On some classes of spaces with the D-property

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Abstract. We shall prove that under CH every regular meta-Lindelöf P-space which is locally D has the D-property. In addition, we shall prove that a regular submeta-Lindelöf P-space is D if it is locally D and has locally extent at most ω_1. Moreover, these results can be extended from the class of locally D-spaces to the wider class of $\mathbb{D}$-scattered spaces. Also, we shall give a direct proof (without using topological games) of the result shown by Peng [*On spaces which are D, linearly D and transitively D*, Topology Appl. **157** (2010), 378–384] which states that every weak $\overline{\theta}$-refinable $\mathbb{D}$-scattered space is D.

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1. Introduction

All spaces under consideration are T_1. Our terminology is standard. Terms not defined here can be found in [4].

A *neighbourhood assignment* for a space X is a function η from X to the topology of X such that $x \in \eta(x)$ for every $x \in X$. If Y is a subset of X, we write $\eta[Y] = \bigcup\{\eta(y) : y \in Y\}$. Then, we say that X is a *D-space*, if for every neighbourhood assignment η for X there is a closed discrete subset D of X such that $\eta[D] = X$.

The notion of a D-space was introduced by van Douwen and Pfeffer in [7], and it is a useful tool in the study of covering properties in topology. It is easy to see that compact spaces and also σ-compact spaces have property D. Also, it is known that (finite unions of) metric spaces and spaces satisfying certain generalized metric properties are D (see [1], [2], [5], [9] and [15]).

However, it is not known whether every regular Lindelöf space is D. Nevertheless, it has been recently shown in [19] that under $\diamondsuit$ there is a Hausdorff hereditarily Lindelöf space which is not D. Also, it is unknown whether the D-property is implied by paracompactness, subparacompactness or metacompactness. These problems have been the subject of much research. It was shown in [6] that on the class of generalized ordered spaces paracompactness is equivalent to the D-property, and it was proved in [9] that for subspaces of finite products of ordinals property D is equivalent to metacompactness. On the other hand, it is known

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that a submetacompact space is D if it is scattered or locally D (see [16]). And recently, it has been shown in [20] that under MA + $\neg$CH every submeta-Lindelöf space of cardinality $\leq \omega_1$ is D. However, it is not known whether it is consistent that every paracompact space of cardinality $\leq \omega_2$ is D. We refer the reader to the survey paper [11] for further results on D-spaces as well as examples and basic facts.

Recall that a topological space X is a *P-space*, if every G_δ-subset of X is open. It is known that regular P-spaces are important in the study of rings of continuous real-valued functions (see [10]). Since in a P-space every countable set of points is closed discrete, it is obvious that every Lindelöf P-space is D. In fact, we have that every Lindelöf P-space is Alster, a property that implies being productively Lindelöf and being D (see [3]). However, it is not known whether every regular meta-Lindelöf P-space is D. In connection with this question, we shall prove here that under CH every regular meta-Lindelöf P-space which is locally D has the D-property. In addition, we shall prove that a regular submeta-Lindelöf P-space is D if it is locally D and every point of the space has a neighbourhood whose extent is at most ω_1. Moreover, these results can be extended from the class of locally D-spaces to the wider class of $\mathbb{D}$-scattered spaces.

Also, by means of stationary winning strategies in topological games, it was shown by Peng in [16, Theorem 18] that every weak $\overline{\theta}$-refinable $\mathbb{D}$-scattered space is D (and so it is obtained as a consequence that every submetacompact $\mathbb{D}$-scattered space is D). Note that this theorem is best possible, because the construction carried out in [8] provides us an example of a locally compact scattered (hereditarily) weak θ-refinable space which is not D. Then, we shall give here an alternative proof of Peng's theorem, which we think is more direct and does not use topological games.

We shall use without explicit mention the well-known facts that "D-space", "P-space", "meta-Lindelöf" and "submeta-Lindelöf" are closed hereditary.

2. P-spaces

Recall that if $\mathcal{U}$ is an open cover of a space X, the *order* of a point $x \in X$ in $\mathcal{U}$ is $\operatorname{ord}(x,\mathcal{U}) = |\{U \in \mathcal{U} : x \in U\}|$. A space X is *meta-Lindelöf*, if for every open cover $\mathcal{U}$ of X there is an open refinement $\mathcal{P}$ such that every element of X has countable order in $\mathcal{P}$.

We say that a space X is *locally D*, if for every point x of X there is a neighbourhood U of x such that U with the relative topology of X is D.

First, our aim is to prove the following result.

Theorem 2.1. *Assume CH and that X is a regular meta-Lindelöf P-space such that X is locally D. Then X is D.*

Proof: Suppose that X is a regular meta-Lindelöf P-space such that X is locally D. To prove that X is D, assume that η is a neighbourhood assignment for X. As X is regular and locally D, we may assume that $\operatorname{Cl}_X(\eta(x))$ is D for every $x \in X$. Since X is meta-Lindelöf, there is a point-countable open refinement $\mathcal{P}$ of η.

