One-dimensional model describing the non-linear viscoelastic response of materials

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Abstract. In this paper we consider a model of a one-dimensional body where strain depends on the history of stress. We show local existence for large data and global existence for small data of classical solutions and convergence of the displacement, strain and stress to zero for time going to infinity.

Keywords: viscoelasticity; integrodifferential equation; classical solution; global existence; implicit constitutive relations

Classification: 45K05, 45G10, 74D10

1. Introduction

If we want to describe deformations of a body, relations between stress and strain (so called constitutive relations) play a crucial role. Classical models expressed stress as a function of strain. In the last years, models where strain is a function of stress or where their dependence is given by a relation (neither of them is a function of the other one) started to be studied more intensively (see e.g. Rajagopal [12], Málek [8], Málek, Průša and Rajagopal [9], Průša and Rajagopal [11], Bulíček et al. [1]).

Muliana, Rajagopal and Wineman have introduced in [10] a model of a viscoelastic body where strain is a function of the history of stress (in contrast to classical integral viscoelastic models where stress was considered as a function of the history of strain). According to [10], this new model was tested on human patellar tendons and fits very well the measured data.

The equations of this model read as follows

$$\rho u_{tt}(t,x) = \sigma_x(t,x) + f(t,x), \quad (t,x) \in Q, \tag{1}$$

$$u_x(t,x) = \alpha(\sigma(0,x))\beta(t) + \int_0^t \frac{d}{d\sigma}\alpha(\sigma(s,x))\sigma_t(s,x)\beta(t-s)\,\mathrm{d}s, \quad (t,x) \in Q \tag{2}$$

where $Q = (0,+\infty) \times [0,1]$, u is displacement, σ is stress, f is an external force, $\rho > 0$ is constant density and $\alpha : \mathbb{R} \to \mathbb{R}$, $\beta : \mathbb{R}_+ \to \mathbb{R}$ are material functions.

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In [10], some examples of function α, β are given, namely β has the following form

$$\beta(t) = J_0 + \sum_{i=1}^{n} J_i(1 - e^{-t/\tau_i}), \text{ e.g., } \beta(t) = J_0 + J_1(1 - e^{-t/\tau_1}),$$

with J_i, τ_i positive constants (more generally, β is a generalized creep function) and α is

$$\alpha(\sigma) = C_1\sigma + C_2\sigma^2 \text{ or } \alpha(\sigma) = a\left[1 - \exp\left(-\frac{\lambda\sigma}{\sqrt{1+b\sigma^2}}\right)\right] + \frac{\mu\sigma}{\sqrt{1+\gamma\sigma^2}},$$

where C_1, C_2, a, b, γ, λ, μ are positive constants.

In this paper we show local existence of classical solutions for large data and global existence for small data for this kind of equations supplemented by certain initial and boundary conditions. In fact, we show that equation (2) can be solved for σ, so we can express stress as a function of the strain. By inserting the obtained formula for σ into (1) we get an equation of the form

$$(3)\quad u_{tt}(t,x) = \psi(\tilde{u}_x(t,x))u_{xx}(t,x) - \psi(\tilde{u}_x(t,x))\int_0^t r(t-s)u_{xx}(s,x)\,\mathrm{d}s + f(t,x),$$

where

$$\tilde{u}(t,x) := u(t,x) - \int_0^t r(t-s)u(s,x)\,\mathrm{d}s.$$

So, we are coming back to a model where stress is depending on the history of strain. Such equations were studied in the eighties by many authors (MacCamy [7], Staffans [14], Dafermos and Nohel [2], Hrusa and Nohel [5], Hrusa and Renardy [6]). In contrast to these works, in our case function ψ depends on the whole history of u and not only on the present value $u(t)$.[1] This is the reason why some estimates must be done more carefully and therefore we present proofs of local and global existence for (3) in this paper, even if the method of the proofs remains unchanged.

Section 2 contains the main results — local and global existence, uniqueness and asymptotics for (1), (2). In Section 3 we present reduction to equation (3). Sections 4, 5 and 6 are devoted to the reduced equation (3), namely some notations and technical lemmas in Section 4 and proofs of local (resp. global) existence in Section 5 (resp. Section 6).

[1] There is also a very general result of Hrusa [4] which covers this type of dependence, however that result deals with the history to $-\infty$, so the solutions are smooth on $(-\infty, T]$. In our case, if we considered zero history for negative times, there would be typically a jump in zero.

2. Main results

We supplement our model with Dirichlet boundary conditions

$$u(t,0) = u(t,1) = 0, \quad t \geq 0 \tag{4}$$

and initial conditions

$$u(0,x) = u_0(x), \quad \dot{u}(0,x) = u_1(x), \quad x \in [0,1]. \tag{5}$$

Our assumptions for local existence and large data are the following (H^k denotes the Sobolev space $W^{k,2}$, $T \in (0,+\infty]$).

(L1) $\alpha \in C^3(\mathbb{R})$ with $0 < \alpha'(z) < \alpha_M$ on $\mathbb{R}$.
(L2) $\beta \in C^3([0,T])$ with $\beta(0) \neq 0$.
(L3) $u_0 \in H^3([0,1])$ with $u_0'(x)/\beta(0) \in \alpha(\mathbb{R})$ and $u_1 \in H^2([0,1])$.
(L4) $f \in C^1([0,T], L^2([0,1])) \cap C([0,T], H^1([0,1]))$, $f_{tt} \in L^2([0,T], L^2([0,1]))$.
(L5) $u_k(0) = u_k(1) = 0$ for $k = 0,1,2$, where

$$u_2(x) := \frac{u_0''(x)}{\rho\beta(0)\alpha'(\alpha^{-1}(\frac{u_0'(x)}{\beta(0)}))} + f(0,x). \tag{6}$$

Let us remark that condition $u'(x)/\beta(0) \in \alpha(\mathbb{R})$ is necessary, otherwise the equation (2) could not have a solution for small t and condition (L5) is also necessary since it is a compatibility condition for initial and boundary values.

We say that a solution (classical) defined on $[0,T')$ is *maximal*, if it cannot be extended to a (classical) solution on a larger interval $[0,\tilde{T}')$, $\tilde{T}' > T'$.

Theorem 2.1 (Local existence). *Let* (L1)–(L5) *be satisfied. Then there exist* $T' \leq T$ *and functions*

$$u \in \bigcap_{k=0}^{3} C^{3-k}([0,T'], H^k([0,1])), \quad \sigma \in \bigcap_{k=0}^{2} C^{2-k}([0,T'], H^k([0,1]))$$

which solve (1), (2), (4), (5). *Moreover, if* $u : [0,T') \to \mathbb{R}$ *is a maximal solution and*

$$\sup_{t\in[0,T')} \int_{[0,1]} u^2(t)+u_x^2(t)+u_{xx}^2(t)+u_{xxx}^2(t)+u_t^2(t)+u_{tx}^2(t)+u_{txx}^2(t)\,dx < +\infty, \tag{7}$$

then $T' = T$.

Before we formulate the assumptions for global existence, let us say that $\varphi : \mathbb{R}_+ \to \mathbb{R}$ is called a *Bernstein function* if $\varphi(x) > 0$ for all $x > 0$ and the k-th derivative satisfies $(-1)^k\varphi^{(k)}(x) \leq 0$ for all $x > 0$ and all $k = 1,2,3,\ldots$. For a Bernstein function, we define $\varphi_\infty := \lim_{t\to+\infty}\varphi(t)$ if it is finite and $\varphi_1(t) := \varphi_\infty - \varphi(t)$, $t \geq 0$.

Our assumptions for global existence and small data are the following.

