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Existence of periodic solutions for first-order totally nonlinear neutral differential equations with variable delay

Abdelouaheb Ardjouni, Ahcène Djoudi

Abstract. We use a modification of Krasnoselskii's fixed point theorem due to Burton (see [*Liapunov functionals, fixed points and stability by Krasnoselskii's theorem*, Nonlinear Stud. **9** (2002), 181–190], Theorem 3) to show that the totally nonlinear neutral differential equation with variable delay

$$x'(t) = -a(t)h(x(t)) + c(t)x'(t-g(t))Q'(x(t-g(t))) + G(t, x(t), x(t-g(t))),$$

has a periodic solution. We invert this equation to construct a fixed point mapping expressed as a sum of two mappings such that one is compact and the other is a large contraction. We show that the mapping fits very nicely for applying the modification of Krasnoselskii's theorem so that periodic solutions exist.

Keywords: periodic solution; nonlinear neutral differential equation; large contraction; integral equation

Classification: 34K20, 45J05, 45D05

1. Introduction

In recent years, there have been a few papers written on the existence of periodic solutions, nontrivial periodic solutions and positive periodic solutions for several classes of functional differential equations with delays, which arise from a number of mathematical ecological models, economical and control models, physiological and population models and other models, see [1]–[16] and the references therein.

Motivated by the papers [1]–[16] and the references therein, we consider the following totally nonlinear neutral differential equation with variable delay

$$x'(t) = -a(t)h\left(x(t)\right) + c(t)x'(t-g(t))Q'\left(x\left(t-g(t)\right)\right) + G(t, x(t), x(t-g(t))), \tag{1.1}$$

where a is positive continuous real valued function, c is continuously differentiable, g is twice continuously differentiable, $h, Q : \mathbb{R} \to \mathbb{R}$ and $G : \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous with respect to its arguments. Our purpose here is to use a modification of Krasnoselskii's fixed point theorem due to Burton (see [7, Theorem 3]) to show the existence of periodic solutions for equation (1.1). Clearly, the present problem is totally nonlinear so that the variation of parameters can not be applied directly. Then, we resort to the idea of adding and subtracting a linear term. As noted by Burton in [7], the added term destroys a contraction already present in part of the equation but it replaces it with the so-called large contraction mapping which is suitable for fixed point theory. During the process we transform (1.1) into an integral equation written as a sum of two mappings, one is a large contraction and the other is compact. After that, we use a variant of Krasnoselskii's fixed point theorem, to show the existence of a periodic solution. For details on Krasnoselskii's theorem we refer the reader to [17]. A particular case of equation (1.1) has been recently studied in [1]. The authors in [1] have proved that, under some restrictions, the nonlinear delay differential equation

$$x'(t) = -a(t)h(x(t)) + G(t, x(t - g(t))),$$

has a periodic solution.

In Section 2, we present the inversion of equation (1.1) and the modification of Krasnoselskii's fixed point theorem. We present our main results on periodicity in Section 3. The final result in this paper is the existence of periodic solutions for (1.1).

2. Inversion of equation (1.1)

Let $T > 0$ and define $C_T = \{\varphi : \mathbb{R} \to \mathbb{R} : \varphi \in C \text{ and } \varphi(t+T) = \varphi(t)\}$ where C is the space of continuous real valued functions. C_T is a Banach space endowed with the norm

$$\|\varphi\| := \max_{0 \le t \le T} |\varphi(t)|.$$

If $L > 0$ is an arbitrary constant then define

$$M_L := \{\varphi \in C_T : \|\varphi\| \le L,\ \varphi' \text{ is bounded}\}. \tag{2.1}$$