Proceeding by induction on $0 < n < \omega$, it is not difficult to construct a closed discrete subset E_n of X such that $\{x \in X : \operatorname{ord}(x, \mathcal{P}) \leq n\} \subseteq \eta[E_n]$. However, if we put $V = \eta[\bigcup\{E_n : n < \omega\}]$, we cannot assume that $X \setminus V$ is a discrete union of D-subspaces, because if $x, y \in X \setminus V$ with $x \neq y$, it may happen that $\{U \in \mathcal{P} : x \in U\} \supsetneq \{U \in \mathcal{P} : y \in U\}$. Then, in order to construct the required closed discrete subset D for η, first we fix an enumeration $\{x_\alpha : \alpha < \gamma\}$ of X. For each $\alpha < \gamma$, we define $\mathcal{P}_\alpha = \{U \in \mathcal{P} : x_\alpha \in U\}$ and $Z_\alpha = \bigcap\{U \in \mathcal{P} : x_\alpha \in U\} \cap \bigcap\{X \setminus U : U \in \mathcal{P}, x_\alpha \notin U\}$. Our purpose is to define for each $\alpha < \gamma$ a closed discrete subset D_α of X such that the following conditions hold:

(1) $D_\alpha \cap \eta[\bigcup\{D_\beta : \beta < \alpha\}] = \emptyset$;
(2) for every $\delta < \gamma$ with $\mathcal{P}_\delta \subseteq \mathcal{P}_\alpha$, $Z_\delta \subseteq \eta[\bigcup\{D_\beta : \beta \leq \alpha\}]$.

So, assume that $\alpha < \gamma$ and D_β has been constructed for every $\beta < \alpha$. Let

$$U_\alpha = \eta[\bigcup\{D_\beta : \beta < \alpha\}].$$

If for every $\delta < \gamma$ with $\mathcal{P}_\delta \subseteq \mathcal{P}_\alpha$ we have that $Z_\delta \subseteq U_\alpha$, we define $D_\alpha = \emptyset$. So, suppose that there is a $\delta < \gamma$ with $\mathcal{P}_\delta \subseteq \mathcal{P}_\alpha$ and $Z_\delta \not\subseteq U_\alpha$. Since we are assuming CH, there is an enumeration $\{\mathcal{P}_{\delta_\xi} : \xi < \zeta\}$ with $\zeta \leq \omega_1$ of $\{\mathcal{P}_\delta : \delta < \gamma, \mathcal{P}_\delta \subseteq \mathcal{P}_\alpha, Z_\delta \not\subseteq U_\alpha\}$. Without loss of generality, we may assume that $\zeta = \omega_1$. Note that for $\delta < \gamma$ with $\mathcal{P}_\delta \subseteq \mathcal{P}_\alpha$, $\operatorname{Cl}_X(Z_\delta)$ is D because Z_δ is contained in some element of the cover $\{\eta(x) : x \in X\}$ and $\operatorname{Cl}_X(\eta(x))$ is D for every $x \in X$. Then, proceeding by transfinite induction on $\xi < \omega_1$, it is easy to construct a closed discrete subset $E_\xi^{(\alpha)}$ of X such that the following holds:

(a) $E_\xi^{(\alpha)} \cap (U_\alpha \cup \eta[\bigcup\{E_\mu^{(\alpha)} : \mu < \xi\}]) = \emptyset$;
(b) $E_\xi^{(\alpha)} \subseteq \operatorname{Cl}_X(Z_{\delta_\xi}) \subseteq U_\alpha \cup \eta[\bigcup\{E_\mu^{(\alpha)} : \mu \leq \xi\}]$.

We define $D_\alpha = \bigcup\{E_\xi^{(\alpha)} : \xi < \omega_1\}$. We need to show that D_α is closed discrete in X. For this, assume that $D_\alpha \neq \emptyset$ and $x \in X$. If $x \in \eta[\bigcup\{D_\beta : \beta < \alpha\}]$, we are done by condition (a). And if $x \in \eta[D_\alpha]$, we are done by condition (a) and the assumption that X is a P-space. So, assume that $x \notin \eta[\bigcup\{D_\beta : \beta \leq \alpha\}]$. Let $\delta < \gamma$ such that $x = x_\delta$. Clearly $Z_\delta \not\subseteq U_\alpha$, and hence $\mathcal{P}_\delta \not\subseteq \mathcal{P}_\alpha$ by condition (b). Let $U_x = \bigcap\{U \in \mathcal{P} : x \in U\}$. Then since $\mathcal{P}_\delta \not\subseteq \mathcal{P}_\alpha$, $U_x \cap \operatorname{Cl}_X(Z_{\delta_\xi}) = \emptyset$ for every $\xi < \omega_1$, hence $U_x \cap E_\xi^{(\alpha)} = \emptyset$ for every $\xi < \omega_1$, and so $U_x \cap D_\alpha = \emptyset$.

