jk}\varphi_k\varphi_i\\ &= \Delta(\varphi_i\varphi_j) - 2\varphi_{ik}\varphi_{jk} - R_{ik}\varphi_k\varphi_j - R_{jk}\varphi_k\varphi_i.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{\partial}{\partial t}h_{ij} &= \frac{\partial}{\partial t}R_{ij} - \alpha\frac{\partial}{\partial t}(\varphi_i\varphi_j)\\ &= \Delta R_{ij} - 2R_{ik}R_{jk} - 2R_{lk}R_{lijk} + 2\alpha\varphi_k\varphi_l R_{lijk}\\ &\quad + 2\alpha(\Delta\varphi)\varphi_{ij} - 2\alpha\varphi_{ki}\varphi_{kj} - \alpha\Delta(\varphi_i\varphi_j) + 2\alpha\varphi_{ik}\varphi_{jk}\\ &\quad + \alpha R_{ik}\varphi_k\varphi_j + \alpha R_{jk}\varphi_k\varphi_i\\ &= \Delta h_{ij} - R_{ik}h_{jk} - R_{jk}h_{ik} - 2R_{lijk}h_{lk} + 2\alpha(\Delta\varphi)\varphi_{ij}.\end{aligned}$$

On the other hand, using Lemma 2.1, it holds that

$$\begin{aligned}\frac{\partial}{\partial t}(\mathrm{tr_g}h) &= \frac{\partial}{\partial t}R - \alpha\frac{\partial}{\partial t}(|\nabla\varphi|^2)\\ &= 2\Delta R - 4\alpha|\nabla^2\varphi|^2 - 4\alpha\varphi_j\varphi_{jii}\\ &\quad - \Delta R + 2\alpha[2\varphi_j\varphi_{jii} + (\Delta\varphi)^2 + |\nabla^2\varphi|^2 - R_{ij}\varphi_i\varphi_j]\\ &\quad + 2R_{ik}h_{ik} - 2(ar+b)R\\ &\quad - \alpha[\Delta(|\nabla\varphi|^2) - 2|\nabla^2\varphi|^2 - 2\alpha|\nabla\varphi|^4 - 2(ar+b)|\nabla\varphi|^2]\\ &= \Delta(\mathrm{tr_g}h) + 2|h_{ij}|^2 + 2\alpha(\Delta\varphi)^2 - 2(ar+b)\mathrm{tr_g}h,\end{aligned}$$

where we used

$$\begin{aligned}h_{ik,ki} &= \frac{1}{2}\Delta R - \alpha[2\varphi_j\varphi_{jii} + (\Delta\varphi)^2 + |\nabla^2\varphi|^2 - R_{ij}\varphi_i\varphi_j],\\ \frac{\partial}{\partial t}(|\nabla\varphi|^2) &= (\frac{\partial}{\partial t}g^{ij})\varphi_i\varphi_j + g^{ij}\frac{\partial}{\partial t}(\varphi_i\varphi_j)\\ &= \Delta(|\nabla\varphi|^2) - 2|\nabla^2\varphi|^2 - 2\alpha|\nabla\varphi|^4 - 2(ar+b)|\nabla\varphi|^2.\end{aligned}$$

We complete the proof of Lemma 2.2. □

3. L^2-estimates and some applications

For any a, b, we will prove that the nonnegativity of $\mathrm{tr_g}h(t)$ is preserved by the evolution equation (1.3).

Theorem 3.1. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on a compact manifold M^n. If $\mathrm{tr_g}h \geq 0$ holds at $t = 0$, then also for all $t > 0$ as long as the solution exists.*

Proof: Let

$$\beta = e^{2\int_0^t [ar(s)+b]\,ds}(\mathrm{tr_g}h).$$

Then

$$\begin{aligned}(\frac{\partial}{\partial t} - \Delta)\beta &= e^{2\int_0^t [ar(s)+b]\,ds}[2(ar+b)(\mathrm{tr_g}h) + \frac{\partial}{\partial t}(\mathrm{tr_g}h) - \Delta(\mathrm{tr_g}h)]\\ &= e^{2\int_0^t [ar(s)+b]\,ds}[2|h_{ij}|^2 + 2\alpha(\Delta\varphi)^2]\\ &\geq 0,\end{aligned}$$

where the second equality used Lemma 2.2. From the maximum principle of parabolic equations, we obtain that $\mathrm{tr_g}h(t)$ remains nonnegative if $\mathrm{tr_g}h(0)$ is nonnegative under the initial metric $g(0)$. □

Next, we begin with an elementary lemma.