Since we are searching for the existence of periodic solutions for equation (1.1), it is natural to assume that

$$a(t+T) = a(t),\ c(t+T) = c(t),\ G(t+T, x, y) = G(t, x, y),\ g(t+T) = g(t), \tag{2.2}$$

with $g(t) \ge g^* > 0$. Also, we assume

$$\int_0^T a(s)\, ds > 0. \tag{2.3}$$

Functions $Q(x)$, $Q'(x)$ and $G(t, x, y)$ are locally Lipschitz continuous in x, x and in x and y, respectively. That is, we assume that there are positive constants k_1, k_2, k_3, k_4 so that $|x|, |y|, |z|, |w| \le L$ imply

$$|Q(x) - Q(y)| \le k_1 \|x - y\|, \tag{2.4}$$

$$|Q'(x) - Q'(y)| \le k_2 \|x - y\|, \tag{2.5}$$

and

$$|G(t, x, y) - G(t, z, w)| \le k_3 \|x - z\| + k_4 \|y - w\|. \tag{2.6}$$

Also, we suppose that for all t, $0 \le t \le T$,

$$g'(t) \neq 1. \tag{2.7}$$

Since g is periodic, condition (2.7) implies that $g'(t) < 1$.

Let us begin by integrating (1.1) to determine a fixed point mapping from which we obtain a desired solution.

Lemma 1. *Suppose* (2.2), (2.3) *and* (2.7) *hold. If* $x \in C_T$, *then* x *is a solution of* (1.1) *if and only if*

(2.8)

$$\begin{aligned} x(t) = &\left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} a(u)H(x(u))e^{-\int_u^t a(s)\,ds}\,du \\ &+ \frac{c(t)}{(1-g'(t))}Q(x(t-g(t))) + \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \\ &\times \int_{t-T}^{t} \left[G(u, x(u), x(u-g(u))) - r(u)Q(x(u-g(u)))\right] e^{-\int_u^t a(s)\,ds}\,du, \end{aligned}$$

where

$$r(u) := \frac{(c'(u) + a(u)c(u))\,(1-g'(u)) + c(u)g''(u)}{(1-g'(u))^2}, \tag{2.9}$$

and

$$H(x(u)) := x(u) - h(x(u)). \tag{2.10}$$

Proof: Let $x \in C_T$ be a solution of (1.1). Rewrite (1.1) as

$$x'(t)+a(t)x(t) = a(t)H(x(t))+c(t)x'(t-g(t))Q'(x(t-g(t)))+G(t,x(t),x(t-g(t))).$$

Multiply both sides of the above equation by $e^{\int_0^t a(s)\,ds}$ and then integrate from $t-T$ to t to obtain

$$\begin{aligned} \int_{t-T}^{t} \left[x(u)e^{\int_0^u a(s)\,ds}\right]' du = &\int_{t-T}^{t} a(u)H(x(u))e^{\int_0^u a(s)\,ds}\,du \\ &+ \int_{t-T}^{t} G(u, x(u), x(u-g(u)))e^{\int_0^u a(s)\,ds}\,du \\ &+ \int_{t-T}^{t} c(u)x'(u-g(u))Q'(x(u-g(u)))e^{\int_0^u a(s)\,ds}\,du. \end{aligned}$$

Rewrite the last term as

$$\begin{aligned} &\int_{t-T}^{t} c(u)x'(u-g(u))Q'(x(u-g(u)))e^{\int_0^u a(s)\,ds}\,du \\ &= \int_{t-T}^{t} \frac{c(u)(1-g'(u))x'(u-g(u))Q'(x(u-g(u)))}{(1-g'(u))}e^{\int_0^u a(s)\,ds}\,du. \end{aligned}$$

Using integration by parts, and that c, g and x are periodic we obtain

$$\begin{aligned}
&\int_{t-T}^{t} c(u)x'(u-g(u))Q'(x(u-g(u)))e^{\int_0^u a(s)\,ds}\,du \\
&= \frac{c(t)}{(1-g'(t))}Q(x(t-g(t)))e^{\int_0^t a(s)\,ds}\left(1-e^{-\int_{t-T}^{t} a(s)\,ds}\right) \\
&\quad - \int_{t-T}^{t} r(u)Q(x(u-g(u)))e^{\int_0^u a(s)\,ds}\,du
\end{aligned}$$

where r is given by (2.9). We arrive at

$$\begin{aligned}
&x(t)e^{\int_0^t a(s)\,ds} - x(t-T)e^{\int_0^{t-T} a(s)\,ds} \\
&= \int_{t-T}^{t} a(u)H\left(x(u)\right)e^{\int_0^u a(s)\,ds}\,du \\
&\quad + \frac{c(t)}{(1-g'(t))}Q\left(x(t-g(t))\right)e^{\int_0^t a(s)\,ds}\left(1-e^{-\int_{t-T}^{t} a(s)\,ds}\right) \\
&\quad + \int_{t-T}^{t} G(u,x(u),x(u-g(u)))e^{\int_0^u a(s)\,ds}\,du \\
&\quad - \int_{t-T}^{t} r(u)Q(x(u-g(u)))e^{\int_0^u a(s)\,ds}\,du.
\end{aligned}$$