Finally, we put $D = \bigcup\{D_\alpha : \alpha < \gamma\}$. By condition (2), for every $\alpha < \gamma$, $x_\alpha \in \eta[\bigcup\{D_\beta : \beta \leq \alpha\}] \subseteq \eta[D]$, and hence $\eta[D] = X$. To check that D is closed discrete in X, consider a point $x \in X$. Let $\alpha < \gamma$ such that $x \in \eta[D_\alpha]$. Since D_α is closed discrete in X, there is a neighbourhood V of x such that $(V \setminus \{x\}) \cap D_\alpha = \emptyset$. Also, by the argument given above, $\bigcap\{U \in \mathcal{P} : x \in U\} \cap D_\beta = \emptyset$ for every $\beta < \alpha$. And by condition (1), we see that $\eta[D_\alpha] \cap D_\beta = \emptyset$ for every $\beta > \alpha$. Therefore, x is not an accumulation point of D. □

Recall that a space X is *submeta-Lindelöf*, if for every open cover $\mathcal{U}$ of X there is a sequence of open refinements $\{\mathcal{P}_n : n \in \omega\}$ such that every element of X has countable order in some $\mathcal{P}_n$.

For every space X, we denote by $e(X)$ the *extent* of X, i.e. the supremum of the cardinalities of the closed discrete subsets of X. If λ is an infinite cardinal, we say that a space X has *locally extent* $\leq \lambda$, if for every $x \in X$ there is a neighbourhood U of x with $e(U) \leq \lambda$.

Now, our purpose is to prove the following result.

Theorem 2.2. *Suppose that X is a regular submeta-Lindelöf P-space such that X is locally D and has locally extent $\leq \omega_1$. Then X is D.*

In fact, we can show the following more general result, whose proof is a refinement of the argument given in [17, Lemma 25]. In the proof, we will use ideas given in the construction carried out in [14, Section 3].

Lemma 2.1. *Assume that there is a sequence $\{\mathcal{P}_k : k \in \omega\}$ of open covers of a P-space X such that every point of X has order $\leq \omega_1$ in some $\mathcal{P}_k$ and in such a way that for every $U \in \bigcup\{\mathcal{P}_k : k \in \omega\}$, $Cl_X(U)$ is a D-subspace whose extent is at most ω_1. Then X is D.*

Proof: In order to show that X is D, suppose that η is a neighbourhood assignment for X. Put $X = \{x_\alpha : \alpha < \gamma\}$. For each $\alpha < \gamma$, we construct a closed discrete subset D_α of X such that the following conditions are satisfied:

(1) $x_\alpha \in \eta[\bigcup\{D_\beta : \beta \leq \alpha\}]$;
(2) $D_\alpha \cap \eta[\bigcup\{D_\beta : \beta < \alpha\}] = \emptyset$;
(3) if $x \in X \setminus \eta[\bigcup\{D_\beta : \beta \leq \alpha\}]$ and $V = \bigcap\{V_k : k \in \omega\}$ where $x \in V_k \in \mathcal{P}_k$ for each $k \in \omega$, then $V \cap \bigcup\{D_\beta : \beta \leq \alpha\} = \emptyset$.

Note that conditions (2) and (3) imply that $\bigcup\{D_\beta : \beta < \alpha\}$ is closed for limit α.

So, assume that $\alpha < \gamma$ and D_β has been constructed for every $\beta < \alpha$. Our aim is to construct D_α. Put $U_\alpha = \eta[\bigcup\{D_\beta : \beta < \alpha\}]$. If $x_\alpha \in U_\alpha$, we put $D_\alpha = \emptyset$. So, assume that $x_\alpha \notin U_\alpha$. Let $\{I_\xi : \xi < \omega_1\}$ be a partition of ω_1 into subsets of size ω_1. In order to construct D_α, we will define a sequence $\{E_\xi : \xi < \omega_1\}$ of closed discrete subsets of X, we will define a strictly increasing function $s : \omega_1 \to \omega_1$ and we will construct a sequence $\{\mathcal{V}_\xi : \xi < \omega_1\}$ of collections of open sets of $\bigcup\{\mathcal{P}_k : k \in \omega\}$ such that each $\mathcal{V}_\xi$ is enumerated by $\{V_i : i \in I_\xi, i > s(\xi)\}$ and in such a way that, for every $\xi < \omega_1$, the following conditions hold:

(a) $E_\xi \cap (U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \xi\}]) = \emptyset$;
(b) if $V_i \in \mathcal{V}_\zeta$ for $\zeta < \xi$ and $i \leq s(\xi)$, then $V_i \subseteq U_\alpha \cup \eta[\bigcup\{E_\mu : \mu \leq \xi\}]$;
(c) if $V \in \mathcal{V}_\xi$, then there is some $k < \omega$ such that $V \in \mathcal{P}_k$ and $V \cap \{x \in E_\xi : \operatorname{ord}(x, \mathcal{P}_k) \leq \omega_1\} \neq \emptyset$.