(G1) $\alpha \in C^3(V)$ with $\alpha(0) = 0$, $\alpha'(0) > 0$ (V is a neighborhood of 0).

(G2) β is a Bernstein function with $\beta(0) > 0$, β_1, $\beta' \in L^1(\mathbb{R}_+)$.
(G3) $u_0 \in H^3([0,1])$ and $u_1 \in H^2([0,1])$.
(G4) $f \in C^1(\mathbb{R}_+, L^2([0,1])) \cap C(\mathbb{R}_+, H^1([0,1]))$, $f_{tt} \in L^2(\mathbb{R}_+, L^2([0,1]))$.
(G5) $u_k(0) = u_k(1) = 0$ for $k = 0, 1, 2$ where u_2 is defined by (6).

Let us define the following quantities

$$\begin{aligned} U_0 &:= \|u_0\|_{H^3}^2 + \|u_1\|_{H^2}^2, \\ F &:= \sup_{t\geq 0} \int_0^1 f^2 + f_x^2 + f_t^2 \,\mathrm{d}x + \int_0^{+\infty} \int_0^1 f^2 + f_x^2 + f_t^2 + f_{tt}^2 \,\mathrm{d}x \,\mathrm{d}t. \end{aligned}$$

Theorem 2.2 (Global existence)**.** *Let* (G1)–(G5) *be satisfied. Then there exists* $\mu > 0$ *such that if* $U_0, F < \mu$ *then there exists a solution* $u \in \bigcap_{k=0}^3 C^{3-k}(\mathbb{R}_+, H^k([0,1]))$, $\sigma \in \bigcap_{k=0}^2 C^{2-k}(\mathbb{R}_+, H^k([0,1]))$ *to* (1), (2), (4), (5). *Moreover,*

(8)
$$u, u_x, u_t, u_{xx}, u_{tx}, u_{tt}, u_{xxx}, u_{txx}, u_{ttx}, u_{ttt} \in C_b(\mathbb{R}_+, L^2([0,1])) \cap L^2(\mathbb{R}_+, L^2([0,1])),$$

and

$$u, u_x, u_t, u_{xx}, u_{tx}, u_{tt} \to 0 \quad \text{and} \quad \sigma, \sigma_x, \sigma_t \to 0 \tag{9}$$

uniformly on $[0,1]$ *as* $t \to +\infty$.

Let us remark that functions α, β from [10] mentioned in the beginning of this paper satisfy the assumptions of the Theorems (more precisely, if α is an odd function defined as above for non-negative arguments).

3. Reduction to a single equation

Integration by parts in equation (2) yields

$$u_x(t,x) = \alpha(\sigma(0,x))\beta(t) + [\beta(t-s)\alpha(\sigma(s,x))]_0^t + \int_0^t \alpha(\sigma(s,x))\beta'(t-s)\,\mathrm{d}s,$$

i.e.,

$$u_x(t,x) = \beta(0)\alpha(\sigma(t,x)) + \int_0^t \alpha(\sigma(s,x))\beta'(t-s)\,\mathrm{d}s. \tag{10}$$

We solve this equation for $\alpha(\sigma(t,x))$. It is easy to show that the solution is

$$\alpha(\sigma(t,x)) := \frac{1}{\beta(0)} u_x(t,x) - \frac{1}{\beta(0)} \int_0^t r(t-s) u_x(s,x)\,\mathrm{d}s,$$

where r satisfies

$$r + \frac{1}{\beta(0)} r * \beta' = \frac{1}{\beta(0)} \beta'. \tag{11}$$

This last equation is linear and scalar and its solution $r \in L^1_{loc}(\mathbb{R}_+)$ (resp. $L^1([0,T])$) exists by Theorem 2.3.1 in [3] (resp. its proof), whenever $\beta' \in L^1_{loc}(\mathbb{R}_+)$ (resp. $L^1([0,T])$). Then, if α is invertible, we can write

$$\sigma(t,x) := \alpha^{-1}\left(\frac{1}{\beta(0)}u_x(t,x) - \frac{1}{\beta(0)}\int_0^t r(t-s)u_x(s,x)\,\mathrm{d}s\right) \tag{12}$$

and insert this into equation (1)

$$\begin{aligned}\rho u_{tt} &= \frac{d}{dx}\alpha^{-1}\left(\frac{1}{\beta(0)}u_x(t,x) - \frac{1}{\beta(0)}\int_0^t r(t-s)u_x(s,x)\,\mathrm{d}s\right) + f \\ &= \frac{d}{dx}\alpha^{-1}(\tilde{u}_x/\beta(0)) + f,\end{aligned}$$

where

$$\tilde{u} := u_x(t,x) - \int_0^t r(t-s)u_x(s,x)\,\mathrm{d}s.$$

If α is differentiable with nonzero derivative, we can write

$$u_{tt}(t,x) = \psi(\tilde{u}_x(t,x))u_{xx}(t,x) - \psi(\tilde{u}_x(t,x))\int_0^t r(t-s)u_{xx}(s,x)\,\mathrm{d}s + f(t,x), \tag{13}$$

where

$$\psi(\tilde{u}) = \frac{1}{\rho\beta(0)\alpha'(\alpha^{-1}(\tilde{u}/\beta(0)))}.$$

It was mentioned above that such equations were studied by many authors. So, we will formulate the assumptions and existence results in the spirit of their results and show that the assumptions are satisfied if (L1)–(L5), resp. (G1)–(G5) hold.

Assume

(l1) there exists $U \subset \mathbb{R}$ open, $\psi \in C^2(U)$, $\psi(u) > 0$ for all $u \in U$.
(l2) $r \in C^2([0,T])$.
(l3) $u_0 \in H^3([0,1])$, $u_1 \in H^2([0,1])$ such that $u_0'(x) \in U$ for all $x \in [0,1]$.
(l4) $f \in \bigcap_{k=0}^1 C^{1-k}([0,T], H^k([0,1]))$, $f_{tt} \in L^2([0,T], L^2([0,1]))$.
(l5) $u_k(0) = u_k(1) = 0$ for $k = 0,1,2$, where u_2 is given by

$$u_2(x) := \psi(u_0'(x))u_0''(x) + f(0,x) \tag{14}$$

Theorem 3.1. *Let* (l1)–(l5) *hold. Then there exists* $T' \in (0,T]$ *such that* (13), (4), (5) *has a maximal solution* $u \in \bigcap_{k=0}^3 C^{3-k}([0,T'], H^k([0,1]))$. *Moreover, if* $u : [0,T')$ *is a maximal solution and*

$$\sup_{t\in[0,T')}\int_{[0,1]} u^2(t)+u_x^2(t)+u_{xx}^2(t)+u_{xxx}^2(t)+u_t^2(t)+u_{tx}^2(t)+u_{txx}^2(t)\,dx < +\infty, \tag{15}$$

then $T' = T$.

The global existence result is based on the fact that the kernel is strongly positive definite. We say that a function $a : \mathbb{R}_+ \to \mathbb{R}$ is positive definite if

$$Q(a,T,v) := \int_0^T v(t) \int_0^t a(t-s)v(s)\,\mathrm{d}s\,\mathrm{d}t \geq 0$$

for all $T > 0$ and $v \in L^2(0,T)$. It is strongly positive definite, if there exists $\varepsilon > 0$ such that $t \mapsto a_\varepsilon(t) := a(t) - \varepsilon e^{-t}$ is positive definite. Let us define $R(t) := \int_t^{+\infty} r(s)\,\mathrm{d}s$ and formulate the assumptions.

