Intersections of essential minimal prime ideals

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Abstract. Let $\mathcal{Z}(\mathcal{R})$ be the set of zero divisor elements of a commutative ring R with identity and $\mathcal{M}$ be the space of minimal prime ideals of R with Zariski topology. An ideal I of R is called strongly dense ideal or briefly *sd*-ideal if $I \subseteq \mathcal{Z}(\mathcal{R})$ and I is contained in no minimal prime ideal. We denote by $R_K(\mathcal{M})$, the set of all $a \in R$ for which $\overline{D(a)} = \overline{\mathcal{M} \setminus V(a)}$ is compact. We show that R has property (A) and $\mathcal{M}$ is compact if and only if R has no *sd*-ideal. It is proved that $R_K(\mathcal{M})$ is an essential ideal (resp., *sd*-ideal) if and only if $\mathcal{M}$ is an almost locally compact (resp., $\mathcal{M}$ is a locally compact non-compact) space. The intersection of essential minimal prime ideals of a reduced ring R need not be an essential ideal. We find an equivalent condition for which any (resp., any countable) intersection of essential minimal prime ideals of a reduced ring R is an essential ideal. Also it is proved that the intersection of essential minimal prime ideals of $C(X)$ is equal to the socle of C(X) (i.e., $C_F(X) = O^{\beta X \setminus I(X)}$). Finally, we show that a topological space X is pseudo-discrete if and only if $I(X) = X_L$ and $C_K(X)$ is a pure ideal.

Keywords: essential ideals; *sd*-ideal; almost locally compact space; nowhere dense; Zariski topology

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1. Introduction

In this paper, R is assumed to be a commutative ring with identity, and $\mathcal{M}$ is the space of minimal prime ideals of R. For $a \in R$ let $V(a) = \{P \in \mathcal{M} : a \in P\}$. It is easy to see that for any R, the set $\{D(a) = \mathcal{M} \setminus V(a) : a \in R\}$ forms a basis of open sets on $\mathcal{M}$. This topology is called the Zariski topology. For $A \subseteq R$, we use $V(A)$ to denote the set of all $P \in \mathcal{M}$ where $A \subseteq P$ (see [8]). For a subset H of $\mathcal{M}$ we denote by $\overline{H}$ the closure points of H in $\mathcal{M}$. The intersection of all minimal prime ideals containing a is denoted by P_a. An ideal I of R is called a z^0-ideal, if $P_b \subseteq P_a$ and $a \in I$ implies $b \in I$ (see [2] and [5]). For any subset S of a ring R, $\mathrm{ann}(S) = \{a \in R : aS = 0\}$.

We denote by $C(X)$ the ring of real-valued, continuous functions on a completely regular Hausdorff space X, βX is the Stone-Čech compactification of X and for any $p \in \beta X$, O^p (resp., M^p) is the set of all $f \in C(X)$ for which $p \in \mathrm{int}_{\beta X}\, cl_{\beta X} Z(f)$ (resp., $p \in cl_{\beta X} Z(f)$). Also, for $A \subseteq \beta X$, O^A is the intersection of all O^p where $p \in A$. It is well known that O^p is the intersection of all minimal prime ideals contained in M^p. We denote the socle of $C(X)$ by $C_F(X)$; it is characterized in [12] as the set of all functions which vanish everywhere except on a finite number of points of X. The known ideal $C_K(X)$ in $C(X)$, is the set of functions with compact support, and the generalization of this ideal is defined in [16]. The reader is referred to [7] for undefined terms and notations.

A non-zero ideal in a commutative ring is said to be essential if it intersects every non-zero ideal non-trivially, and the intersection of all essential ideals, or the sum of all minimal ideals, is called the socle (see [14]).

We denote by $R_K(\mathcal{M})$ the set of all $a \in R$ for which $\overline{D(a)}$ is compact as a subspace of $\mathcal{M}$. In Section 2, by algebraic properties of the ideal $R_K(\mathcal{M})$, we find topological properties of the space $\mathcal{M}$ of minimal prime ideals of R, namely locally compactness and almost locally compactness. Also we call I a strongly dense ideal or briefly sd-ideal if $I \subseteq \mathcal{Z}(\mathcal{R})$ and I is contained in no minimal prime ideal. We characterize commutative reduced rings R which have no sd-ideals. It is proved that $R_K(\mathcal{M})$ is contained in the intersection of all strongly dense fd-ideals (i.e., such ideals I that, if $\operatorname{ann}(F) \subseteq \operatorname{ann}(a)$ for some finite subset F of I and $a \in R$, then $a \in I$).