Lemma 3.1. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on a compact manifold M^n. Then for any $f \in C^2(\mathbb{R}_+ \times M^n)$, it holds that*

$$\begin{aligned}\frac{d}{dt}\int_M f\,dV_g &= \int_M (\frac{\partial}{\partial t} - \Delta) f\,dV_g \\ &\quad - \int_M (\mathrm{tr_g} h) f\,dV_g + n(ar+b)\int_M f\,dV_g\end{aligned} \tag{3.1}$$

and

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla f|^2\,dV_g &= -2\int_M |\nabla^2 f|^2\,dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g \\ &- 2\alpha\int_M \langle\nabla\varphi, \nabla f\rangle^2\,dV_g \\ &- 2\int_M (\Delta f)(\frac{\partial}{\partial t} - \Delta) f\,dV_g + (n-2)(ar+b)\int_M |\nabla f|^2\,dV_g.\end{aligned} \tag{3.2}$$

Proof: By (4) of Lemma 2.1, we have

$$\frac{\partial}{\partial t} dV_g = -(\mathrm{tr_g}\tilde{h})\,dV_g = [-\mathrm{tr_g} h + n(ar+b)]\,dV_g.$$

Therefore,

$$\begin{aligned}\frac{d}{dt}\int_M f\,dV_g &= \int_M \{\frac{\partial}{\partial t} f + [-\mathrm{tr_g} h + n(ar+b)] f\}\,dV_g \\ &= \int_M \frac{\partial}{\partial t} f\,dV_g - \int_M (\mathrm{tr_g} h) f\,dV_g + n(ar+b)\int_M f\,dV_g \\ &= \int_M (\frac{\partial}{\partial t} - \Delta) f\,dV_g - \int_M (\mathrm{tr_g} h) f\,dV_g + n(ar+b)\int_M f\,dV_g,\end{aligned}$$

which completes the proof of (3.1).

According to the Bochner formula, we have

$$\Delta|\nabla f|^2 = 2|\nabla^2 f|^2 + 2R^{ij} f_i f_j + 2\langle\nabla\Delta f, \nabla f\rangle. \tag{3.3}$$

By virtue of (1) of Lemma 2.1, we get

$$\frac{\partial}{\partial t}|\nabla f|^2 = 2\tilde{h}^{ij} f_i f_j + 2\langle\nabla(\frac{\partial}{\partial t} f), \nabla f\rangle,$$

so that

$$(3.4)\qquad \begin{aligned}&(\frac{\partial}{\partial t}-\Delta)|\nabla f|^2\\ &\quad=-2|\nabla^2 f|^2+2\langle\nabla(\frac{\partial}{\partial t}-\Delta)f,\nabla f\rangle+2(\tilde{h}^{ij}-R^{ij})f_if_j\\ &\quad=-2|\nabla^2 f|^2+2\langle\nabla(\frac{\partial}{\partial t}-\Delta)f,\nabla f\rangle-2\alpha\langle\nabla\varphi,\nabla f\rangle^2-2(ar+b)|\nabla f|^2\end{aligned}$$

and

$$\begin{aligned}\frac{d}{dt}\int_M|\nabla f|^2\,dV_g&=\int_M\{\frac{\partial}{\partial t}|\nabla f|^2+[-\mathrm{tr_g}h+n(ar+b)]|\nabla f|^2\}\,dV_g\\ &=\int_M\{(\frac{\partial}{\partial t}-\Delta)|\nabla f|^2+[-\mathrm{tr_g}h+n(ar+b)]|\nabla f|^2\}\,dV_g\\ &=-2\int_M|\nabla^2 f|^2\,dV_g-\int_M(\mathrm{tr_g}h)|\nabla f|^2\,dV_g-2\alpha\int_M\langle\nabla\varphi,\nabla f\rangle^2\,dV_g\\ &\quad-2\int_M(\Delta f)(\frac{\partial}{\partial t}-\Delta)f\,dV_g+(n-2)(ar+b)\int_M|\nabla f|^2\,dV_g.\end{aligned}$$

We complete the proof of Lemma 3.1. □

Theorem 3.2. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on a compact manifold M^n. Then for any $f\in C^2(\mathbb{R}_+\times M^n)$,*

$$(3.5)\qquad \begin{aligned}\frac{d}{dt}\int_M|\nabla f|^2\,dV_g\le&-\int_M(\mathrm{tr_g}h)|\nabla f|^2\,dV_g+(n-2)(ar+b)\int_M|\nabla f|^2\,dV_g\\ &+\frac{n}{8}\int_M[(\frac{\partial}{\partial t}-\Delta)f]^2\,dV_g.\end{aligned}$$