(g1) $\exists m > 0$ such that $\psi \in C^2(-m,m)$ with $\psi(0) > 0$.
(g2) r, $r' \in L^1(\mathbb{R}_+) \cap C^1(\mathbb{R}_+)$ and $R \in L^1(\mathbb{R}_+)$ is strongly positive definite and $R(0) < 1$.
(g3) $u_0 \in H^3([0,1])$ and $u_1 \in H^2([0,1])$.
(g4) $f \in C^1(\mathbb{R}_+, L^2([0,1])) \cap C(\mathbb{R}_+, H^1([0,1]))$, $f_{tt} \in L^2(\mathbb{R}_+, L^2([0,1]))$.
(g5) $u_k(0) = u_k(1) = 0$ for $k = 0,1,2$ where u_2 is defined by (14).

Theorem 3.2. *Let* (g1)–(g5) *hold. Then there exists* $\mu > 0$ *such that if* U_0, $F \leq \mu$, *then there exists a (unique) global solution* $u \in \bigcap_{k=0}^3 C^{3-k}(\mathbb{R}_+, H^k([0,1]))$ *to* (13), (4), (5). *Moreover,* u *satisfies*
(16)
$$u, u_x, u_t, u_{xx}, u_{tx}, u_{tt}, u_{xxx}, u_{txx}, u_{ttx}, u_{ttt} \in C_b(J, L^2([0,1])) \cap L^2(J, L^2([0,1]))$$

and

$$u, u_x, u_t, u_{xx}, u_{tx}, u_{tt} \to 0 \tag{17}$$

uniformly on $[0,1]$ *as* $t \to +\infty$.

Now, we show that Theorems 2.1 and 2.2 follow from Theorems 3.1 and 3.2. The latter theorems will be proved in the next sections.

We show that (l1)–(l5) follow from (L1)–(L5). From (L1) it follows that α is invertible on $\alpha(\mathbb{R})$ and from (L3) it follows that there exists a neighborhood U of $\{u_0'(x)/\beta(0); x \in [0,1]\}$ which lies in $\alpha(\mathbb{R})$. Then ψ is well defined on U and it is $C^2(U)$ ($\alpha' \neq 0$) and positive. It remains to show that $r \in C^2([0,T])$. Since $\beta \in C^3([0,T])$, we have $\beta' \in L^1([0,T])$, hence $r \in L^1([0,T])$ (by proof of Theorem 2.3.1 in [3]). It follows from $\beta' \in C^2([0,T])$ and properties of convolution that r satisfying (11) belongs to $C^2([0,T])$.

We show that (g1)–(g5) follow from (G1)–(G5). From (G1) there is a neighborhood of zero where α is invertible and $\alpha' > 0$. Then ψ is well defined, positive and C^2 on a neighborhood of zero. Since β is a Bernstein's function, β' is *completely monotone* (it means $(-1)^k\beta^{(k)} \geq 0$ on $\mathbb{R}_+$ for all $k = 1,2,3,\dots$). Therefore r is also completely monotone by Theorem 5.3.1 in [3]. Moreover, by the same Theorem, $r \in L^1$ and $R(0) = \int_0^{+\infty} r(t)\,\mathrm{d}t < 1$. Since R is positive and $R' = -r$, it follows that R is completely monotone and by Proposition 16.4.3. in [3], R is strongly positive definite (equivalently "of strong positive type"). We show that

$R \in L^1(\mathbb{R}_+)$. Compute convolution of (11) with constant 1 and multiply by $\beta(0)$. We obtain

$$\beta(0)(R(0) - R(t)) + r * (\beta(\cdot) - \beta(0))(t) = \beta(t) - \beta(0).$$

We can rewrite this equation as

$$\beta_\infty(R(0) - R(t)) + r * (\beta(\cdot) - \beta_\infty)(t) = \beta(t) - \beta(0),$$

i.e.

$$\beta_\infty R(t) = r * (\beta(\cdot) - \beta_\infty)(t) - (\beta(t) - \beta_\infty) - \beta_\infty + \beta(0) + \beta_\infty R(0). \tag{18}$$

Then $R \in L^1(\mathbb{R}_+)$ since $\beta - \beta_\infty = \beta_1 \in L^1(\mathbb{R}_+)$, $r \in L^1(\mathbb{R}_+)$ and $-\beta_\infty + \beta(0) + \beta_\infty R(0) = 0$ (this follows immediately by taking the limit for $t \to +\infty$ in (18)).

Clearly, the assertions of Theorem 3.1, resp. 3.2 imply the assertions of Theorem 2.1, resp. 2.2 with function σ defined by (12). So, to prove Theorems 2.1, 2.2 it is sufficient to prove Theorems 3.1, 3.2.

Let us mention that in the case introduced in [10] ($\beta(t) = J_0 + \sum_{i=1}^n J_i(1 - e^{-t/\tau_i})$) we can compute r explicitly. In fact, we have $\beta' = \sum J_i/\tau_i e^{-t/\tau_i}$. If we had $\tilde{\beta}'(t) = ce^{-\lambda t}$, then $\tilde{r}(t) = ce^{-\lambda(1+c/\lambda)t}$ (applying the Laplace transform to (11)). Hence, by linearity we have $r(t) = \sum_{i=1}^n \mu_i e^{-\lambda_i t}$ for appropriate positive constants μ_i, λ_i.

Concerning properties of

$$\psi = \frac{1}{\rho\beta(0)(\alpha^{-1})'}$$

in case of α's from the examples above, the first example (quadratic dependence) yields that $\gamma := \alpha^{-1}$ is concave with $\gamma'(0) > 0$ and $\gamma(z) \to 0$ for $z \to +\infty$. Then ψ is positive and $\psi(z) \to \infty$ for $z \to +\infty$. In the second example, $\alpha(\sigma)$ has a finite limit for $\sigma \to +\infty$, so ψ is defined (and positive) on a bounded interval $[-A, A]$ with zero limits at the endpoints. However, since we will prove just local existence for arbitrary data and global existence for small data, the shape of ψ will not be very important for us. These functions ψ satisfy the assumptions (l1), (g1).

4. Notation and preliminaries

Let us introduce some notation. In the following, we denote by $\|\cdot\|_q$ the norm on $L^q([0,1])$, by $\|\cdot\|_{S,p,q}$ the norm on $L^p([0,S], L^q([0,1]))$ (sometimes we will write shortly $\|\cdot\|_{p,q}$ if S is clear from the context). Partial derivatives of v will be denoted by v_t, v_x, v_{tx}, ..., or $\partial_t^2\partial_x^3 v$. If $u : [0,S] \times [0,1] \to \mathbb{R}$, we will often write $u(t)$ instead of $u(t,\cdot)$. We denote $Q_S := [0,S] \times [0,1]$.

P will be a generic function of one or more variables with values in $\mathbb{R}_+$ that maps bounded sets on bounded sets. By Z we will denote a generic function that maps bounded sets to bounded sets and is continuous in 0 and $Z(0) = 0$. $C > 0$

will be a generic constant. P, Z, and C may vary from expression to expression. Let us define the difference operator Δ_h by

$$(\Delta_h f)(t) := f(t+h) - f(t),$$

where f is any function defined on $\mathbb{R}_+$ with values in a Banach space.

Lemma 4.1 ([13, Lemma IV.9])**.** *Let $a \in L^1(\mathbb{R}_+)$ be strongly positive definite and $a', a'' \in L^1(\mathbb{R}_+)$. Then there exists $\kappa > 0$ such that for every $S > 0$ and every $w \in C([0,S],X)$ we have*

$$\int_0^t \|w(s)\|^2\, ds \le \kappa \|w(0)\|^2 + \kappa Q(a,t,w) + \kappa \liminf_{h\to 0+} \frac{1}{h^2} Q(a,t,\Delta_h w)$$

for all $t \in [0,S)$.