In Section 3, we show that the intersection of essential minimal prime ideals in any ring need not be an essential ideal. In a reduced ring R, we prove that every intersection of essential minimal prime ideals is an essential ideal if and only if the set of isolated points of $\mathcal{M}$ is dense in $\mathcal{M}$. Also it is proved that every countable intersection of essential minimal prime ideals of a reduced ring R is an essential ideal if and only if every first category subset of $\mathcal{M}$ is nowhere dense in $\mathcal{M}$. We characterize the intersection of essential minimal prime ideals in $C(X)$, i.e., the intersection of essential minimal prime ideals of $C(X)$ is equal to the ideal $C_F(X)$ (i,e., the socle of $C(X)$). Finally, we prove that the intersection of essential minimal prime ideals of $C(X)$ is equal to the ideal $C_K(X)$ if and only if $I(X) = X_L = \bigcup_{f \in C_K(X)} \overline{X \setminus Z(f)}$, i.e., $I(X) = X_L$ and $C_K(X)$ is a pure ideal. By this result and Theorem 4.5 in [3], we see that X is a pseudo-discrete space if and only if $I(X) = X_L$ and $C_K(X)$ is a pure ideal.

2. $R_K(\mathcal{M})$ and strongly dense ideals

In this section we introduce the ideal $R_K(\mathcal{M})$ and the class of strongly dense ideals as a subclass of dense ideals. We show that a reduced ring R has no sd-ideal if and only if $T(R)$ (i.e., the total quotient ring of R) is a von Neumann regular ring. By this, we have $C(X)$ has no sd-ideal if and only if X is a cozero-complemented space. It is proved that $R_K(\mathcal{M})$ is an essential ideal (resp., sd-ideal) if and only if $\mathcal{M}$ is an almost locally compact space (resp., locally compact non-compact space).

Definition 2.1. Let R be a commutative ring with identity and $D(a)$ be the set of all prime ideals which do not contain a. We define the family $R_K(\mathcal{M})$ to be the set of all $a \in R$ for which $\overline{D(a)}$ is compact (as a subspace of $\mathcal{M}$).

Example 2.2. If $\mathcal{M}$ is a discrete space, then $R_K(\mathcal{M}) = \{a \in R : D(a) \text{ is finite}\}$. For example, let R be the weak (discrete) direct sum of countably many copies of the integers. R may be regarded as the ring of all sequences of integers that are ultimately zero. Then $\mathcal{M}$ is a countable discrete space (see [8, 2.11]).

Lemma 2.3. (i) *$R_K(\mathcal{M})$ is a z^0-ideal of R.*
(ii) *$R_K(\mathcal{M}) = R$ if and only if $\mathcal{M}$ is compact.*
(iii) *$R_K(\mathcal{M}) = 0$ if and only if $\mathcal{M}$ is nowhere compact (i.e., the interior of every compact set is empty).*

Proof: (i) For $a, b \in R_K(\mathcal{M})$, we have $\overline{D(a+b)} \subseteq \overline{D(a)} \cup \overline{D(b)}$, and if $a \in R$, $b \in R_K(\mathcal{M})$, then $\overline{D(ab)} \subseteq \overline{D(b)}$. Therefore $R_K(\mathcal{M})$ is an ideal of R. Now let $P_b \subseteq P_a$ and $a \in R_K(\mathcal{M})$. Then $V(a) \subseteq V(b)$, hence $\overline{D(b)} \subseteq \overline{D(a)}$ so $\overline{D(b)}$ is compact, i.e., $b \in R_K(\mathcal{M})$.

(ii) By definition, it is obvious.

(iii) $\Rightarrow$ Let K be compact subset of $\mathcal{M}$ and $P \in \operatorname{int}(K)$. Then there is a non-zero element $f \in R$ such that $P \in D(f) \subseteq \operatorname{int}(K)$, so $f \in R_K(\mathcal{M}) = 0$, i.e., $D(f) = \phi$, which is a contradiction.