Let $F = \mathrm{Cl}_X(V)$ where $x_\alpha \in V \in \bigcup\{\mathcal{P}_k : k \in \omega\}$. Since F is a D-subspace whose extent is at most ω_1, there is a closed discrete subset E_0 of $F \setminus U_\alpha$ such that $F \setminus U_\alpha \subseteq \eta[E_0]$ and $|E_0| \leq \omega_1$. For every $k \in \omega$, let $E_0^{(k)} = \{x \in E_0 : \operatorname{ord}(x, \mathcal{P}_k) \leq \omega_1\}$. Clearly, $E_0 = \bigcup\{E_0^{(k)} : k \in \omega\}$. Now, for $k \in \omega$ let $\mathcal{V}_0^{(k)} =$

$\{V \in \mathcal{P}_k : V \cap E_0^{(k)} \neq \emptyset\}$. Let $\mathcal{V}_0 = \bigcup\{\mathcal{V}_0^{(k)} : k \in \omega\}$. Since every point of $E_0^{(k)}$ has order $\leq \omega_1$ in $\mathcal{P}_k$ and $|E_0^{(k)}| \leq \omega_1$, we deduce that $|\mathcal{V}_0^{(k)}| \leq \omega_1$ for every $k \in \omega$, and hence $|\mathcal{V}_0| \leq \omega_1$. We enumerate $\mathcal{V}_0$ by $\{V_i : i \in I_0 \setminus \{0\}\}$. We put $s(0) = 0$.

Now, assume that $0 < \xi < \omega_1$, E_μ and $\mathcal{V}_\mu$ have been constructed for $\mu < \xi$ and s has been defined for every $\mu < \xi$. If $V \subseteq U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \xi\}]$ for every $V \in \bigcup\{\mathcal{V}_\mu : \mu < \xi\}$, we define $D_\alpha = \bigcup\{E_\mu : \mu < \xi\}$. Otherwise, we consider the least ordinal $i \in \bigcup\{I_\mu : \mu < \xi\}$ such that $V_i \not\subseteq U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \xi\}]$. We put $V = V_i$ and $s(\xi) = i$. We show that $s(\mu) < s(\xi)$ for every $\mu < \xi$. First, assume that $\xi = \zeta + 1$ is a successor ordinal. Note that if $V \in \mathcal{V}_\zeta$, then $s(\zeta) < s(\xi)$ because $\mathcal{V}_\zeta$ is enumerated by $\{V_j : j \in I_\zeta, j > s(\zeta)\}$, and hence $s(\mu) < s(\xi)$ for every $\mu < \xi$ by the induction hypotheses. So, suppose that $V \in \mathcal{V}_\mu$ for some $\mu < \zeta$. Note that, by condition (b), $V_{s(\zeta)} \subseteq U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \xi\}]$, and hence $s(\zeta) \neq s(\xi)$. Then, by the election of V and the definition of $s(\zeta)$, it follows that $s(\zeta) < s(\xi)$. Now, assume that ξ is a limit ordinal. Let $\zeta < \xi$ such that $V = V_i \in \mathcal{V}_\zeta$. As $V \not\subseteq U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \xi\}]$, it follows that $i = s(\xi) > s(\mu)$ for $\zeta < \mu < \xi$, and so $s(\xi) > s(\mu)$ for every $\mu < \xi$ by the induction hypotheses. Now, let $F = \mathrm{Cl}_X(V)$. Since F is a D-subspace whose extent is at most ω_1, there is a closed discrete subset E_ξ of $F \setminus (U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \xi\}])$ such that $F \setminus (U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \xi\}]) \subseteq \eta[E_\xi]$ and $|E_\xi| \leq \omega_1$. For $k < \omega$, let $E_\xi^{(k)} = \{x \in E_\xi : \mathrm{ord}(x, \mathcal{P}_k) \leq \omega_1\}$ and $\mathcal{V}_\xi^{(k)} = \{V \in \mathcal{P}_k : V \cap E_\xi^{(k)} \neq \emptyset \text{ and } V \notin \mathcal{V}_\mu \text{ for } \mu < \xi\}$. Let $\mathcal{V}_\xi = \bigcup\{\mathcal{V}_\xi^{(k)} : k \in \omega\}$. Since every point of $E_\xi^{(k)}$ has order $\leq \omega_1$ in $\mathcal{P}_k$ and $|E_\xi^{(k)}| \leq \omega_1$, we deduce that $\mathcal{V}_\xi$ has size at most ω_1. We enumerate $\mathcal{V}_\xi$ by $\{V_i : i \in I_\xi, i > s(\xi)\}$.