Proof: Using the Cauchy inequality, we have

$$|\nabla^2 f|^2\ge\frac{1}{n}(\Delta f)^2,$$

so that by (3.2)

$$\begin{aligned}\frac{d}{dt}\int_M|\nabla f|^2\,dV_g\le&-\frac{2}{n}\int_M(\Delta f)^2\,dV_g-\int_M(\mathrm{tr_g}h)|\nabla f|^2\,dV_g\\ &-2\int_M(\Delta f)(\frac{\partial}{\partial t}-\Delta)f\,dV_g+(n-2)(ar+b)\int_M|\nabla f|^2\,dV_g\\ \le&-\int_M(\mathrm{tr_g}h)|\nabla f|^2\,dV_g+(n-2)(ar+b)\int_M|\nabla f|^2\,dV_g\\ &+\frac{n}{8}\int_M[(\frac{\partial}{\partial t}-\Delta)f]^2\,dV_g,\end{aligned}$$

which completes the proof of Theorem 3.2. □

Together with the evolution for $\mathrm{tr_g}h$ in Lemma 2.2, one can establish the following

Theorem 3.3. *Let $g(t)$ be a solution to the evolution equation* (1.3) *on a compact manifold M^n. Then*

$$\begin{aligned} \frac{d}{dt}\int_M |\nabla(\mathrm{tr_g}h)|^2\,dV_g \leq &-\int_M (\mathrm{tr_g}h)|\nabla(\mathrm{tr_g}h)|^2\,dV_g \\ &+(n-2)(ar+b)\int_M |\nabla(\mathrm{tr_g}h)|^2\,dV_g \\ &+\frac{n}{8}\int_M [2|h_{ij}|^2+2\alpha(\Delta\varphi)^2-2(ar+b)\mathrm{tr_g}h]^2\,dV_g. \end{aligned} \tag{3.6}$$

Next, we give the evolution equation of eigenvalues with respect to the Laplacian.

Theorem 3.4. *Let $g(t)$ be a solution to the evolution equation* (1.2) *on a compact manifold M^n.*

(1) *If $\lambda(t)$ denotes the non-negative eigenvalue of the Laplacian on M^n, then*

$$\frac{d}{dt}\lambda = 2\int_M h^{ij}f_if_j\,dV_g - \int_M (\mathrm{tr_g}h)|\nabla f|^2\,dV_g + \lambda\int_M (\mathrm{tr_g}h)f^2\,dV_g.$$

In particular, if $\mathrm{tr_g}h \geq 0$ holds at $t=0$, then

$$\frac{d}{dt}\lambda \leq 2\int_M h^{ij}f_if_j\,dV_g + \lambda\int_M (\mathrm{tr_g}h)f^2\,dV_g.$$

(2) *If $\lambda(t)$ denotes the non-negative eigenvalue of $\Delta - \frac{1}{2}\mathrm{tr_g}h$ on M^n, then*

$$\frac{d}{dt}\lambda = 2\int_M h^{ij}f_if_j\,dV_g + \int_M f^2[|h_{ij}|^2+\alpha(\Delta\varphi)^2]\,dV_g.$$

In particular, if $h_{ij}(t)\geq 0$, then the eigenvalue of the operator $-\Delta+\frac{1}{2}\mathrm{tr_g}h$ are nondecreasing.

(3) *If $\lambda(t)$ denotes the non-negative eigenvalue of $\Delta + \frac{1}{4}\mathrm{tr_g}h$ on M^n, then*

$$\frac{d}{dt}\lambda \leq \int_M [\frac{n-1}{2n}(\mathrm{tr_g}h)^2+\frac{5}{2}\lambda(\mathrm{tr_g}h)]f^2\,dV_g + 2\lambda^2.$$

Remark 3.1. In particular, taking $\alpha = 2$, then (2) in Theorem 3.4 becomes Theorem 1.10 in [4].

Remark 3.2. We remark that the second and third inequalities of p. 160 in [9] is slight incorrect. The right version of the third inequality of p. 160 in [9] should be

$$\frac{d}{dt}\lambda = -2\int_M |\nabla^2 f|^2\, dV_g + (2\varphi - 1)\int_M R|\nabla f|^2\, dV_g$$
$$+ \int_M (2\varphi S - \frac{2}{3}\varphi AR + \varphi' R) f^2\, dV_g$$
$$- \varphi \int_M R(R - 4\varphi R - \frac{2}{3}A + 2\lambda) f^2\, dV_g$$
$$+ \lambda \int_M R(R - 4\varphi R - \frac{2}{3}A + 2\lambda) f^2\, dV_g.$$

As a result, for $n = 3$, the estimate $\frac{d}{dt}\lambda \leq \frac{19}{8}\lambda^2$ of Theorem 6.4 in [9] cannot be obtained.