Lemma 4.2 ([13, Lemma III.7(i)])**.** *Let $F \in C^k(\mathbb{R})$. Then $F \circ w \in W^{k,2}([0,1])$ whenever $w \in W^{k,2}([0,1])$ and for every $K > 0$ there exists $C(K) > 0$ such that*

$$\sup\{\|F(w)\|_{W^{k,2}} :\ w \in W^{k,2}([0,1]),\ \|w\|_{W^{k,2}} \le K\} \le C(K).$$

Lemma 4.3. *Let $r \in W^{1,1} \cap W^{1,\infty}(\mathbb{R}_+)$ and set $I^v(t) := \int_0^t r(t-s)v(s)\, ds$ for any $v \in L^1_{loc}(\mathbb{R}_+, X)$ (where X is a Banach space). Then there exists $C > 0$ and a function $P : [0,+\infty) \to [0,+\infty)$ bounded on bounded sets such that*

(i) $\|v\|_{S,\infty,2} \le M$ *implies* $\|I^v\|_{S,\infty,2} \le S \cdot P(M)$,
(ii) $\|v_x\|_{S,\infty,2} \le M$ *implies* $\|I^v_x\|_{S,\infty,2} \le S \cdot P(M)$,
(iii) $\|v_t\|_{S,\infty,2} \le M$ *implies* $\|I^v_t\|_{S,\infty,2} \le C\|v(0)\|_2 + S \cdot P(M)$,
(iv) $\|v\|_{S,\infty,2} \le M$ *implies* $\|I^v_t\|_{S,2,2} \le S \cdot P(M)$,
(v) $\|v\|_{S,\infty,2}$, $\|v_t\|_{S,\infty,2} \le M$ *implies* $\|I^v_{tt}\|_{S,2,2} \le S \cdot P(M)$,
(vi) $\|v_x\|_{S,\infty,2} \le M$ *implies* $\|I^v\|_{S,1,\infty} \le S \cdot P(M)$.

Proof: The proof is straightforward using Hölder inequality. The only tricks are writing $v(t) = v(0) + \int_0^t v_t$ in (iii) and using Sobolev embedding in (vi), which gives $\|v\|_{S,\infty,\infty} \le CM$. In (v) one has to differentiate r once and v once in the integral term. □

Lemma 4.4. *Let k, $m \in \mathbb{N} \cup \{0\}$, $p \in [1,\infty]$, $r^{(l)} \in L^1(\mathbb{R}_+) \cap L^\infty(\mathbb{R}_+)$ for $l = 0,\dots,k$ and define the mapping $u \to \tilde{u}$ by*

$$\tilde{u}(t,x) := u(t,x) + \int_0^t r(t-s)u(s,x)\, ds.$$

Then

$$\|\partial_t^k \partial_x^m \tilde{u}(t)\|_p \le c \sum_{l=0}^k \|\partial_t^l \partial_x^m u(t)\|_p + c\|\partial_x^m u\|_{t,2,p},$$

$$\|\partial_t^k \partial_x^m \tilde{u}(t)\|_p \le c \sum_{l=0}^k \|\partial_t^l \partial_x^m u(t)\|_p + c\|\partial_x^m u(s)\|_{t,\infty,p},$$

$$\|\partial_t^k \partial_x^m \tilde{u}\|_{t,\infty,p} \leq c \sum_{l=0}^{k} \|\partial_t^l \partial_x^m u\|_{t,\infty,p}$$

and

$$\|\partial_t^k \partial_x^m \tilde{u}\|_{t,2,p} \leq c \sum_{l=0}^{k} \|\partial_t^l \partial_x^m u\|_{t,2,p} \tag{19}$$

with c depending on the norm of r and independent of

$$u \in W^{k,\infty}([0,S], W^{m,p}([0,1]))$$

(resp. $u \in W^{k,2}([0,S], W^{m,p}([0,1]))$ in case of the last inequality), $t \in [0,S]$ and $S > 0$.

Proof: The proof is straightforward. $\square$

5. Local existence

In this section we will prove Theorem 3.1. Let us first consider the linearized equation

$$u_{tt} = \psi(\tilde{w}_x) u_{xx} + \psi(\tilde{w}_x) \int_0^t r(t-s) w_{xx}(s) ds + f(t).$$

Let us denote

$$B_w := \psi(\tilde{w}_x) \quad \text{and} \quad g_w := \psi(\tilde{w}_x) \int_0^t r(t-s) w_{xx}(s)\, \mathrm{d}s + f(t).$$

Then, by Lemma III.3 in [13], the linear equation has a unique solution $u \in \bigcap_{k=0}^3 C^{3-k}([0,T], H^k([0,1]))$ for each $w \in \bigcap_{k=0}^3 C^{3-k}([0,T], H^k([0,1]))$. In the following two lemmas we show that the mapping $w \mapsto u$ is a contraction on $X(T', M)$ if T' is small and M large enough, where

$$\begin{aligned} X(T', M) = \{w \in \bigcap_{k=0}^{3} W^{3-k,\infty}([0,T'], H^k([0,1])),\ \partial_t^k w(0,\cdot) = u_k(\cdot), \\ k = 0,1,2, \quad \text{and } M(w) := \sum_{k=0}^{3} \sum_{l=0}^{3-k} \sup_{t \in [0,T']} \|\partial_x^k \partial_t^l w(t)\|_2^2 \leq M\}. \end{aligned}$$

So, the existence and uniqueness will be proved by the following two lemmas and the Banach contraction theorem. The moreover-part follows from standard continuation arguments.

In the following, we will often use $S^2 \leq S$ assuming that $S \leq T' \leq 1$.

Lemma 5.1. *The mapping $w \mapsto u$ maps $X(T', M)$ into itself if T' is small enough and M large enough.*

Proof: Let us drop the subscript w and write $B := B_w$ and $g := g_w$. Let $S \in (0,T)$ be arbitrary. From Lemma III.3 in [13] we have

$$E_m(u(t)) \le \Gamma(U,K,L_0)\exp(S\cdot\Lambda(U,K,L,S)), \text{ for all } t\in[0,S],$$

where

$$E_m(u(t)) := \sum_{k=0}^{3}\sum_{l=0}^{3-k}\|\partial_t^k\partial_x^l u(t)\|_2^2,$$

$$U := \max_{t\in[0,S]}\|g(t)\|_2^2 + \max_{t\in[0,S]}\|\partial_t g(t)\|_2^2 + \max_{t\in[0,S]}\|\partial_x g(t)\|_2^2 + \int_0^S \|\partial_t^2 g(t)\|_2\,\mathrm{d}t,$$

$$K := \max_{t\in[0,S]}\|B(t)\|_2^2 + \max_{t\in[0,S]}\|\partial_x B(t)\|_2^2,$$

$$L := \sum_{k=0}^{2}\sum_{l=0}^{2-k}\sup_{t\in[0,S]}\|\partial_t^k\partial_x^l B(t)\|_2^2,$$

$$L_0 := \sum_{k=0}^{1}\sum_{l=0}^{2-k}\|\partial_t^k\partial_x^l B(0)\|_2^2$$

and Γ, Λ are functions that are bounded on bounded sets. We estimate these terms by a function of $M(w)$.

Working on $[0,S]$, by Lemma 4.4 we have $M(\tilde w) \le CM(w)$. Now, from Lemma 4.2 it follows that $L \le P_L(M)$. L_0 is constant since the initial values are fixed. Further,

$$B(t,x) = \psi(u_0'(x)) + \int_0^t \frac{d}{dt}(\psi(w_x))(s,x)\,\mathrm{d}s,$$

$$B_x(t,x) = \psi(u_0'(x))_x + \int_0^t \frac{d}{dt}(\psi(w_x))_x(s,x)\,\mathrm{d}s,$$

which implies $K \le C_K + S\cdot P_K(M)$. We postpone the proof of $U \le C_U + S\cdot P_U(M)$ and believe for a while that it holds.