Without loss of generality, we may assume that for every $\xi < \omega_1$ there is an element $V \in \bigcup\{\mathcal{V}_\mu : \mu < \xi\}$ such that $V \not\subseteq U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \xi\}]$. Otherwise, D_α is a countable union $\bigcup\{E_\mu : \mu < \xi\}$ for some $\xi < \omega_1$ and the argument is easier. We define $D_\alpha = \bigcup\{E_\xi : \xi < \omega_1\}$. We show that D_α is a closed discrete subset of X. First, assume that $x \in U_\alpha \cup \eta[D_\alpha]$. Then since X is a P-space, by condition (a), it follows that x is not an accumulation point of D_α. So, assume that $x \notin U_\alpha \cup \eta[D_\alpha]$. For each $k \in \omega$ let $W_k \in \mathcal{P}_k$ with $x \in W_k$. Let $W = \bigcap\{W_k : k \in \omega\}$. We show that $W \cap D_\alpha = \emptyset$. For this, assume on the contrary that $W \cap D_\alpha \neq \emptyset$. Let ξ be the least $\mu < \omega_1$ such that $W \cap E_\mu \neq \emptyset$. So, there is an $m \in \omega$ such that $W \cap E_\xi^{(m)} \neq \emptyset$. Therefore, $W_m \cap E_\xi^{(m)} \neq \emptyset$. It follows that $W_m \in \mathcal{V}_\zeta$ for some $\zeta \leq \xi$. Let $i \in I_\zeta$ such that $W_m = V_i$. Since s is strictly increasing, there is an ordinal $\delta < \omega_1$ such that $i < s(\delta)$. Note that $\zeta < \delta$, because if $\delta \leq \zeta$, then $s(\delta) \leq s(\zeta) < i$, which contradicts the election of δ. Now, by the definition of $s(\delta)$, we deduce that $V_i = W_m \subseteq U_\alpha \cup \eta[\bigcup\{E_\mu : \mu < \delta\}] \subseteq U_\alpha \cup \eta[D_\alpha]$, which contradicts our assumption on x.

So, we have shown that D_α is a closed discrete subset of X. Clearly, the family $\{D_\beta : \beta \leq \alpha\}$ satisfies conditions (1) and (2). And by using the induction hypotheses and the argument given in the preceding paragraph, we can see that it also satisfies condition (3).

Now, we can check that $D = \bigcup\{D_\alpha : \alpha < \gamma\}$ is as required. By condition (1), $\eta[D] = X$. To verify that D is closed discrete in X, consider a point $x \in X$. Let $\alpha < \gamma$ such that $x \in \eta[D_\alpha]$. Since D_α is closed discrete in X, by using conditions (2) and (3), it is easy to see that there is a neighbourhood U of x such that $(U \setminus \{x\}) \cap D = \emptyset$. □

Proof of Theorem 2.2: Since X is regular, locally D and has locally extent $\leq \omega_1$, for every $x \in X$ there is an open neighbourhood U_x of x such that $\mathrm{Cl}_X(U_x)$ is a D-subspace whose extent is at most ω_1. As X is submeta-Lindelöf, there is a sequence $\{\mathcal{P}_k : k \in \omega\}$ of open refinements of $\{U_x : x \in X\}$ such that every element of X has countable order in some $\mathcal{P}_k$. Clearly, the sequence $\{\mathcal{P}_k : k \in \omega\}$ satisfies the requirements of Lemma 2.1. □

Also, since every Lindelöf P-space is D and has countable extent, the following result is an immediate consequence of Theorem 2.2.

Corollary 2.1. *Every regular submeta-Lindelöf locally Lindelöf P-space is D.*

3. $\mathbb{D}$-scattered spaces

Recall that a space X is *scattered*, if every non-empty closed subspace of X has an isolated point. And a space X is $\mathbb{C}$-*scattered*, if every non-empty closed subspace Y of X has a point with a compact neighbourhood in Y. Clearly, the class of $\mathbb{C}$-scattered spaces contains every locally compact space and every scattered space. And we say that a space X is $\mathbb{D}$-*scattered*, if every non-empty closed subspace Y of X has a point with a D-neighbourhood in Y. Note that the class of $\mathbb{D}$-scattered spaces contains every locally D-space and every $\mathbb{C}$-scattered space. These notions of scattered spaces have been useful in the study of D-spaces (see [12], [13], [15], [16] and [17]).