$\Leftarrow$ Suppose that $f \in R_K(\mathcal{M})$. Then $D(f)$ is contained in the interior of $\overline{D(f)}$ so $D(f) = \phi$, i.e., $f = 0$. □

Definition 2.4. An ideal I of R is called a strongly dense ideal or briefly an *sd*-ideal if $I \subseteq \mathcal{Z}(\mathcal{R})$ and I is contained in no minimal prime ideal ($V(I) = \phi$).

Example 2.5. (i) Every prime ideal of a ring R which is not a minimal prime and is contained in $\mathcal{Z}(\mathcal{R})$ is an *sd*-ideal.
(ii) For a set X, let $R = \mathbb{R}^X$ (i.e., the ring of real valued functions). Then we can see that any element of R is an unit or a zero-divisor. So any ideal I of R for which $V(I) = \phi$ is an *sd*-ideal.
(iii) Let p, q be two non-isolated points in an almost P-space X (i.e., every zero-set has nonempty interior (see [13] and [17])). Then the ideal $I = M_p \cap M_q$ is an *sd*-ideal.

Recall that an ideal I of R is a dense ideal if $\operatorname{ann}(I) = 0$. We observe that in any commutative reduced ring R the ideal $I \oplus \operatorname{ann}(I)$ is an essential ideal. Hence an ideal I in a reduced ring R is an essential ideal if and only if it is a dense ideal [14].

In the following, we see that a non-minimal prime ideal need not be an *sd*-ideal.

Remark 2.6. Every *sd*-ideal in a reduced ring R is a dense ideal (essential ideal), but there is a dense ideal which is not *sd*-ideal. To see this, let I be an *sd*-ideal and $g \in \operatorname{ann}(I)$. Then $gf = 0$ for each $f \in I$, therefore we have $\bigcap_{f\in I} V(gf) = \mathcal{M}$. Hence $V(g) \cup (\bigcap_{f\in I} V(f)) = \mathcal{M}$, i.e., $V(g) = \mathcal{M}$, and we get $g = 0$. Now let x be a non-isolated point in compact space X. Then by [4, Remark 3.2], the ideal O_x is an essential ideal of $C(X)$ which is not a minimal prime ideal. By [4, Theorem 3.1], $\operatorname{ann}(O_x) = 0$, i.e., O_x is dense ideal. However, this ideal is not an *sd*-ideal. Because there is a minimal prime ideal in $C(X)$ which contains O_x, i.e., $V(O_x) \neq \phi$.

We denote by I_z, the intersection of all z-ideals that contain I. An ideal I of R is called a rez-ideal if there is an ideal J for which $I \nsubseteq J$ and $I_z \cap J \subseteq I$. For more see [2] and [5].

Proposition 2.7. *Every ideal I in a reduced ring R is a rez-ideal or a dense ideal.*

Proof: Let I be a non-rez-ideal in R. By [2, Corollary 2.8], $\mathrm{ann}(I) = 0$, so I is a dense ideal. □

Lemma 2.8. *Let R be a reduced ring.*

(i) $\bigcap_{i=1}^{n} V(f_i) = \phi$ *if and only if* $\bigcap_{i=1}^{n} \mathrm{ann}(f_i) = 0$.
(ii) *If F is a finite subset of R, then $V(F) = \mathcal{M} \setminus V(\mathrm{ann}(F))$.*
(iii) *If $I \subseteq \mathcal{Z}(\mathcal{R})$ is a finitely generated ideal, then I is an sd-ideal if and only if I is a dense ideal.*
(iv) *If R has finitely many minimal prime ideals, then R has no sd-ideal.*

Proof: Trivial. □

Recall that a ring R has property (A) (resp., property $(a.c.)$), if for every finitely generated ideal $I \subseteq \mathcal{Z}(\mathcal{R})$, $\mathrm{ann}(I) \neq 0$ (resp., for any finitely generated ideal I of R there is $c \in R$ such that $\mathrm{ann}(I) = \mathrm{ann}(c)$), see [8] and [11].

In the following theorem we characterize a class of reduced rings which have no *sd*-ideal.