Now, we obtain (2.8) by dividing both sides of the above equation by $e^{\int_0^t a(s)\,ds}$ and using the fact that $x(t) = x(t-T)$. Since each step is reversible, the converse follows easily by differentiating (2.8). This completes the proof. □

Krasnoselskii (see [8] or [17]) combined the contraction mapping theorem and Shauder's theorem and formulated the following hybrid and attractive result.

Theorem 1. *Let M be a closed convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A and B map M into S such that*

(i) $\forall x, y \in M \Rightarrow Ax + By \in M$,

(ii) *A is continuous and AM is contained in a compact set,*

(iii) *B is a contraction with constant $\alpha < 1$.*

Then there is a $z \in M$ with $z = Az + Bz$.

This is a captivating result and has a number of interesting applications. In recent years much attention has been paid to this theorem. Burton [8] observed that Krasnoselskii's result can be more useful in applications with certain changes and formulated Theorem 3 below (see [9] for the proof).

Definition 1. Let (M, d) be a metric space and $B : M \to M$. Then B is said to be a large contraction if for $\varphi, \psi \in M$ with $\varphi \neq \psi$ we have $d(B\varphi, B\psi) < d(\varphi, \psi)$ and for all $\varepsilon > 0$ there exists $\delta < 1$ such that

$$[\varphi, \psi \in M,\ d(\varphi, \psi) \geq \varepsilon] \Rightarrow d(B\varphi, B\psi) \leq \delta d(\varphi, \psi).$$

It has been shown in [8, Example 1.2.7] that if we let

$$M = \left\{ \varphi : [0, +\infty) \to \mathbb{R} : \varphi \text{ continuous and } \|\varphi\| \le \frac{\sqrt{3}}{3} \right\},$$

then the mapping $(Hx)(t) = x(t) - x^3(t)$ is a large contraction on M endowed with the supremum norm.

Theorem 2. *Let (M, d) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and $\kappa > 0$, such that $d(x, B^n x) \le \kappa$ for all $n \ge 1$. Then B has a unique fixed point in M.*

Theorem 3 (Burton-Krasnoselskii)**.** *Let M be a closed bounded convex non-empty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A, B map M into M and that*

(i) $\forall x, y \in M \Rightarrow Ax + By \in M$,
(ii) *A is continuous and AM is contained in a compact subset of M,*
(iii) *B is a large contraction.*

Then there is a $z \in M$ with $z = Az + Bz$.

We will use this theorem to prove the existence of periodic solutions for (1.1). We begin with the following proposition (see [1]) and for convenience we present, below, its proof. Let L be a fixed number. In the next proposition we prove that, for a well chosen function h, the mapping H in (2.10) is a large contraction on M_L (see (2.1)). So, let us make the following assumptions on the function $h : \mathbb{R} \to \mathbb{R}$.

(H1) h is continuous on $U_L = [-L, L]$ and differentiable on $(-L, L)$.
(H2) h is strictly increasing on U_L.
(H3) $\sup_{s \in (-L,L)} h'(s) \le 1$.