We define the D-*derivative* X^* of a space X as the set of all $x \in X$ such that x does not have a D-neighbourhood in X. We can extend the Cantor-Bendixson process for topological spaces by using the notion of D-derivative. If X is a space and α is an ordinal, we define X^α as follows. $X^0 = X$; if $\alpha = \beta + 1$, $X^\alpha = (X^\beta)^*$; and if α is a limit, $X^\alpha = \bigcap\{X^\beta : \beta < \alpha\}$. It is easy to check that a space X is $\mathbb{D}$-scattered iff there is an ordinal α such that $X^\alpha = \emptyset$. Then, we define the *rank* of a $\mathbb{D}$-scattered space X by $\mathrm{rank}(X) =$ the least ordinal α such that $X^\alpha = \emptyset$. Note that if $X \neq \emptyset$, then X is locally D iff $\mathrm{rank}(X) = 1$.

In [17], Peng proved that if X is a regular meta-Lindelöf $\mathbb{D}$-scattered space with locally countable extent, then X is a D-space.

We want to remark that the notion of a locally D-space is strictly stronger than the notion of a $\mathbb{D}$-scattered space. For this, we construct a regular scattered space X which is not locally D. First, for each $n < \omega$ we consider a copy X_n of the ordinal ω_1 with the order topology in such a way that $X_n \cap X_m = \emptyset$ if $n \neq m$. The underlying set of X is $\{x^*\} \cup \bigcup\{X_n : n < \omega\}$ where $x^* \notin X_n$ for $n < \omega$. If $x \in X_n$, a basic neighbourhood of x in X is a basic neighbourhood of x in X_n.

And a basic neighbourhood of x^* in X is a set of the form $\{x^*\} \cup \bigcup\{X_n : n > m\}$ where $m < \omega$. It is easy to check that X is as required.

The following proposition will be useful to extend Theorems 2.1 and 2.2 from the class of locally D-spaces to the class of $\mathbb{D}$-scattered spaces.

Proposition 3.1. *Let K be a class of regular spaces that is closed hereditary such that every locally D-space in K is a D-space. Then, every $\mathbb{D}$-scattered space in K is a D-space.*

PROOF: Let X be a $\mathbb{D}$-scattered space in K. We proceed by transfinite induction on the rank α of X. If $\alpha = 0$, then $X = \emptyset$, and so we are done. Suppose that $\alpha = \beta + 1$ is a successor ordinal. Let η be a neighbourhood assignment for X. Put $Z = X^\beta$. We see that Z is a closed locally D-subspace of X, hence Z is a locally D-space of K, and so Z is a D-space. Let D be a closed discrete subset of Z such that $\bigcup\{\eta(x) \cap Z : x \in D\} = Z$. Let $Y = X \setminus \eta[D]$. Since Y is closed in X and $\operatorname{rank}(Y) < \alpha$, it follows that Y is D by the induction hypotheses. Let E be a closed discrete subset of Y such that $\bigcup\{\eta(x) \cap Y : x \in E\} = Y$. Clearly, $D \cup E$ is as required.

Now, assume that α is a limit ordinal. Since X is regular, for every $x \in X$ there is an open neighbourhood U_x of x such that $\mathrm{Cl}_X(U_x)$ is a subset of $X \setminus X^\beta$ for some $\beta < \alpha$. By the induction hypotheses, $\mathrm{Cl}_X(U_x)$ is D for every $x \in X$. It follows that X is a locally D-space of K, and hence X is D. □

Corollary 3.1. *Assume CH and that X is a regular meta-Lindelöf P-space such that X is $\mathbb{D}$-scattered. Then X is D.*

Corollary 3.2. *Assume that X is a regular submeta-Lindelöf P-space such that X is $\mathbb{D}$-scattered and has locally extent $\leq \omega_1$. Then X is D.*

We do not know whether Proposition 3.1 can be extended to classes of T_1 spaces. Note that in the proof we carried out regularity is used in an essential way in the limit case.

A space X is *weak $\overline{\theta}$-refinable*, if for every open cover $\mathcal{U}$ of X there is an open refinement $\bigcup\{\mathcal{P}_k : k \in \omega\}$ such that the following two conditions hold:

(1) $\{\bigcup \mathcal{P}_k : k \in \omega\}$ is a point-finite open cover of X;
(2) for every $x \in X$ there is a $k \in \omega$ such that $0 < \operatorname{ord}(x, \mathcal{P}_k) < \omega$.

It is known that every submetacompact space is weak $\overline{\theta}$-refinable (see [18]). In [16, Theorem 18], it was shown by means of stationary winning strategies in topological games that every weak $\overline{\theta}$-refinable $\mathbb{D}$-scattered space is D. This theorem has been used in [17, Sections 2 and 4] to obtain further results on D-spaces. Then, our aim is to show without using topological games (an extension of) Peng's theorem proved in [16].

