Hardy and Cowling-Price theorems for a Cherednik type operator on the real line

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Abstract. This paper is aimed to establish Hardy and Cowling-Price type theorems for the Fourier transform tied to a generalized Cherednik operator on the real line.

Keywords: differential-difference operator; generalized Fourier transform; Hardy and Cowling-Price theorems

Classification: 33C45, 43A15, 43A32, 44A15

1. Introduction

In his 1933 paper [8], Hardy obtained the following famous theorem:

Theorem 1.1. *Let $1 \le p,\ q \le \infty$ with at least one of them finite. Let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2} f \in L^p(\mathbb{R}) \quad \text{and} \quad e^{b\lambda^2}\mathcal{F}_u(f) \in L^q(\mathbb{R}), \tag{1}$$

for some positive constants a and b. Then

- *if $ab \ge 1/4$, we have $f = 0$ almost everywhere;*
- *if $ab < 1/4$, there are infinitely many nonzero functions satisfying* (1).

Above mentioned $\mathcal{F}_u$ stands for the ordinary Fourier transform on $\mathbb{R}$ given by

$$\mathcal{F}_u(f)(\lambda) = \int_{\mathbb{R}} f(x) e^{-i\lambda x}\, dx.$$

Later, Cowling and Price [4] obtained the following L^p version of Theorem 1.1:

Theorem 1.2. *Let f be a measurable function on $\mathbb{R}$ such that*

$$e^{ax^2} f \in L^\infty(\mathbb{R}) \quad \text{and} \quad e^{b\lambda^2}\mathcal{F}_u(f) \in L^\infty(\mathbb{R}), \tag{2}$$

for some positive constants a and b. Then

- *if $ab > 1/4$, we have $f = 0$ almost everywhere;*
- *if $ab = 1/4$, the function f is of the form $f(x) = c_0\, e^{-ax^2}$, $c_0 \in \mathbb{C}$;*
- *if $ab < 1/4$, there are infinitely many nonzero functions satisfying* (2).

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Many generalizations of Theorems 1.1 and 1.2 to new contexts have been discovered. For instance, these theorems have been obtained in [2] for semi-simple Lie groups, in [5] for the motion group and in [15] for Chébli-Trimèche hypergroups.

The intention of this paper is to establish analogues of Theorems 1.1 and 1.2 when in (1) and (2) the usual Fourier transform $\mathcal{F}_u$ is substituted by a generalized Fourier transform $\mathcal{F}_\Lambda$ on $\mathbb{R}$ associated with the first-order singular differential-difference operator:

$$\Lambda f(x) = \frac{df}{dx} + \frac{A'(x)}{A(x)}\left(\frac{f(x)-f(-x)}{2}\right) - \rho f(-x),$$

where

$$A(x) = |x|^{2\alpha+1} B(x), \quad \alpha > -\frac{1}{2},$$

B being a positive C^∞ even function on $\mathbb{R}$, and $\rho > 0$. In addition we suppose that

(i) A is increasing on $[0,\infty[$ and $\lim_{x\to\infty} A(x) = \infty$;
(ii) A'/A is decreasing on $]0,\infty[$ and $\lim_{x\to\infty} A'(x)/A(x) = 2\rho$;
(iii) there exists a constant $\delta > 0$ such that the function $e^{\delta x}(A'(x)/A(x) - 2\rho)$ is bounded for large $x > 0$ together with its derivatives.

Notice that the differential-difference operator

$$\begin{aligned} D_{\alpha,\beta} f(x) = {} & \frac{df}{dx} + \left[(2\alpha+1)\coth x + (2\beta+1)\tanh x\right]\left(\frac{f(x)-f(-x)}{2}\right) \\ & - (\alpha+\beta+1) f(-x), \end{aligned}$$

which is referred to as the Jacobi-Cherednik operator (see [7]) is of the same type as Λ with

$$\begin{cases} A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}; \quad \alpha \geq \beta > -1/2; \\ \rho = \alpha+\beta+1; \quad \delta = 2. \end{cases}$$

The one-dimensional Cherednik operator (see [3]) is a particular case of $D_{\alpha,\beta}$. Such operators have been used by Heckmann and Opdam to develop a theory generalizing harmonic analysis on symmetric spaces (see [9], [12]). For recent important results in this direction we refer to [13], [16], [17].

In [11] the author has initiated a quite new commutative harmonic analysis on the real line related to the differential-difference operator Λ in which several analytic structures on $\mathbb{R}$ were generalized. The tools actually required for the discussion in the present paper, are essentially the Fourier transform and the Gaussian kernel tied to Λ.

2. Preliminaries

In [11] we have shown that for each $\lambda \in \mathbb{C}$, the differential-difference equation

$$\Lambda u = i\lambda u, \qquad u(0) = 1,$$

admits a unique C^∞ solution on $\mathbb{R}$, denoted Φ_λ and given by

$$\Phi_\lambda(x) = \begin{cases} \varphi_\lambda(x) + \frac{1}{i\lambda-\rho}\frac{d}{dx}\varphi_\lambda(x) & \text{if } \lambda \neq -i\rho, \\ 1 + \frac{2\rho}{A(x)}\int_0^x A(t)\,dt & \text{if } \lambda = -i\rho, \end{cases} \tag{3}$$

where φ_λ denotes the solution of the differential equation

$$\Delta u = -\left(\lambda^2+\rho^2\right)u, \quad u(0) = 1, \quad u'(0) = 1, \tag{4}$$

Δ being the second-order singular differential operator defined by

$$\Delta = \frac{1}{A(x)}\frac{d}{dx}\left(A(x)\frac{d}{dx}\right). \tag{5}$$

Moreover, $\Phi_\lambda(x)$ is entire in λ.