Take $M_1 > \max(C_K, C_U, L_0)$ and $M>0$ such that $\Gamma([0,M_1]^3) \subset [0, M/100]$. Take any $S\in(0,T)$. Then U, K, $L \le P(M)$ for all $w\in X(S,M)$. Hence, $\tilde N := \Lambda([0,P(M)]^3\times[0,S])$ is bounded and we can take $T'\in(0,S)$ such that $T'\cdot\max\tilde N < \ln 2$. Moreover, we can take T' so small, that $[U,K,L_0]\in[0,M_1]^3$ for all $w\in X(T',M)$. Hence, for all $w\in X(T',M)$ we have

$$E_m(u(t)) \le \Gamma(U,K,L_0)\exp(T'\cdot\Lambda(U,K,L,T')) \le \frac{M}{100}\exp(T'\cdot\max\tilde N) < M/50$$

for all $t\in[0,T']$, so $u\in X(T',M)$.

It remains to estimate U, where

$$U = \max_{t\in[0,S]} \|g(t)\|_2^2 + \max_{t\in[0,S]} \|g_x(t)\|_2^2 + \max_{t\in[0,S]} \|g_t(t)\|_2^2 + \int_0^S \|g_{tt}(t)\|_2 \,\mathrm{d}t.$$

Denoting $I := r * w_{xx}$ we can write

$$U = \max_{t\in[0,S]} \|BI(t)\|_2^2 + \max_{t\in[0,S]} \|B_x I + BI_x(t)\|_2^2 + \max_{t\in[0,S]} \|B_t I + BI_t(t)\|_2^2 + \int_0^S \|B_{tt} I + B_t I_t + BI_{tt}(t)\|_2 \,\mathrm{d}t.$$

Since

$$B, B_t, B_{tt}, B_x, B_{tx}, B_{xx} \text{ are bounded in } L^\infty(L^2) \text{ by } P(M) \tag{20}$$

we have by Sobolev embeddings

$$B, B_t, B_x \text{ are bounded in } L^\infty(L^\infty) \text{ by } P(M) \tag{21}$$

and as a consequence

$$B, B_t \text{ are bounded in } L^2(L^\infty) \text{ by } S \cdot P(M) \tag{22}$$

Moreover,

$$\|B\|_\infty \le \|B(0)\|_\infty + \int_0^t \|B_t\|_\infty \le \|B(0)\|_\infty + S \cdot \|B_t\|_\infty. \tag{23}$$

Hence,

$$B \text{ is bounded in } L^\infty(L^\infty) \text{ by } C + S \cdot P(M). \tag{24}$$

Now, (21) and Lemma 4.3(i), (ii) yield

$$BI, B_x I, BI_x, B_t I \text{ are bounded in } L^\infty(L^2) \text{ by } S \cdot P(M),$$

(24) and Lemma 4.3(iii) yield

$$BI_t \text{ is bounded in } L^\infty(L^2) \text{ by } C + S \cdot P(M),$$

(22) and Lemma 4.3(iv), (v) yield

$$B_t I_t, BI_{tt} \text{ are bounded in } L^1(L^2) \text{ by } C + S \cdot P(M),$$

and (20) and Lemma 4.3(vi) yield

$$B_{tt} I \text{ is bounded in } L^1(L^2) \text{ by } S \cdot P(M).$$

Hence, $U \le C_U + S \cdot P_U(M)$. □

Lemma 5.2. *The mapping $w \mapsto u$ is a contraction on $X(T', M)$ with respect to the metric*

$$d(w^1, w^2) := \left(\int_0^T \sum_{k=0}^{2} \|\partial_x^k \partial_t^{2-k} (w^1 - w^2)(s)\|_2^2 \, ds \right)^{1/2}$$

if T' is small enough.

Proof: In this proof we will write $\|\cdot\|_{\infty,2}$ instead of $\|\cdot\|_{T',\infty,2}$. The function d is a metric and $X(T', M)$ is complete with respect to this metric (see [13], p. 92–93). For w^1, $w^2 \in X(T', M)$ and their images u^1, u^2 we define $U := u^1 - u^2$, $W := w^1 - w^2$ and $\Psi := \psi(\tilde{w}_x^1) - \psi(\tilde{w}_x^2)$. Subtracting the equations for u^1 and u^2 we get

$$U_{tt} - \psi(\tilde{w}_x^1) U_{xx} = J \tag{25}$$

where

$$J := \Psi u_{xx}^2 + \Psi \int_0^t r(t-s) w_{xx}^1(s) \, \mathrm{d}s + \psi(\tilde{w}_x^2) \int_0^t r(t-s) W_{xx}(s) \, \mathrm{d}s.$$

Hence (differentiate (25) w.r.t. t and multiply U_{tt}),

$$U_{ttt} U_{tt} - \psi(\tilde{w}_x^1) U_{xxt} U_{tt} = (\psi(\tilde{w}_x^1))_t U_{xx} U_{tt} + J_t U_{tt}. \tag{26}$$

Integrating (26) over Q_t and integrating the second term by parts we obtain

$$\begin{aligned} &\frac{1}{2}(\|U_{tt}(t)\|_2^2 + \|U_{tx}(t)\|_2^2) \\ &\qquad \le \int_0^t \|(\psi(\tilde{w}_x^1))_t U_{xx}(s) + (\psi(\tilde{w}_x^1))_x U_{tx}(s)\|_2 \|U_{tt}(s)\|_2 \, \mathrm{d}s + \int_{Q_t} |J_t U_{tt}|, \end{aligned}$$

since $U_{tt}(0) = U_{xx}(0) = 0$. The first integral on the right is bounded by

$$P(M) \int_0^t \|U_{xx}(s)\|_2^2 + \|U_{tx}(s)\|_2^2 + \|U_{tt}(s)\|_2^2 \, \mathrm{d}s$$

since $(\psi(\tilde{w}_x^1))_x$, $(\psi(\tilde{w}_x^1))_t$ are pointwise bounded by $P(M)$. To estimate the second integral, we use $|\Psi(t,x)|$, $|\psi'(\tilde{w}_x^1(t,x)) - \psi'(\tilde{w}_x^2(t,x))| \le L \sup_{t\in[0,S]} |W_x(t,x)|$ (since $\tilde{w}_x \in L^\infty(L^\infty)$ and ψ, ψ' are locally Lipschitz continuous) and the fact that $\|u_{xx}^2\|_\infty$, $\|w_{xx}^1\|_\infty$, $\|u_{xxt}\|_{\infty,2}$ and other norms of derivatives of these functions are bounded by $P(M)$. In fact, since

$$\begin{aligned} J_t = \Psi_t u_{xx}^2 + \Psi u_{xxt}^2 + \Psi_t (r * w_{xx}^1) + \Psi r(0) w_{xx}^1 + \Psi (r' * w_{xx}^1) \\ + \psi(\tilde{w}_x^2)_t (r * W_{xx}) + \psi(\tilde{w}_x^2) r(0) W_{xx} + \psi(\tilde{w}_x^2)(r' * W_{xx}) \end{aligned}$$

we have

$$\begin{aligned}\int_{Q_t} |J_t||U_{tt}| &\le \int_0^t \|J_t(s)\|_2^2 + \|U_{tt}(s)\|_2^2 \,\mathrm{d}s \\ &\le T' \cdot P(M)(\|W_{xt}\|_{\infty,2}^2 + \|W_{xx}\|_{\infty,2}^2 + \|W_x\|_{\infty,2}^2) + \|U_{tt}^2\|_{2,2}^2.\end{aligned}$$