We say that X is a *generalized weak $\overline{\theta}$-refinable* space, if for every open cover $\mathcal{U}$ of X there are an infinite cardinal λ and an open refinement $\bigcup\{\mathcal{P}_\xi : \xi < \lambda\}$ such that the following two conditions hold:

(*) (1) $\{\bigcup \mathcal{P}_\xi : \xi < \lambda\}$ is a point-finite open cover of X;
(2) for every $x \in X$ there is a $\xi < \lambda$ such that $0 < \mathrm{ord}(x, \mathcal{P}_\xi) < \omega$.

Note that the class of generalized weak $\overline{\theta}$-refinable $\mathbb{D}$-scattered spaces is hereditary with respect to closed subspaces.

Theorem 3.1. *Every generalized weak $\overline{\theta}$-refinable $\mathbb{D}$-scattered space is D.*

PROOF: Let X be a generalized weak $\overline{\theta}$-refinable $\mathbb{D}$-scattered space. We will proceed by transfinite induction on the rank α of X. Note that since we are not assuming that X is regular, we cannot argue as in Proposition 3.1 in the limit case. Then, suppose that either $\alpha = 1$ or α is a limit. Let η be a neighbourhood assignment for X. For every $x \in X$, we take an open neighbourhood U_x of x as follows. If $\alpha = 1$, U_x is a subset of some D-neighbourhood of x. And if α is a limit, $U_x \subseteq X \setminus X^\beta$ for some $\beta < \alpha$. As X is a generalized weak $\overline{\theta}$-refinable space, there are an infinite cardinal λ and an open refinement $\bigcup\{\mathcal{P}_\xi : \xi < \lambda\}$ of $\{U_x : x \in X\}$ satisfying conditions (*)(1)–(2). Let $\mathcal{P} = \{\bigcup \mathcal{P}_\xi : \xi < \lambda\}$. For $0 < n < \omega$ let

$$H_n = \{x \in X : \mathrm{ord}(x, \mathcal{P}) \leq n\}.$$

Note that, by condition $(*)(1)$, $X = \bigcup\{H_n : 0 < n < \omega\}$. Proceeding by induction on $n \geq 1$ we construct a closed discrete subset D_n of X such that $D_n \cap \eta[D_1 \cup \cdots \cup D_{n-1}] = \emptyset$ and $H_n \subseteq \eta[D_1 \cup \cdots \cup D_n]$. Hence, it follows that $D = \bigcup\{D_n : 0 < n < \omega\}$ is as required.

So, assume that $n \geq 1$ and D_m has been constructed for $m < n$. Put

$$U = \eta[D_1 \cup \cdots \cup D_{n-1}].$$

Note that $H_n \setminus U = \bigcup \mathcal{H}_n$ where $\mathcal{H}_n$ is a closed discrete family of subsets of X. In order to define D_n, we will construct a closed discrete subset E_H for each $H \in \mathcal{H}_n$. So, fix $H \in \mathcal{H}_n$. Note that

$$H = \Big(\bigcup \mathcal{P}_{i_1} \cap \cdots \cap \bigcup \mathcal{P}_{i_n} \cap \bigcap \Big\{X \setminus \bigcup \mathcal{P}_i : i \notin \{i_1, \ldots, i_n\}\Big\}\Big) \setminus U$$

where $i_1, \ldots, i_n$ are pairwise different elements of λ. As $H_{n-1} \subseteq U$, we infer that H is closed in X. For $1 \leq m \leq n$ and $1 \leq l < \omega$, we define

$$F_{m,l} = \{x \in H : \mathrm{ord}(x, \mathcal{P}_{i_m}) \leq l\}.$$

Now, we fix a bijection $h : \omega \to \{1, \ldots, n\} \times (\omega \setminus \{0\})$. In order to define E_H, we construct by induction on $k \in \omega$ a closed discrete subset E_k of X such that $E_k \cap (U \cup \eta[E_0 \cup \cdots \cup E_{k-1}]) = \emptyset$ and in such a way that if $h(k) = (m, l)$ then $F_{m,l} \subseteq U \cup \eta[E_0 \cup \cdots \cup E_k]$.