Theorem 2.9. *Let R be a reduced ring with total quotient $T(R)$. The following conditions are equivalent.*

(i) *$T(R)$ is a von Neumann regular ring.*
(ii) *R satisfies property (A) and $\mathcal{M}$ is compact.*
(iii) *R has no sd-ideal.*
(iv) *R satisfies property $(a.c)$ and $\mathcal{M}$ is compact.*

Proof: For (i)⇔(ii)⇔(iv), see [10, Theorem 4.5].

(ii)⇒(iii) Let I be an *sd*-ideal. Then $I \subseteq \mathcal{Z}(R)$ and $\bigcap_{f \in I} V(f) = \phi$. Hence $\mathcal{M} = \bigcup_{f \in I} D(f)$. Compactness of $\mathcal{M}$ implies that there are $f_1, \ldots, f_n \in I$ such that $\bigcap_{i=1}^{n} V(f_i) = \phi$. By Lemma 2.8, we have $\bigcap_{i=1}^{n} \mathrm{ann}(f_i) = \mathrm{ann}(F) = 0$, where $F = \{f_1, \ldots, f_n\}$. This is a contradiction, for R has property (A) and $F \subseteq \mathcal{Z}(\mathcal{R})$.

(iii)⇒(ii) Suppose that $I \subseteq \mathcal{Z}(\mathcal{R})$ is a finitely generated ideal and $\mathrm{ann}(I) = 0$. Then by Lemma 2.8, I is an *sd*-ideal, which contradicts the hypothesis. Thus R has property (A). Now, let $\mathcal{M} = \bigcup_{f \in S} D(f)$ where S is a proper subset of R. If $(S) \subseteq \mathcal{Z}(\mathcal{R})$, then $\bigcap_{f \in (S)} V(f) = \phi$ implies that the ideal generated by S is an *sd*-ideal, a contradiction. Hence $(S) \nsubseteq \mathcal{Z}(\mathcal{R})$, so there is $H = \{f_1, \ldots, f_n\} \subseteq S$ such that $\mathrm{ann}(H) = 0$. By Lemma 2.8, $V(H) = \bigcap_{i=1}^{n} V(f_i) = \mathcal{M} \setminus V(\mathrm{ann}(H)) = \emptyset$, thus $\mathcal{M} = \bigcup_{i=1}^{n} D(f_i)$, i.e., $\mathcal{M}$ is compact. □

Henriksen and Woods have introduced cozero complemented spaces. Such a space X is defined by the property that, for every cozero-set V of X, there is a disjoint cozero-set V' of X such that $V \cup V'$ is a dense subset of X (see [9]). In [8], they have proved that the space of minimal prime ideals of $C(X)$ is compact if and only if X is a cozero complemented space. Now by Theorem 2.9, and the fact that $C(X)$ satisfies property $(a.c)$ we have the following corollary.

Corollary 2.10. *$C(X)$ has no sd-ideal if and only if X is a cozero complemented space.*

Recall that a ring R has property $(c.a.c)$, if for any countably generated ideal I of R, there exists $c \in R$ such that $\operatorname{ann}(I) = \operatorname{ann}(c)$, see [8]. If R is a ring with property $(c.a.c)$, then by [8, Theorem 4.9] $\mathcal{M}$ is countably compact. But in a ring with property (A) this need not be true.

Proposition 2.11. *Let R be a reduced ring. Then R satisfies property (A) and $\mathcal{M}$ is countably compact if and only if R has no countably generated sd-ideal.*

Proof: The proof is similar to that of Theorem 2.9 step by step. □

Recall that an ideal I of R is called an fd-ideal, if for each finite subset F of I and $x \in R$, $\operatorname{ann}(F) \subseteq \operatorname{ann}(x)$ implies that $x \in I$. For more details see [15].