Putting the two estimates together (and using $\|W_x\|_{\infty,2} \le c\|W_{xx}\|_{\infty,2}$) we have

$$\begin{aligned}&\frac{1}{2}(\|U_{tt}(t)\|_2^2 + \|U_{tx}(t)\|_2^2) \\ &\le P(M)\int_0^t \|U_{tt}(s)\|_2^2 + \|U_{tx}(s)\|_2^2 + \|U_{xx}(s)\|_2^2 \,\mathrm{d}s \\ &\quad + T' \cdot P(M)(\|W_{xt}\|_{\infty,2}^2 + \|W_{xx}\|_{\infty,2}^2).\end{aligned} \tag{27}$$

Rewriting (25) as

$$U_{xx} = \frac{1}{\psi(\tilde{w}_x^1)}(J - U_{tt}),$$

we obtain

$$\|U_{xx}(t)\|_2^2 \le P(M)(\|J(t)\|_2^2 + \|U_{tt}(t)\|_2^2). \tag{28}$$

Since $\|u_{xx}^2(t)\|_\infty \le P(M)$, $\|(r * w_{xx}^1)(t)\|_\infty \le T'P(M)$, $\|\psi(\tilde{w}_x^2)(t)\|_\infty \le P(M)$ and

$$\|(r * W_{xx})(t)\|_2^2 \le T'\|r\|_\infty^2 \|W_{xx}\|_{\infty,2}^2$$

we have

$$\begin{aligned}\|J(t)\|_2^2 &\le P(M)\|W_x\|_{\infty,2}^2 + T' \cdot P(M)\|W_{xx}\|_{\infty,2}^2 \\ &\le T' \cdot P(M)(\|W_{xt}\|_{\infty,2}^2 + \|W_{xx}\|_{\infty,2}^2).\end{aligned} \tag{29}$$

Now, (27), (28) and (29) yield

$$\begin{aligned}&\|U_{tt}(t)\|_2^2 + \|U_{tx}(t)\|_2^2 + \|U_{xx}(t)\|_2^2 \\ &\le P(M)\int_0^t \|U_{tt}(s)\|_2^2 + \|U_{tx}(s)\|_2^2 + \|U_{xx}(s)\|_2^2 \,\mathrm{d}s \\ &\quad + T' \cdot P(M)(\|W_{xt}\|_{\infty,2}^2 + \|W_{xx}\|_{\infty,2}^2).\end{aligned}$$

By Gronwall's lemma we have

$$\|U_{tt}(t)\|_2^2 + \|U_{tx}(t)\|_2^2 + \|U_{xx}(t)\|_2^2 \le T' \cdot P(M)(\|W_{xt}\|_{\infty,2}^2 + \|W_{xx}(t)\|_{\infty,2}^2)e^{T'P(M)},$$

so the mapping $w \mapsto u$ is a contraction if T' is sufficiently small. $\square$

6. Proof of global existence

In this section we prove Theorem 3.2. Since the proof is standard, we will make it short and refer to proofs of Theorems IV.5 and IV.3 in [13] for more details.

Let us define $\varphi(z) := 1/\psi(z)$ and multiply (13) by φ. We obtain

$$\varphi(\tilde{u}_x(t,x))u_{tt}(t,x) = u_{xx}(t,x) - \int_0^t r(t-s)u_{xx}(s,x)\,\mathrm{d}s + h(\tilde{u}_x(t,x),t,x), \tag{30}$$

where $h(z,t,x) = \varphi(z)f(t,x)$. After integration by parts we obtain

$$\varphi(\tilde{u}_x(t,x))u_{tt}(t,x) = Au_{xx}(t,x) + \int_0^t R(t-s)u_{xxt}(s,x)ds + g(\tilde{u}(t,x),t,x), \tag{31}$$

where

$$g(z,t,x) = h(z,t,x) + R(t)u_{xx}(0,x) \quad \text{and} \quad A = 1 - R(0) > 0.$$

We define

$$E(t) := \max_{[0,t]} \int_0^1 \{u^2 + u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,s)\,\mathrm{d}x$$

$$+ \int_0^t \int_0^1 \{u^2 + u_x^2 + u_t^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2 + u_{xxx}^2 + u_{xxt}^2 + u_{xtt}^2 + u_{ttt}^2\}(x,s)\,\mathrm{d}x\,\mathrm{d}s$$

and

$$\nu(t) := \max_{[0,1]\times[0,t]} \{u_x^2 + u_{xx}^2 + u_{xt}^2 + u_{tt}^2\}^{1/2}(x,s).$$

Then by Sobolev embeddings we have

$$\nu(t) \le c\sqrt{E(t)}$$

and by Lemma 4.4

$$\begin{aligned} \|\tilde{u}_x\|_{t,\infty,\infty}, \|\tilde{u}_{xt}\|_{t,\infty,\infty}, \|\tilde{u}_{xx}\|_{t,\infty,\infty} &\le c\nu(t) \\ \|\tilde{u}_{\dots}\|_{t,\infty,2}, \|\tilde{u}_{\dots}\|_{t,2,2} &\le c\sqrt{E(t)}, \end{aligned}$$

where $\tilde{u}_{\dots}$ means any derivative (w.r.t. t and x) up to order three. In this section, Z will be a generic function (of one or two variables) that maps bounded sets to bounded sets and is continuous in 0 and $Z(0) = 0$. Function Z may vary from expression to expression.

As in [13] (see pages 141, 142 for the arguments), the estimate

$$E(t) \le Z(U_0,F)(1 + E(T)^{3/2}) + C(1 + \sqrt{E(T)})E^{3/2}(T),\ t \in [0,T] \tag{32}$$

implies global solution and (16) and (17) ($C > 0$ is a constant and $Z(U_0,F)$ is small for small values of U_0 and F). Uniqueness follows from Theorem 3.1, so it remains to show that (32) holds for every $T > 0$ and every solution defined on $[0, T]$.

We will do it in 7 steps, but first we prove a lemma. Let us remark that we can redefine φ and ψ on the complement of small neigborhood of zero without causing any harm (since $\tilde{u}_x(t,x)$ stays in a neighborhood of zero). So, we can assume $M_\varphi := \sup\{|\varphi(z)|, |\varphi'(z)|, |\varphi''(z)|; z \in \mathbb{R}\} < +\infty$, $m_\varphi := \inf\{\varphi(z); z \in \mathbb{R}\} > 0$ and $M_R := \sup\{|R(t)|, |R'(t)|, |R''(t)|; t \in \mathbb{R}_+\} < +\infty$. In the following lemma we will collect some estimates.

Lemma 6.1. *The following estimates hold.*

(i) $\|[\varphi(\tilde{u}_x(t,x))]_{tx}\|_{2,2} \le C(1 + \sqrt{E(t)})\sqrt{E(t)}$.
(ii) $\|g(\tilde{u}_x(t,x), t, x)\|_{2,2} \le Z(U_0, F)$.
(iii) $\|[g(\tilde{u}_x(t,x), t, x)]_t\|_{\infty,2} \le Z(U_0, F)(1 + \sqrt{E(t)})$.
(iv) $\|[g(\tilde{u}_x(t,x), t, x)]_{tt}\|_{2,2} \le Z(U_0, F)(1 + E(t))$.