So, assume that $k \geq 0$ and $E_0, \ldots, E_{k-1}$ have been constructed. Put $h(k) = (m, l)$. Let

$$V = U \cup \eta[E_0 \cup \cdots \cup E_{k-1}].$$

Without loss of generality, we may assume that $V \cap F_{m,j} = \emptyset$ for $1 \le j \le l$. In order to construct E_k we will define closed discrete subsets $B_1, \dots, B_l$ of X such that for $1 \le t \le l$, $B_t \cap (V \cup \eta[B_1 \cup \dots \cup B_{t-1}]) = \emptyset$ and $F_{m,t} \subseteq V \cup \eta[B_1 \cup \dots \cup B_t]$. Suppose that $1 \le t \le l$ and $B_1, \dots, B_{t-1}$ have been constructed. Let

$$V_t = V \, \cup \, \eta[B_1 \cup \dots \cup B_{t-1}].$$

Let $F^*_{m,t} = F_{m,t} \setminus V_t$. Since $H_{n-1} \cup \bigcup\{F_{m,r} : r < t\} \subseteq V_t$, it follows that $F^*_{m,t}$ is the union of a discrete family $\mathcal{F}^*_{m,t}$ of closed subsets of X. Moreover, as every element of $\mathcal{P}_{i_m}$ is contained in some element of the cover $\{U_x : x \in X\}$, by using the induction hypotheses if α is a limit, we deduce that every element of $\mathcal{F}^*_{m,t}$ is D. So, there is a closed discrete subset B_t of X such that $B_t \subseteq F^*_{m,t} \subseteq \eta[B_t]$, and hence $F_{m,t} \subseteq V \cup \eta[B_1 \cup \dots \cup B_t]$. Then, we define $E_k = B_1 \cup \dots \cup B_l$.

Now, we define $E_H = \bigcup\{E_k : k \in \omega\}$. Note that, by condition $(*)(2)$, $H \subseteq \eta[E_H]$. Then, we define $D_n = \bigcup\{E_H : H \in \mathcal{H}_n\}$. We see that $D_n \cap U = \emptyset$ and $H_n \subseteq U \cup \eta[D_n] = \eta[D_1 \cup \dots \cup D_n]$.

Finally, if $\alpha = \beta + 1$ is a successor ordinal, we can proceed as in the proof of Proposition 3.1. □

References

[1] Arhangel'skii A.V., *D-spaces and finite unions*, Proc. Amer. Math. Soc. **132** (2004), 2163–2170.

[2] Arhangel'skii A.V., Buzyakova R.Z., *Addition theorems and D-spaces*, Comment. Math. Univ. Carolin. **43** (2002), 653–663.

[3] Barr M., Kennison J.F., Raphael R., *On productively Lindelöf spaces*, Sci. Math. Jpn. **65** (2007), 319–332.

[4] Burke D.K., *Covering properties*, in Handbook of Set-Theoretic Topology, edited by K. Kunen and J.E. Vaughan, North-Holland, Elsevier, Amsterdam, 1984, pp. 347–422.

[5] Buzyakova R.Z, *On D-property of strong Σ-spaces*, Comment. Math. Univ. Carolin. **43** (2002), 493–495.

[6] van Douwen E.K., Lutzer D., *A note on paracompactness in generalized ordered spaces*, Proc. Amer. Math. Soc. **125** (1997), 1237–1245.

[7] van Douwen E.K., Pfeffer W.F., *Some properties of the Sorgenfrey line and related spaces*, Pacific J. Math. **81** (1979), 371–377.

[8] van Douwen E.K., Wicke H.H., *A real, weird topology on the reals*, Houston J. Math. **3** (1977), 141–152.

[9] Fleissner W.G, Stanley A.M., *D-spaces*, Topology Appl. **114** (2001), 261–271.

[10] Gillman L., Jerison M., *Rings of Continuous Functions*, Graduate Texts in Math., 43, Springer, Berlin-Heidelberg-New York, 1976.

[11] Gruenhage G., *A survey of D-spaces*, Contemporary Math. **533** (2011), 13–28.

[12] Martínez J. C., Soukup L., *The D-property in unions of scattered spaces*, Topology Appl. **156** (2009), 3086-3090.

[13] Martínez J.C., *On finite unions and finite products with the D-property*, Topology Appl. **158** (2011), 223–228.

[14] Mashburn J.D., *A note on irreducibility and weak covering properties*, Topology Proc. **9** (1984), 339–352.

[15] Peng L.-X., *On finite unions of certain D-spaces*, Topology Appl. **155** (2008), 522–526.

[16] Peng L.-X., *On spaces which are D, linearly D and transitively D*, Topology Appl. **157** (2010), 378–384.
[17] Peng L.-X., *The D-property which relates to certain covering properties*, Topology Appl. **159** (2012), 869–876.
[18] Smith J.C., *Properties of weak $\overline{\theta}$-refinable spaces*, Proc. Amer. Math. Soc. **53** (1975), 511–517.
[19] Soukup D.T., Szeptycki P.J., *A counterexample in the theory of D-spaces*, Topology Appl. **159** (2012), 2669–2678.
[20] Zhang H., Shi W.-X., *A note on D-spaces*, Topology Appl. **159** (2012), 248–252.

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