Proposition 2.12. *Let R be a reduced ring.*

(i) *$R_K(\mathcal{M})$ is contained in the intersection of all strongly dense fd-ideals in R.*
(ii) *$R_K(\mathcal{M})$ is an sd-ideal if and only if $\mathcal{M}$ is a locally compact non-compact space.*

Proof: (i) Let I be a strongly dense fd-ideal, $f \in R_K(\mathcal{M})$ and $P \in \overline{D(f)}$. Then there is $g \in I$ such that $P \in D(g)$, and so $\overline{D(f)} \subseteq \bigcup_{g\in I} D(g)$. On the other hand, $\overline{D(f)}$ is compact so there are $g_1, \dots, g_n \in I$ such that $\overline{D(f)} \subseteq \bigcup_{i=1}^n D(g_i)$. Hence $V(f) \supseteq \bigcap_{i=1}^n V(g_i) = V(\{g_1, \dots, g_n\})$. This implies that $\operatorname{ann}(F) \subseteq \operatorname{ann}(f)$ where $F = \{g_1, \dots, g_n\} \subseteq I$. But I is a fd-ideal so $f \in I$.

(ii) Let $R_K(\mathcal{M})$ is an sd-ideal and $P \in \mathcal{M}$. By definition, there is $f \in R_K(\mathcal{M})$ such that $P \in D(f) \subseteq \overline{D(f)}$ so P has a compact neighborhood, i.e., $\mathcal{M}$ is a locally compact space. On the other hand, $\mathcal{M}$ is not compact, since if $\mathcal{M}$ were compact, then by Lemma 2.3, $R_K(\mathcal{M}) = R$, which is a contradiction.

Conversely, first, we see that $R_K(\mathcal{M}) \subseteq Z(R)$. Otherwise, if $f \in R_K(\mathcal{M})$ and $\operatorname{ann}(f) = 0$, then $D(f) = \mathcal{M}$ is compact, which is a contradiction, by hypothesis. Now for every $P \in \mathcal{M}$ there is a compact neighborhood K of P in $\mathcal{M}$. So there is $f \in R$ such that $P \in D(f) \subseteq \operatorname{int} K \subseteq K$, i.e., $f \in R_K(\mathcal{M})$, hence $R_K(\mathcal{M})$ is an sd-ideal. □

A Hausdorff space X is said to be an almost locally compact space if every non-empty open set of X contains a non-empty open set with compact closure (see [3]). The next result is a topological characterization of $R_K(\mathcal{M})$ as an essential ideal.

Theorem 2.13. *Let R be a reduced ring. Then $R_K(\mathcal{M})$ is an essential ideal if and only if $\mathcal{M}$ is an almost locally compact space.*

Proof: $\Rightarrow$ Let $R_K(\mathcal{M})$ be an essential ideal and U be an open subset of $\mathcal{M}$. Then there exists a non-zero element $f \in R$ such that $D(f) \subseteq U$. It is enough to prove that $D(f)$ contains $D(g)$ for some $g \in R_K(\mathcal{M})$. If $D(f) \cap D(g) = \phi$ for each $g \in R_K(\mathcal{M})$, then $D(fg) = \phi$, so $fg = 0$, i.e., $R_K(\mathcal{M}) \cap (f) = 0$, which is a contradiction by essentiality of $R_K(\mathcal{M})$. Hence there is $g \in R_K(\mathcal{M})$ such that $D(fg) = D(f) \cap D(g) \neq \phi$, but $D(fg) \subseteq D(f)$, i.e., U contains an open subset for which the closure is compact.

$\Leftarrow$ Let f be a non-zero element in R. It is enough to prove that $R_K(\mathcal{M}) \cap (f) \neq \phi$. $D(f) \neq \phi$ is an open subset in X. By hypothesis there is an open subset $V \subseteq D(f)$ such that $\overline{V}$ is compact, so there is a non-zero element $g \in R$ such that $D(g) \subseteq V \subseteq D(f)$, i.e., $g \in R_K(\mathcal{M})$. Now $D(fg) = D(f) \cap D(g) = D(g) \neq \phi$, hence $fg \neq 0$ is an element of $R_K(\mathcal{M}) \cap (f)$, i.e., $R_K(\mathcal{M})$ is an essential ideal. $\square$

3. Intersections of essential minimal prime ideals

The intersection of essential minimal prime ideals of a reduced ring R need not be an essential ideal. Even a countable intersection of essential minimal prime ideals need not be an essential ideal. For example, the ideal O_r for any rational $0 \leq r \leq 1$ is an essential ideal in $C(\mathbb{R})$, which is the intersection of minimal prime ideals. Now for any $0 \leq r \leq 1$ let P_r be a minimal prime ideal that contains O_r. Then any P_r is an essential ideal, but by [3, Theorem 3.1], the ideal $I = \bigcap P_r$ is not an essential ideal, for $\bigcap Z[I] = [0,1]$ and $\operatorname{int}[0,1] \neq \phi$. In this section we give a topological characterization of the intersection of essential minimal prime ideals of a reduced ring R (resp., $C(X)$) which is an essential ideal.