Proof: Let us consider an eigenfunction f associated with the eigenvalue λ, that is,

$$-(\Delta + V)f = \lambda f, \qquad \int_M f^2\, dV_g = 1$$

where V is a potential function to be chosen later. Then, we obtain from (3.2) that

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla f|^2\, dV_g = {} & -2\int_M |\nabla^2 f|^2\, dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\, dV_g \\ & - 2\alpha \int_M \langle \nabla\varphi, \nabla f\rangle^2\, dV_g \\ & + 2\int_M (V + \lambda)^2 f^2\, dV_g + \int_M (V + \lambda)\frac{\partial f^2}{\partial t}\, dV_g \\ & + (n-2)(ar + b)\int_M |\nabla f|^2\, dV_g.\end{aligned} \tag{3.7}$$

By virtue of the divergence theorem,

$$\int_M |\nabla f|^2\, dV_g = \int_M f(-\Delta)f\, dV_g = \int_M (V + \lambda) f^2\, dV_g,$$

so that

$$\begin{aligned}\frac{d}{dt}\int_M |\nabla f|^2\, dV_g = {} & \int_M f^2 \frac{\partial}{\partial t}(V + \lambda)\, dV_g + \int_M (V + \lambda)\frac{\partial f^2}{\partial t}\, dV_g \\ & + \int_M [-\mathrm{tr_g} h + n(ar + b)](V + \lambda) f^2\, dV_g.\end{aligned} \tag{3.8}$$

Thus, (3.8) combines with (3.7) to give

$$
\begin{aligned}
\frac{d}{dt}\lambda = &-2\int_M |\nabla^2 f|^2\,dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g - 2\alpha\int_M \langle\nabla\varphi,\nabla f\rangle^2\,dV_g \\
&+ 2\int_M (V+\lambda)^2 f^2\,dV_g - \int_M f^2\frac{\partial V}{\partial t}\,dV_g \\
&+ \int_M [(\mathrm{tr_g} h) - 2(ar+b)](V+\lambda) f^2\,dV_g.
\end{aligned}
\tag{3.9}
$$

Choose $V = c\,\mathrm{tr_g} h$, where c is a constant to be determined. Then

$$
\frac{\partial V}{\partial t} = c\frac{\partial}{\partial t}\mathrm{tr_g} h
$$

so that

$$
\int_M f^2\frac{\partial V}{\partial t}\,dV_g = c\int_M f^2\frac{\partial}{\partial t}\mathrm{tr_g} h\,dV_g.
$$

Using integration by parts again and by use of Lemma 2.2, we have

$$
\begin{aligned}
\int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g &= \int_M (\mathrm{tr_g} h)(c\,\mathrm{tr_g} h+\lambda) f^2\,dV_g + \frac{1}{2}\int_M f^2\Delta(\mathrm{tr_g} h)\,dV_g \\
&= \int_M (\mathrm{tr_g} h)(c\,\mathrm{tr_g} h+\lambda) f^2\,dV_g + \frac{1}{2}\int_M f^2\frac{\partial}{\partial t}\mathrm{tr_g} h\,dV_g \\
&\quad - \int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2 - (ar+b)\mathrm{tr_g} h]\,dV_g.
\end{aligned}
$$

In other words,

$$
\begin{aligned}
-\int_M f^2\frac{\partial V}{\partial t}\,dV_g &= -c\int_M f^2\frac{\partial}{\partial t}\mathrm{tr_g} h\,dV_g \\
&= -2c\int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g + 2c\int_M (\mathrm{tr_g} h)(c\,\mathrm{tr_g} h+\lambda) f^2\,dV_g \\
&\quad - 2c\int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2 - (ar+b)\mathrm{tr_g} h]\,dV_g.
\end{aligned}
$$