Proposition 1. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function satisfying* (H1)–(H3)*. Then the mapping H in* (2.10) *is a large contraction on the set M_L.*

Proof: Let $\phi, \varphi \in M_L$ with $\phi \ne \varphi$. Then $\phi(t) \ne \varphi(t)$ for some $t \in \mathbb{R}$. Define the set

$$D(\phi, \varphi) := \{t \in \mathbb{R} : \phi(t) \ne \varphi(t)\}.$$

Note that $\varphi(t) \in U_L$ for all $t \in \mathbb{R}$ whenever $\varphi \in M_L$. Since h is strictly increasing

$$\frac{h(\varphi(t)) - h(\phi(t))}{\varphi(t) - \phi(t)} = \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > 0 \tag{2.11}$$

holds for all $t \in D(\phi, \varphi)$. On the other hand, for all $t \in D(\phi, \varphi)$, we have

$$\begin{aligned} |(H\phi)(t) - (H\varphi)(t)| &= |\phi(t) - h(\phi(t)) - \varphi(t) + h(\varphi(t))| \\ &= |\phi(t) - \varphi(t)| \left| 1 - \left(\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} \right) \right|. \end{aligned} \tag{2.12}$$

For each fixed $t \in D(\phi, \varphi)$, define the set $U_t \subset U_L$ by

$$U_t = \begin{cases} (\varphi(t), \phi(t)), & \text{if } \phi(t) > \varphi(t), \\ (\phi(t), \varphi(t)), & \text{if } \varphi(t) > \phi(t), \end{cases} \quad \text{for } t \in D(\phi, \varphi).$$

The mean value theorem implies that for each fixed $t \in D(\phi, \varphi)$ there exists a real number $c_t \in U_t$ such that

$$\frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} = h'(c_t).$$

By (H2) and (H3), we have

$$1 \geq \sup_{t \in (-L,L)} h'(t) \geq \sup_{t \in U_t} h'(t) \geq h'(c_t) \geq \inf_{s \in U_t} h'(s) \geq \inf_{t \in (-L,L)} h'(t) \geq 0. \tag{2.13}$$

Consequently, by (2.11)–(2.13), we obtain

$$|(H\phi)(t) - (H\varphi)(t)| \leq \left| 1 - \inf_{u \in (-L,L)} h'(u) \right| |\phi(t) - \varphi(t)|, \tag{2.14}$$

for all $t \in D(\phi, \varphi)$. Hence, the mapping H is a large contraction in the supremum norm. Indeed, fix $\epsilon \in (0,1)$ and assume that ϕ and φ are two functions in M_L satisfying

$$\|\phi - \varphi\| = \sup_{t \in D(\phi,\varphi)} |\phi(t) - \varphi(t)| \geq \epsilon.$$

If $|\phi(t) - \varphi(t)| \leq \epsilon/2$ for some $t \in D(\phi, \varphi)$, then from (2.13) and (2.14), we get

$$|(H\phi)(t) - (H\varphi)(t)| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{2.15}$$

Since h is continuous and strictly increasing, the function $h(u + \frac{\epsilon}{2}) - h(u)$ attains its minimum on the closed and bounded interval $[-L, L]$. Thus, if $\frac{\epsilon}{2} < |\phi(t) - \varphi(t)|$ for some $t \in D(\phi, \varphi)$, then from (H2) and (H3) we conclude that

$$1 \geq \frac{h(\phi(t)) - h(\varphi(t))}{\phi(t) - \varphi(t)} > \lambda,$$

where

$$\lambda := \frac{1}{2L} \min \left\{ h\left(u + \frac{\epsilon}{2}\right) - h(u),\ u \in [-L, L] \right\} > 0.$$

Therefore, from (2.12), we have

$$|(H\phi)(t) - (H\varphi)(t)| \leq (1 - \lambda) \|\phi - \varphi\|. \tag{2.16}$$

Consequently, it follows from (2.15) and (2.16) that

$$|(H\phi)(t) - (H\varphi)(t)| \leq \eta \|\phi - \varphi\|,$$

where

$$\eta = \max\left\{\frac{1}{2}, 1-\lambda\right\} < 1.$$

The proof is complete. □

3. Existence of periodic solutions

To apply Theorem 3, we need to define a Banach space S, a closed bounded convex subset M_L of S and construct two mappings such that one is a large contraction and the other is completely continuous. So, we let $(S, \|\cdot\|) = (C_T, \|\cdot\|)$ and $M_L = \{\varphi \in S : \|\varphi\| \le L,\ \varphi' \text{ is bounded}\}$, where L is a positive constant. So we have to express (2.8) as