Proof: (i) We have $[\varphi(\tilde{u}_x(t,x))]_{tx} = \varphi''\tilde{u}_{tx}\tilde{u}_{xx} + \varphi'\tilde{u}_{txx}$. Hence,

$$\begin{aligned}\|[\varphi(\tilde{u}_x(t,x))]_{tx}\|_{2,2} &\le M_\varphi(\|\tilde{u}_{tx}\|_{\infty,\infty}\|\tilde{u}_{xx}\|_{2,2} + \|\tilde{u}_{txx}\|_{2,2})\\ &\le C(\nu(t)\sqrt{E(t)} + \sqrt{E(t)}).\end{aligned}$$

(ii) is obvious. (iii) We have $[g(\tilde{u}_x(t,x), t, x)]_t = \varphi'\tilde{u}_{tx}f + \varphi f_t + R'u_0''$. Hence,

$$\|[g(\tilde{u}_x(t,x), t, x)]_t\|_{\infty,2} \le M_\varphi F(1 + \|\tilde{u}_{tx}\|_{\infty,2}) + M_R U_0 \le Z(U_0, F)(1 + \sqrt{E(t)}).$$

(iv) We have $[g(\tilde{u}_x(t,x), t, x)]_{tt} = \varphi''\tilde{u}_{tx}^2 f + \varphi'\tilde{u}_{ttx}f + 2\varphi'\tilde{u}_{tx}f_t + \varphi f_{tt} + R''u_0''$. Then

$$\begin{aligned}&\|[g(\tilde{u}_x(t,x), t, x)]_{tt}\|_{2,2}\\ &\qquad\le M_\varphi F(\|\tilde{u}_{tx}\|_{\infty,\infty}\|\tilde{u}_{tx}\|_{2,2} + \|\tilde{u}_{ttx}\|_{2,2} + 2\|\tilde{u}_{tx}\|_{2,2} + 1) + M_R U_0\end{aligned}$$

and using $2\sqrt{E(t)} \le 1 + E(t)$ we complete the proof. □

1st step. We will prove

$$\begin{aligned}(33)\quad &\|u_{tx}(T)\|_2^2 + \|u_{xx}(T)\|_2^2 + Q(R, T, u_{xxt})\\ &\qquad\le Z(U_0) + C\nu(T)E(T) + Z(F, U_0)\sqrt{E(T)}.\end{aligned}$$

Multiply (31) by u_{xxt} and integrate over Q_T. On the left-hand side we get (integration by parts, Dirichlet boundary conditions)

$$\begin{aligned}&\int_{Q_T} \varphi(\tilde{u}_x)u_{tt}u_{xxt} = -\int_{Q_T} [\varphi(\tilde{u}_x)u_{tt}]_x u_{xt}\\ &= -\int_{Q_T} [\varphi(\tilde{u}_x)]_x u_{tt}u_{xt} - \int_{Q_T} \varphi(\tilde{u}_x)u_{ttx}u_{xt}\end{aligned}$$

$$
\begin{aligned}
&= -\int_{Q_T} [\varphi(\tilde{u}_x)]_x u_{tt} u_{xt} - \int_{Q_T} \frac{d}{dt}\left[\frac{1}{2}\varphi(\tilde{u}_x) u_{tx}^2\right] + \int_{Q_T} \frac{1}{2}[\varphi(\tilde{u}_x)]_t u_{tx}^2 \\
&= -\int_0^1 \frac{1}{2}\varphi(\tilde{u}_x) u_{tx}^2(T) + \int_0^1 \frac{1}{2}\varphi(\tilde{u}_x) u_{tx}^2(0) \\
&\quad - \int_{Q_T} [\varphi(\tilde{u}_x)]_x u_{tt} u_{xt} + \int_{Q_T} \frac{1}{2}[\varphi(\tilde{u}_x)]_t u_{tx}^2.
\end{aligned}
$$

On the right-hand side we obtain

$$
\int_0^1 \frac{1}{2} A u_{xx}^2(T) - \int_0^1 \frac{1}{2} A u_{xx}^2(0) + Q(R, T, u_{xxt}) + \int_{Q_T} g u_{xxt}.
$$

Together we have

$$
\begin{aligned}
&\frac{1}{2} m_\varphi \int_0^1 u_{tx}^2(T) + \frac{1}{2} A \int_0^1 u_{xx}^2(T) + Q(R, T, u_{xxt}) \\
&\le \int_0^1 \frac{1}{2} M_\varphi |u_{tx}^2(0)| + \int_{Q_T} M_\varphi |\tilde{u}_{xx}||u_{tt}||u_{xt}| + \int_{Q_T} \frac{1}{2} M_\varphi |\tilde{u}_{xt}||u_{tx}|^2 \\
&\quad + \int_0^1 \frac{1}{2} A |u_{xx}(0)|^2 + \int_{Q_T} |g||u_{xxt}| \le Z(U_0) + C\nu(T) E(T) + Z(F, U_0)\sqrt{E(T)}.
\end{aligned}
$$

2nd step. We will prove

$$
\begin{aligned}
(34) \quad \|u_{ttx}(T)\|_2^2 + \|u_{txx}(T)\|_2^2 + \lim_{h\to 0} \frac{1}{h^2} Q(R, T, \Delta_h \psi(u_x)_{xt}) \\
\le Z(U_0, F)(1 + E(T)^{3/2}) + C(\nu(T) + \nu^2(T)) E(T)
\end{aligned}
$$

Apply Δ_h to (31), then multiply by $\Delta_h u_{txx}$, integrate over Q_T and denote the obtained equation by (E). We integrate the left-hand side of (E) by parts with respect to x and use the identity (easy computations)

$$
\Delta_h [fg]_x = \Delta_h [f_x g + f g_x] = \Delta_h f_x g(t+h) + f_x \Delta_h g + \Delta_h f g_x(t+h) + f \Delta_h g_x
$$

to obtain

$$
-\int_{Q_T} [\Delta_h \varphi(\tilde{u}_x)_x u_{tt}(t+h) + \varphi(\tilde{u}_x)_x \Delta_h u_{tt} + \Delta_h \varphi(\tilde{u}_x) u_{ttx}(t+h) + \varphi(\tilde{u}_x) \Delta_h u_{ttx}] \Delta_h u_{tx}.
$$

Since

$$
\varphi(\tilde{u}_x) \Delta_h u_{ttx} \Delta_h u_{tx} = \frac{1}{2}\left(\frac{d}{dt}[\varphi(\tilde{u}_x)(\Delta_h u_{tx})^2] - \varphi(\tilde{u}_x)_t (\Delta_h u_{tx})^2\right)
$$

we obtain

$$
-\frac{1}{2}\int_0^1 \varphi(\tilde{u}_x)(\Delta_h u_{tx})^2(T) + \frac{1}{2}\int_0^1 \varphi(\tilde{u}_x)(\Delta_h u_{tx})^2(0) + \frac{1}{2}\int_{Q_T} \varphi(\tilde{u}_x)_t (\Delta_h u_{tx})^2
$$

$$-\int_{Q_T}[\Delta_h\varphi(\tilde u_x)_x u_{tt}(t+h)+\varphi(\tilde u_x)_x\Delta_h u_{tt}+\Delta_h\varphi(\tilde u_x)u_{ttx}(t+h)]\Delta_h u_{tx}.$$

Dividing by h^2 and taking $\lim_{h\to0}$ we obtain that the left-hand side is

$$\begin{aligned}(35)\qquad &-\frac12\int_0^1\varphi(\tilde u_x)u_{ttx}^2(T)+\frac12\int_0^1\varphi(\tilde u_x)u_{ttx}^2(0)+\frac12\int_{Q_T}\varphi(\tilde u_x)_t u_{ttx}^2\\ &\quad-\int_{Q_T}[\varphi(\tilde u_x)_{tx}u_{tt}+\varphi(\tilde u_x)_x u_{ttt}+\varphi(\tilde u_x)_t u_{ttx}]u_{ttx}.\end{aligned}$$