For an open subset A of $\mathcal{M}$, suppose that $O_A := \{a \in R : A \subseteq V(a)\}$. Since for any $a, b \in R$, $V(a) \cap V(b) \subseteq V(a-b)$ and for each $r \in R$, $a \in O_A$, we have $V(a) \subseteq V(ra)$, thus O_A is an ideal of R. It is easy to see that $O_A = \bigcap_{P \in A} P$ and $V(O_A) = \overline{A}$, where $\overline{A}$ is the cluster points of A in $\mathcal{M}$.

We need the following lemmas which are easy to prove.

Lemma 3.1. *Let R be a reduced ring. An ideal I of R is an essential ideal if and only if* $\operatorname{int} V(I) = \phi$.

Lemma 3.2. *The intersection of all essential minimal prime ideals in a reduced ring R is equal to the ideal $O_{(\mathcal{M} \setminus I(\mathcal{M}))}$, where $I(\mathcal{M})$ is the set of isolated points of $\mathcal{M}$.*

In [3], Corollary 2.3 and Theorem 2.4, Azarpanah showed that every intersection (resp., countable intersection) of essential ideals of $C(X)$ is essential if and only if the set of isolated points of X is dense in X (resp., every first category subset of X is nowhere dense in X). Now, we generalize these results for the essentiality of the intersection of essential minimal prime ideals in a reduced ring.

Proposition 3.3. *Let R be a reduced ring. Every intersection of essential minimal prime ideals is an essential ideal if and only if the set of isolated points of $\mathcal{M}$ is dense in $\mathcal{M}$.*

Proof: Assume that every intersection of essential minimal prime ideals is an essential ideal. Then Lemma 3.2 implies that $O_{(\mathcal{M}\setminus I(\mathcal{M}))}$ is an essential ideal. By Lemma 3.1, $\operatorname{int} V(O_{\mathcal{M}\setminus I(\mathcal{M})}) = \phi$. On the other hand, we have

$$V(O_{\mathcal{M}\setminus I(\mathcal{M})}) = \overline{\mathcal{M}\setminus I(\mathcal{M})} = (\mathcal{M}\setminus I(\mathcal{M})).$$

Therefore $\operatorname{int}(\mathcal{M}\setminus I(\mathcal{M})) = \operatorname{int}(V(O_{\mathcal{M}\setminus I(\mathcal{M})})) = \phi$. This shows that $\overline{I(\mathcal{M})} = \mathcal{M}$.

Conversely, by hypothesis, $\operatorname{int}(V(O_{\mathcal{M}\setminus I(\mathcal{M})})) = \operatorname{int}(\mathcal{M}\setminus I(\mathcal{M})) = \phi$. So by Lemma 3.1, $O_{(\mathcal{M}\setminus I(\mathcal{M}))}$ is an essential ideal. Since $O_{(\mathcal{M}\setminus I(\mathcal{M}))}$ is contained in every intersection of essential minimal prime ideals, so every intersection of essential minimal prime ideals is an essential ideal. $\square$

Theorem 3.4. *Let R be a reduced ring. Every countable intersection of essential minimal prime ideals of R is an essential ideal if and only if every first category subset of $\mathcal{M}$ is nowhere dense in $\mathcal{M}$.*

Proof: $\Rightarrow$ Let (F_n) be a sequence of nowhere dense subsets of $\mathcal{M}$. Then by Lemma 3.1, for each $n \in \mathbb{N}$, the ideal $O_{F_n} = \bigcap_{P\in F_n} P$, is an essential ideal. By hypothesis, $E = \bigcap_{n=1}^{\infty} O_{F_n} = O_{(\bigcup_{i=1}^{\infty} F_n)}$ is an essential ideal. On the other hand $V(E) = \overline{(\bigcup_{i=1}^{\infty} F_n)}$. So we must have $\operatorname{int}(\overline{\bigcup_{n=1}^{\infty} F_n}) = \phi$, i.e., $\bigcup_{i=1}^{\infty} F_n$ is nowhere dense.