It follows that

$$\begin{aligned}\frac{d}{dt}\lambda &= -2\int_M |\nabla^2 f|^2\,dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g - 2\alpha\int_M \langle\nabla\varphi,\nabla f\rangle^2\,dV_g\\ &\quad + 2\int_M (V+\lambda)^2 f^2\,dV_g - \int_M f^2\frac{\partial V}{\partial t}\,dV_g\\ &\quad + \int_M [(\mathrm{tr_g} h) - 2(ar+b)](V+\lambda)f^2\,dV_g\\ &= -2\int_M |\nabla^2 f|^2\,dV_g - (1+2c)\int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g - 2\alpha\int_M \langle\nabla\varphi,\nabla f\rangle^2\,dV_g\\ &\quad + \int_M [(1+4c)(\mathrm{tr_g} h) + 2\lambda - 2(ar+b)](c\,\mathrm{tr_g} h+\lambda)f^2\,dV_g\\ &\quad - 2c\int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2 - (ar+b)\mathrm{tr_g} h]\,dV_g.\end{aligned}$$

Integrating (3.3) yields

$$\begin{aligned}-2\int_M |\nabla^2 f|^2\,dV_g - 2\alpha\int_M \langle\nabla\varphi,\nabla f\rangle^2\,dV_g\\ = \int_M (2R^{ij} f_i f_j + 2\langle\nabla\Delta f,\nabla f\rangle)\,dV_g\\ - 2\alpha\int_M \langle\nabla\varphi,\nabla f\rangle^2\,dV_g\\ = 2\int_M h^{ij} f_i f_j\,dV_g - 2\int_M (c\,\mathrm{tr_g} h+\lambda)^2 f^2\,dV_g.\end{aligned}\tag{3.10}$$

Therefore, we obtain

$$\begin{aligned}\frac{d}{dt}\lambda &= 2\int_M h^{ij} f_i f_j\,dV_g - (1+2c)\int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g\\ &\quad + \int_M [(1+2c)(\mathrm{tr_g} h) - 2(ar+b)](c\,\mathrm{tr_g} h+\lambda)f^2\,dV_g\\ &\quad - 2c\int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2 - (ar+b)\mathrm{tr_g} h]\,dV_g.\end{aligned}\tag{3.11}$$

Now we are in the position to complete the proof of Theorem 3.4. Notice that under the evolution equation (1.2), we have $a = b = 0$.

If we choose $c = 0$, then (3.11) reduces to

$$\frac{d}{dt}\lambda = 2\int_M h^{ij} f_i f_j\,dV_g - \int_M (\mathrm{tr_g} h)|\nabla f|^2\,dV_g + \lambda\int_M (\mathrm{tr_g} h) f^2\,dV_g$$

and (1) of Theorem 3.4 follows.

If we choose $c = -\frac{1}{2}$, then (3.11) reduces to

$$\frac{d}{dt}\lambda = 2\int_M h^{ij} f_i f_j \, dV_g + \int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2] \, dV_g. \tag{3.12}$$

Hence, we obtain (2) of Theorem 3.4.

If we choose $c = \frac{1}{4}$, then (3.11) reduces to

$$\begin{aligned} \frac{d}{dt}\lambda = {} & 2\int_M h^{ij} f_i f_j \, dV_g - \frac{3}{2}\int_M (\mathrm{tr_g} h)|\nabla f|^2 \, dV_g \\ & + \frac{3}{2}\int_M (\mathrm{tr_g} h)(\frac{1}{4}\,\mathrm{tr_g} h + \lambda) f^2 \, dV_g \\ & - \frac{1}{2}\int_M f^2[|h_{ij}|^2 + \alpha(\Delta\varphi)^2] \, dV_g. \end{aligned} \tag{3.13}$$

It follows from (3.10) that

$$2\int_M h^{ij} f_i f_j \, dV_g \le 2\int_M (\frac{1}{4}\,\mathrm{tr_g} h + \lambda)^2 f^2 \, dV_g.$$

Thus, we deduce from (3.13) the following

$$\begin{aligned} \frac{d}{dt}\lambda \le {} & 2\int_M (\frac{1}{4}\,\mathrm{tr_g} h + \lambda)^2 f^2 \, dV_g \\ & + \frac{3}{2}\int_M (\mathrm{tr_g} h)(\frac{1}{4}\,\mathrm{tr_g} h + \lambda) f^2 \, dV_g - \frac{1}{2}\int_M f^2 |h_{ij}|^2 \, dV_g. \end{aligned} \tag{3.14}$$

Applying the Cauchy inequality

$$|h_{ij}|^2 \ge \frac{1}{n}(\mathrm{tr_g} h)^2$$

into (3.14) gives

$$\frac{d}{dt}\lambda \le \int_M [\frac{n-1}{2n}(\mathrm{tr_g} h)^2 + \frac{5}{2}\lambda(\mathrm{tr_g} h)] f^2 \, dV_g + 2\lambda^2.$$

We complete the proof of Theorem 3.4. □

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