$$\varphi(t) = (B\varphi)(t) + (A\varphi)(t) := (\mathbb{C}\varphi)(t),$$

where $A, B : S \to S$ are defined by

$$(B\varphi)(t) := \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} a(u)H(\varphi(u))e^{-\int_{u}^{t} a(s)\,ds}\,du, \tag{3.1}$$

and

$$\begin{aligned}(A\varphi)(t) :={}& \frac{c(t)}{(1-g'(t))}Q\left(\varphi(t-g(t))\right) + \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \\ &\times \int_{t-T}^{t} \left[G(u,\varphi(u),\varphi(u-g(u))) - r(u)Q(\varphi(u-g(u)))\right] e^{-\int_{u}^{t} a(s)\,ds}\,du.\end{aligned} \tag{3.2}$$

In our analysis we need the following assumptions

$$\left[(k_3+k_4)\,L + |G(t,0,0)|\right] \le \beta L a(t), \tag{3.3}$$

$$|r(t)|\left(k_1 L + |Q(0)|\right) \le \delta L a(t), \tag{3.4}$$

$$\max_{t\in[0,T]} \left|\frac{c(t)}{(1-g'(t))}\right| = \alpha, \tag{3.5}$$

$$J\left[\alpha\left(k_1 + \frac{|Q(0)|}{L}\right) + \beta + \delta\right] \le 1, \tag{3.6}$$

$$\max\left(|H(-L)|, |H(L)|\right) \le \frac{(J-1)L}{J}, \tag{3.7}$$

where α, β, δ and J are constants with $J \ge 3$.

We shall prove that the mapping $\mathbb{C}$ has a fixed point which solves (1.1), whenever its derivative exists.

Lemma 2. *For A defined in* (3.2), *suppose that* (2.2)–(2.7) *and* (3.3)–(3.6) *hold. Then $A : M_L \to M_L$ is continuous in the supremum norm and maps M_L into a compact subset of M_L.*

Proof: Clearly, if φ is continuous then $A\varphi$ is. A change of variable in (3.2) shows that $(A\varphi)(t+T)=\varphi(t)$. Observe that

$$|Q(x)| \leq k_1|x| + |Q(0)|,\ |Q'(x)| \leq k_2|x| + |Q'(0)|,$$

and

$$|G(t,x,y)| \leq |G(t,x,y)-G(t,0)| + |G(t,0,0)| \leq k_3|x| + k_4|y| + |G(t,0,0)|.$$

So, for any $\varphi \in M_L$, we have

$$\begin{aligned}
&|(A\varphi)(t)| \\
&\leq \left|\frac{c(t)}{(1-g'(t))}\right| |Q(\varphi(t-g(t)))| + \left(1-e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \\
&\quad \times \int_{t-T}^{t} \left[|G(u,\varphi(u),\varphi(u-g(u)))| + |r(u)|\,|Q(\varphi(u-g(u)))|\right] e^{-\int_u^t a(s)\,ds}\,du \\
&\leq \alpha\left(k_1 L + |Q\,(0)|\right) + \left(1-e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \\
&\quad \times \int_{t-T}^{t} \left((k_3+k_4)\,L + |G(u,0,0)| + |h(u)|\,(k_1 L + |Q(0)|)\right) e^{-\int_u^t a(s)\,ds}\,du \\
&\leq \alpha\left(k_1 + \frac{|Q(0)|}{L}\right) L + \left(1-e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} (\beta+\delta) L \int_{t-T}^{t} a(u) e^{-\int_u^t a(s)\,ds}\,du \\
&\leq \left[\alpha\left(k_1 + \frac{|Q(0)|}{L}\right) + \beta + \delta\right] L \leq \frac{L}{J} < L.
\end{aligned}$$

That is $A\varphi \in M_L$.