$\Leftarrow$ Let (I_n) be a sequence of essential minimal prime ideals in R. Letting $\{I_n\} = F_n$, then $\operatorname{int}(F_n) = \operatorname{int} V(I_n) = \phi$, i.e., each F_n is a nowhere dense subset of $\mathcal{M}$. $O_{F_n} \subseteq I_n$ implies that $O_A \subseteq \bigcap_{n=1}^{\infty} I_n$, where $A = \bigcup_{n=1}^{\infty} F_n$. Now we have $V(O_A) = \overline{A}$, and since A is a first category subset, then $\operatorname{int}(\overline{A}) = \phi$, i.e., O_A is an essential ideal. Thus $\bigcap_{n=1}^{\infty} I_n$ is also an essential ideal. $\square$

The following lemma is a characterization of the intersection of all essential minimal prime ideals of $C(X)$.

Lemma 3.5. *The intersection of all essential minimal prime ideals of $C(X)$ is equal to $O^{\beta X\setminus I(X)}$, where $I(X)$ is the set of isolated points of topological space X.*

Proof: Let P be an essential minimal prime ideal of $C(X)$. Then by [4, Corollary 3.3], there is $p \in \beta X \setminus I(X)$ such that $O^p \subseteq P$ and so $O^{\beta X\setminus I(X)}$ is contained in the intersection of essential minimal prime ideals. Now let f be an element of the intersection of essential minimal prime ideals and $p \in \beta X \setminus I(X)$. Then by [4, Theorem 3.1], O^p is an essential ideal which is the intersection of some essential minimal prime ideals, therefore $f \in O^p$. Hence the intersection of essential minimal prime ideals is contained in $O^{\beta X\setminus I(X)}$. $\square$

An ideal I of R is called a pure ideal, if for any $f \in I$, there is a $g \in I$ such that $f = fg$ (see [1]). The set of all points in a topological space X which have compact neighborhoods is denoted by X_L. It is easily seen that $X_L = coz(C_K(X)) = \bigcup_{f\in C_K(X)} coz(f)$. Since $\beta X \setminus X \subseteq \beta X \setminus I(X)$, we have, $C_F(X) \subseteq O^{\beta X\setminus I(X)} \subseteq C_K(X)$, where $C_K(X) = O^{\beta X\setminus X}$, see [7, 7.F]. In the following theorem we show that the intersection of essential minimal prime ideals in $C(X)$ is equal to the socle of $C(X)$. However, it need not be equal to $C_K(X)$.

Theorem 3.6. (i) *The intersection of all essential minimal prime ideals of $C(X)$ is equal to the socle of $C(X)$ (i.e., $C_F(X)$).*
(ii) *Every intersection of essential minimal prime ideals of $C(X)$ is an essential ideal if and only if the set of isolated points of X is dense in X.*
(iii) *The intersection of all essential minimal prime ideals of $C(X)$ is equal to $C_K(X)$ if and only if $I(X) = X_L = \bigcup_{f\in C_K(X)} \overline{X \setminus Z(f)}$, i.e., $I(X) = X_L$ and $C_K(X)$ is a pure ideal.*

PROOF: (i) By Lemma 3.5, the intersection of essential minimal prime ideals is $O^{\beta X\setminus I(X)}$. Hence $C_F(X) \subseteq O^{\beta X\setminus I(X)}$. Now let $f \in O^{\beta X\setminus I(X)}$. Then $\beta X \setminus \mathrm{int}_{\beta X}\, cl_{\beta X} Z(f) \subseteq I(X)$. By [7, 6.9 d], any isolated point of X is isolated in βX, so $\beta X \setminus \mathrm{int}_{\beta X}\, cl_{\beta X} Z(f)$ is a compact subset of βX consisting of some isolated points. Therefore $\beta X \setminus \mathrm{int}_{\beta X}\, cl_{\beta X} Z(f)$ is finite, which implies that $X \setminus Z(f)$ is finite. Thus $f \in C_F(X)$.

(ii) By (i), this is [3, Corollary 3.3].