We divide the right-hand side of (E) by h^2 and take $\lim_{h\to0}$. The first term on the right-hand side of (E) easily yields

$$(36)\qquad \frac12A\int_0^1|u_{txx}|^2(T)-\frac12A\int_0^1|u_{txx}|^2(0)$$

and the second term (the convolution term) due to

$$\Delta_h(f*g)(t)=[f*(\Delta_h g)](t)+\int_t^{t+h}f(s)g(t+h-s)\,\mathrm{d}s$$

and

$$\int_0^T f(t)\Delta_h g(t)\,\mathrm{d}t=\int_0^h f(t)g(t)\,\mathrm{d}t-\int_T^{T+h}f(t)g(t)\,\mathrm{d}t-\int_0^T\Delta f(t)g(t+h)\,\mathrm{d}t,$$

leads to

$$\begin{aligned}(37)\qquad &\int_0^1 u_{txx}^2(0)\,\mathrm{d}x-\int_0^1 R(T)u_{txx}(0)u_{txx}(T)\,\mathrm{d}t\\ &\quad+\nu\int_0^1(R(0)-R(T))u_{txx}(0)+\liminf_{h\to0}Q(R,T,\Delta_h u_{txx}).\end{aligned}$$

The last term on the right-hand side of (E) gives after integration by parts with respect to t

$$(38)\qquad -\int_{Q_T}g_{tt}u_{txx}.$$

Putting (36), (37), (38), (35) together we obtain

$$\begin{aligned}&\int_0^1 u_{txx}^2(T)+\int_0^1 u_{ttx}^2(T)+\liminf_{h\to0}Q(R,T,\Delta_h u_{txx})\\ &\le C\int_0^1|u_{txx}(0)||u_{txx}(T)|\,\mathrm{d}t+CU_0+C\int_{Q_T}|\tilde u_{xt}|u_{ttx}^2\\ &\quad+C\int_{Q_T}|\varphi(\tilde u_x)_{tx}||u_{tt}||u_{ttx}|+C\int_{Q_T}M_\varphi|\tilde u_{xx}||u_{ttt}||u_{ttx}|\end{aligned}$$

$$+ C\int_{Q_T} |\tilde{u}_{xt}|u_{ttx}^2 + C\int_{Q_T} |g_{tt}||u_{txx}|.$$

So, we have that

$$\begin{aligned}
&\int_0^1 u_{txx}^2(T) + \int_0^1 u_{ttx}^2(T) + \liminf_{h\to 0} Q(R,T,\Delta_h u_{txx}) \\
&\le Z(U_0)\sqrt{E} + Z(U_0) + C\nu E + C(\nu+\nu^2)E + C\nu E + C\nu E \\
&\quad + Z(U_0,F)(1+E)\sqrt{E} \\
&\le Z(U_0,F)(1+E^{3/2}) + C(\nu+\nu^2)E,
\end{aligned}$$

where E, ν means $E(T)$, $\nu(T)$ respectively.

3rd step. The estimates from 1st and 2nd steps and Lemma 4.1 give

$$\begin{aligned}
\|u_{ttx}(T)\|_2^2 + \|u_{txx}(T)\|_2^2 + \|u_{tx}(T)\|_2^2 + \|u_{xx}(T)\|_2^2 + \|u_{txx}\|_2^2 \\
\le Z(U_0,F)(1+E(T)^{3/2}) + C(\nu(T)+\nu^2(T))E(T)
\end{aligned} \tag{39}$$

4th step. We will prove

$$\begin{aligned}
\|u_{tt}(T)\|_2^2 + \|u_{ttt}(T)\|_2^2 + \|u_{ttt}\|_2^2 \\
\le Z(U_0,F)(1+E(T)^{3/2}) + C(\nu(T)+\nu^2(T))E(T)
\end{aligned} \tag{40}$$

Taking L^2-norms of (13) we have

$$\|u_{tt}(t)\|_2^2 \le C\|u_{xx}(t)\|_2^2 + C\|u_{xx}\|_{\infty,2}^2 + Z(U_0,F).$$

Differentiating the equation (30) with respect to t, moving the term with φ' to the right-hand side, squaring and integrating over $[0,1]$ w.r.t. x yields

$$\begin{aligned}
\|u_{ttt}(t)\|_2^2 \le C(&\|u_{txx}(t)\|_2^2 + \|r(t)u_{xx}(0)\|_2^2 + \|(r*u_{xxt})(t)\|_2^2 \\
&+ \|[h(\tilde{u}_x,t,x)]_t\|_2^2 + \|\varphi'(\tilde{u}_x(t))\|_\infty^2\|\tilde{u}_{tx}(t)\|_\infty^2\|u_{tt}(t)\|_2^2).
\end{aligned} \tag{41}$$

This is estimated by

$$C\|u_{xxt}(t)\|_2^2 + Z(U_0) + C\|u_{xxt}\|_{2,2}^2 + Z(F)(1+\sqrt{E(t)}) + C\nu(t)^2E(t).$$

Integrating (41) over $[0,T]$ we get

$$\|u_{ttt}\|_{2,2}^2 \le C\|u_{xxt}\|_{2,2}^2 + Z(U_0) + C\|r\|_1^2\|u_{xxt}\|_{2,2}^2 + Z(F)(1+\sqrt{E(t)} + C\nu(t)^2E(t)).$$

Since all the terms on the right-hand sides are estimated in (39), estimate (40) is proven.

5th step. We will prove

$$\|u_{ttx}\|_{2,2}^2 \le Z(U_0, F)(1 + E(T)^{3/2}) + C(\nu(T) + \nu^2(T))E(T). \tag{42}$$

Using difference operators one can derive

$$\int_0^T \int_0^1 u_{ttx}^2 = \int_0^T \int_0^1 u_{ttt}u_{txx} + \int_0^1 u_{txx}u_{tt}(0) - \int_0^1 u_{txx}u_{tt}(T).$$

Then apply estimates of $\|u_{ttt}(T)\|_2^2$ and $\|u_{txx}(T)\|_2^2$ proved above.

6th step. We will prove

$$\|u_{xxx}(T)\|_2^2 + \|u_{xxx}\|_{2,2}^2 \le Z(U_0, F)(1 + E(T)^{3/2}) + C(\nu(T) + \nu^2(T))E(T) \tag{43}$$

Rewrite (31) as

$$u_{xx} - \int_0^t r(t-s)u_{xx}(s)\,\mathrm{d}s = G \tag{44}$$

with

$$G(t,x) := \varphi(\tilde{u}_x(t,x))u_{tt}(t,x) - \varphi(\tilde{u}_x(t,x))f(t,x).$$

Solving this equation for u_{xx} we get

$$u_{xx} := G + k * G \quad \text{and} \quad u_{xxx} = G_x + k * G_x, \tag{45}$$

where k satisfies $k + k * r = r$. Then $k \in L^1(\mathbb{R}_+)$ by Theorem 5.3.1 in [3]. Taking $\|\cdot\|_{2,2}$ and $\|\cdot\|_{\infty,2}$ norms of the second equality in (45) and using

$$\|G_x(t)\|_2^2 \le C\nu^2(t)\|u_{tt}(t)\|_2^2 + C\|u_{ttx}(t)\|_2^2 + CZ(F)(1 + \|u_{xx}(t)\|_2^2)$$

and the estimates derived in the previous steps we get (43).

7th step. It remains to show the estimates for u_x, u_t, u in $L^\infty(L^2)$ and u_{tt}, u_{tx}, u_{xx}, u_x, u_t, u in $L^2(L^2)$. All these estimates except u_{xx} follow from Poincaré inequality, the estimate for u_{xx} can be easily derived from the first equality in (45), since we already have estimates for u_{tt}.

The estimate (32) is proved and the proof of global existence is complete.

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