We show that A is continuous in the supremum norm. Let $\varphi, \psi \in M_L$, and let (3.8)

$$\gamma = \max_{t\in[0,T]} \left(1-e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1},\ \theta = \max_{u\in[t-T,t]} e^{-\int_u^t a(s)\,ds},\ \sigma = \max_{t\in[0,T]} \{a(t)\},$$
$$\rho = \max_{t\in[0,T]} |G(t,0,0)|,\ \mu = \max_{t\in[0,T]} \left|\frac{c'(t)}{(1-g'(t))}\right|,\ \vartheta = \max_{t\in[0,T]} \left|\frac{g''(t)c(t)}{(1-g'(t))^2}\right|.$$

Then

$$\begin{aligned}
&|(A\varphi)(t) - (A\psi)(t)| \\
&\leq \left|\frac{c(t)Q(\varphi(t-g(t)))}{(1-g'(t))} - \frac{c(t)Q(\psi(t-g(t)))}{(1-g'(t))}\right| + \left(1-e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \\
&\quad \times \int_{t-T}^{t} \{|G(u,\varphi(u),\varphi(u-g(u))) - G(u,\psi(u),\psi(u-g(u)))| \\
&\quad + |h(u)|\,|Q(\varphi(u-g(u))) - Q(\psi(u-g(u)))|\}\, e^{-\int_u^t a(s)\,ds}\,du
\end{aligned}$$

$$\le \alpha k_1 \|\varphi - \psi\| + (k_3 + k_4)\|\varphi - \psi\| \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} e^{-\int_{u}^{t} a(s)\,ds}\,du$$
$$+ k_1 \delta \|\varphi - \psi\| \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} a(u) e^{-\int_{u}^{t} a(s)\,ds}\,du$$
$$\le (\alpha k_1 + (k_3 + k_4) T\gamma\theta + \delta k_1)\|\varphi - \psi\|.$$

Let $\varepsilon > 0$ be arbitrary. Define $\eta = \frac{\varepsilon}{K}$, with $K = \alpha k_1 + (k_3 + k_4)T\gamma\theta + \delta k_1$, where k_1, k_3 and k_4 are given by (2.4) and (2.6). Then, for $\|\varphi - \psi\| < \eta$, we obtain

$$\|A\varphi - A\psi\| \le K\|\varphi - \psi\| < \varepsilon.$$

It remains to show that A is compact. Let $\varphi_n \in M_L$, where n is a positive integer. Then, as above, we can see that

$$\|A\varphi_n\| \le L. \tag{3.9}$$

Moreover, a direct calculation shows that

$$\begin{aligned}(A\varphi_n)'(t) = {} & \frac{c'(t)Q(\varphi_n(t-g(t))) + c(t)\varphi_n'(t-g(t))Q'(\varphi_n(t-g(t)))}{1-g'(t)} \\ & + \frac{g''(t)c(t)Q(\varphi_n(t-g(t)))}{(1-g'(t))^2} + G(t, \varphi_n(t), \varphi_n(t-g(t))) \\ & - h(t)Q(\varphi_n(t-g(t))) - a(t)\left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \\ & \times \int_{t-T}^{t} [G(t, \varphi_n(t), \varphi_n(t-g(t))) - h(u)Q(\varphi_n(u-g(u)))] e^{-\int_{u}^{t} a(s)\,ds}\,du.\end{aligned}$$

Let L' be the norm bound of φ'. By invoking the conditions (2.4)–(2.6), (3.3), (3.5), (3.8) and (3.9) we obtain

$$\begin{aligned}|(A\varphi_n)'(t)| \le {} & \mu(k_1 L + |Q(0)|) + \alpha L'(k_2 L + |Q'(0)|) + \vartheta(k_1 L + |Q(0)|) \\ & + (k_3 + k_4)L + \rho + \delta a(t)(k_1 L + |Q(0)|) \\ & + a(t)\gamma T\theta[(k_3 + k_4)L + \rho + \delta a(t)(k_1 L + |Q(0)|)] \\ \le {} & (1 + \sigma\gamma T\theta)(k_3 + k_4)L + (\mu + \vartheta + (1 + \sigma\gamma T\theta)\delta\sigma)k_1 L \\ & + \alpha L'(k_2 L + |Q'(0)|) + (1 + \sigma\gamma T\theta)\rho \\ & + \left(\mu + \vartheta + \delta\sigma + \sigma^2\gamma T\theta\delta\right)|Q(0)| \\ \le {} & D,\end{aligned}$$

for some positive constant D. Hence the sequence $(A\varphi_n)$ is uniformly bounded and equicontinuous. The Ascoli-Arzela theorem implies that a subsequence $(A\varphi_{n_k})$ of $(A\varphi_n)$ converges uniformly to a continuous $T-$periodic function. Thus, A is continuous and AM_L is contained in a compact subset of M_L. $\square$