(iii) Let $C_K(X) = O^{\beta X\setminus I(X)}$. It is easily seen that $I(X) \subseteq X_L$. Now let $x \in X_L$, then there exists a compact subset U in X such that $x \in \mathrm{int}\, U$, i.e., $x \notin X \setminus \mathrm{int}\, U$. By complete regularity of X there is $f \in C(X)$ such that $x \in X\setminus Z(f) \subseteq U \subseteq cl_X U$, hence $x \in X\setminus Z(f)$, where $f \in C_K(X)$. By hypothesis, $X \setminus I(X) \subseteq Z(f)$ so $x \in I(X)$. Therefore $I(X) = X_L$, hence $C_K(X) = O^{\beta X\setminus I(X)} = O^{\beta X\setminus coz(C_K(X))}$. By [1, Theorem 3.2], $X_L = coz(C_K(X)) = \bigcup_{f\in C_K(X)} \overline{X \setminus Z(f)}$.

Conversely, we have $I(X) = X_L = \bigcup_{f\in C_K(X)} \overline{X \setminus Z(f)}$. By [1, Theorem 3.2], $C_K(X) = O^{\beta X\setminus coz(C_K(X))} = O^{\beta X\setminus X_L} = O^{\beta X\setminus I(X)}$. □

Recall that a completely regular space X is said to be a pseudo-discrete space if every compact subset of X has finite interior. Clearly the class of pseudo-discrete spaces contains the class of P-spaces (see [3]).

Corollary 3.7. *A topological space X is pseudo-discrete if and only if $I(X) = X_L$ and $C_K(X)$ is a pure ideal.*

PROOF: This is a consequence of Theorem 4.5 in [3] and Theorem 3.6. □

By using the above theorem, we give examples of topological spaces X for which $C_K(X)$ is equal to the intersection of essential minimal prime ideals (i.e., X is a pseudo-discrete space).

Example 3.8. (i) If X is a locally compact space and $C_K(X) = O^{\beta X\setminus I(X)}$, then X is a discrete space. For if X is a locally compact, then $C_K(X)$ is a pure ideal and $X_L = X$. Since $C_K(X) = O^{\beta X\setminus I(X)}$, then $I(X) = X_L = X$, i.e., X is a discrete space.

(ii) Let X be the set of rational numbers with the topology such that all points have their usual neighborhoods except for $x = 0$ which is isolated point. Then

$X_L = I(X) = \{0\}$ and $C_K(X) = \{f \in C(X) : f = 0$ except for $x = 0\}$ is a pure ideal and so $C_K(X) = O^{\beta X \setminus I(X)}$.

In the following, we give an example of a space X for which $C_K(X)$ is not equal to the intersection of essential minimal prime ideals.

Example 3.9. Let $X = [-1, 1]$ with the topology in which $x = 0$ has the usual neighborhoods and all other points are isolated. Then $X_L = X \setminus \{0\} = I(X)$ but $C_K(X)$ is not equal to $O^{\beta X \setminus I(X)}$, for $C_K(X)$ is not a pure ideal, see [1, Example 3.3].

By Proposition 3.3 and Theorem 3.6, we have the following corollary.

Corollary 3.10. *The set of isolated points of X is dense in X if and only if the set of isolated points of $\mathcal{M}(C(X))$ is dense in $\mathcal{M}(C(X))$.*

By [3, Theorem 2.4] and Theorem 3.4, we have the following corollary.

Corollary 3.11. *Let X be a compact space. Every first category subset of X is nowhere dense in X if and only if every first category subset of $\mathcal{M}(C(X))$ is nowhere dense in $\mathcal{M}(C(X))$.*

Question 3.12.
- (i) For $R = C(X)$, determine X for which $R_K(\mathcal{M}) = C_K(X)$. Note that in case X and $\mathcal{M}$ are compact or nowhere compact we have $R_K(\mathcal{M}) = C_K(X)$.
- (ii) When is the intersection of *sd*-ideals in a reduced ring R an *sd*-ideal?
- (iii) When is the intersection of *sd*-ideals in $C(X)$ an *sd*-ideal?
- (iv) When is $R_K(\mathcal{M})$ equal to the intersection of all strongly dense *fd*-ideals?

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