Lemma 3. *For B defined in* (3.2), *suppose that* (H1)–(H3) *and* (3.7) *hold. Then* $B : M_L \to M_L$ *is a large contraction.*

Proof: Obviously, $B\varphi$ is continuous and it is easy to show that $(B\varphi)(t+T) = (B\varphi)(t)$. So, for any $\varphi \in M_L$, we get by (3.7) that

$$\begin{aligned}|(B\varphi)(t)| &\le \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} a(u)\,|H(\varphi(u))|\, e^{-\int_u^t a(s)\,ds}\,du \\ &\le \max\left(|H(-L)|, |H(L)|\right) \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} a(u) e^{-\int_u^t a(s)\,ds}\,du \\ &\le \frac{(J-1)L}{J} < L.\end{aligned}$$

Thus $B\varphi \in M_L$. Consequently, we have $B : M_L \to M_L$.

It remains to show that B is a large contraction. From the proof of Proposition 1 we have for $\phi, \varphi \in M_L$, with $\phi \neq \varphi$,

$$\begin{aligned}&|(B\phi)(t) - (B\varphi)(t)| \\ &\le \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} a(u)\,|H(\phi(u)) - H(\varphi(u))|\, e^{-\int_u^t a(s)\,ds}\,du \\ &\le \|\phi - \varphi\| \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} a(u) e^{-\int_u^t a(s)\,ds}\,du \\ &= \|\phi - \varphi\|.\end{aligned}$$

Then $\|B\phi - B\varphi\| \le \|\phi - \varphi\|$. Now, let $\epsilon \in (0,1)$ be given and let $\phi, \varphi \in M_L$ with $\|\phi - \varphi\| \ge \epsilon$. From the proof of Proposition 1 we have found $\eta < 1$ such that

$$\begin{aligned}&|(B\phi)(t) - (B\varphi)(t)| \\ &\le \left(1 - e^{-\int_{t-T}^{t} a(s)\,ds}\right)^{-1} \int_{t-T}^{t} a(u)\eta\|\phi - \varphi\| e^{-\int_u^t a(s)\,ds}\,du \\ &\le \eta\|\phi - \varphi\|.\end{aligned}$$

Then $\|B\phi - B\varphi\| \le \eta\|\phi - \varphi\|$. Consequently, B is a large contraction. □

Theorem 4. *Let* $(S, \|\cdot\|)$ *be the Banach space of continuous* T*-periodic real functions and* $M_L = \{\varphi \in S : \|\varphi\| \le L,\ \varphi' \text{ is bounded}\}$, *where* L *is a positive constant. Suppose* (H1)–(H3), (2.2)–(2.7) *and* (3.3)–(3.7) *hold. Then equation* (1.1) *possesses a* T*-periodic solution* φ *in the subset* M_L.

Proof: By Lemma 2, the operator $A : M_L \to M_L$ is continuous and AM_L is contained in compact subset of M_L. By Lemma 3, $B : M_L \to M_L$ is a large contraction. Moreover, if $\phi, \varphi \in M_L$, we see that

$$\|A\phi + B\varphi\| \le \|A\phi\| + \|B\varphi\| \le \frac{L}{J} + \frac{(J-1)\,L}{J} = L.$$

Thus $A\phi + B\varphi \in M_L$.

Clearly, all the hypotheses of the Burton-Krasnoselskii theorem are satisfied. Thus, there exists a fixed point $\varphi \in M_L$ such that $\varphi = A\varphi + B\varphi$. By Lemma 1 this fixed point is a solution of (1.1) and the proof is complete. □

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Department of Mathematics, Faculty of Sciences, University of Annaba, P.O. Box 12 Annaba, Algeria

E-mail: abd_ardjouni@yahoo.fr
adjoudi@yahoo.com

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