Remark 2.1. For $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$, $\alpha \geq \beta > -1/2$, the differential operator Δ reduces to the so-called Jacobi operator. The eigenfunction φ_λ is given by

$$\varphi_\lambda(x) = {}_2F_1\left(\frac{\alpha+\beta+1+i\lambda}{2}, \frac{\alpha+\beta+1-i\lambda}{2}; \alpha+1; -(\sinh x)^2\right)$$

where ${}_2F_1$ is the Gauss hypergeometric function [10].

Lemma 2.1. (i) *For every $x \in \mathbb{R}$,*

$$e^{-\rho|x|} \leq \varphi_0(x) \leq 1. \tag{6}$$

(ii) *There is a constant $C > 0$ such that*

$$\left|\frac{d^n}{d\lambda^n}\varphi_\lambda(x)\right| \leq C(1+|x|)\,|x|^n\,e^{(|\mathrm{Im}\lambda|-\rho)|x|} \tag{7}$$

for all $x \in \mathbb{R}$, $\lambda \in \mathbb{C}$ and $n = 0, 1, \ldots$.

Proof: Assertion (i) may be found in [14, p. 99]. Let us prove (ii). By [14, Equation (I.2)] we know that for $x \neq 0$,

$$\varphi_\lambda(x) = \int_0^{|x|} \mathcal{K}(x,y)\cos\lambda y\,dy,$$

where $\mathcal{K}(x,\cdot):\mathbb{R}\to\mathbb{R}$ is an even positive C^∞ function on $]-|x|,|x|[$, with support in $[-|x|,|x|]$. So using the derivation theorem under the integral sign we find

$$\begin{aligned}\left|\frac{d^n}{d\lambda^n}\varphi_\lambda(x)\right| &= \left|\int_0^{|x|}\mathcal{K}(x,y)\,y^n\cos(\lambda y+n\pi/2)\,dy\right|\\ &\le \int_0^{|x|}\mathcal{K}(x,y)\,y^n\,e^{|\mathrm{Im}\lambda||y|}\,dy\\ &\le |x|^n\,e^{|\mathrm{Im}\lambda||x|}\int_0^{|x|}\mathcal{K}(x,y)\,dy\\ &= |x|^n\,e^{|\mathrm{Im}\lambda||x|}\,\varphi_0(x).\end{aligned}$$

To conclude, recall from [14, p. 99] that there is a constant $C>0$ such that

$$\varphi_0(x)\le C(1+|x|)\,e^{-\rho|x|}$$

for all $x\in\mathbb{R}$. □

Analogous estimates for $\Phi_\lambda(x)$ are provided by the next statement.

Proposition 2.1. *There is a constant $C>0$ such that*

$$\left|\frac{d^n}{d\lambda^n}\Phi_\lambda(x)\right|\le C(1+|\lambda|)(1+|x|)^2\,|x|^n\,e^{(|\mathrm{Im}\lambda|-\rho)|x|}, \tag{8}$$

for all $x\in\mathbb{R}$, $\lambda\in\mathbb{C}$ and $n=0,1,\dots$.

Proof: By (3),

$$\frac{d^n}{d\lambda^n}\Phi_\lambda(x)=\frac{d^n}{d\lambda^n}\varphi_\lambda(x)+\frac{d^n}{d\lambda^n}\left(\frac{1}{i\lambda-\rho}\frac{d}{dx}\varphi_\lambda(x)\right).$$

As by (4),

$$\frac{d}{dx}\varphi_\lambda(x)=-\mathrm{sgn}(x)\frac{\lambda^2+\rho^2}{A(x)}\int_0^{|x|}\varphi_\lambda(t)A(t)\,dt, \tag{9}$$

we obtain

$$\frac{d^n}{d\lambda^n}\left(\frac{1}{i\lambda-\rho}\frac{d}{dx}\varphi_\lambda(x)\right)=\frac{\mathrm{sgn}(x)}{A(x)}\int_0^{|x|}\frac{d^n}{d\lambda^n}\big[(i\lambda+\rho)\varphi_\lambda(t)\big]A(t)\,dt.$$

The result follows now from (7) and Leibniz formula. □

Note 2.1. *For a function f on $\mathbb{R}$, write $f_e(x)=(f(x)+f(-x))/2$ and $f_o(x)=(f(x)-f(-x))/2$ respectively for its even and odd parts. We denote by*

- *$\mathcal{S}(\mathbb{R})$ the space of $\mathcal{C}^\infty$ functions f on $\mathbb{R}$ which are rapidly decreasing together with their derivatives, i.e., such that for all $m, n = 0, 1, \dots,$*

$$P_{m,n}(f) = \sup_{x\in\mathbb{R}} \left(1+x^2\right)^m \left|\frac{d^n}{dx^n} f(x)\right| < \infty.$$

The topology of $\mathcal{S}(\mathbb{R})$ is defined by the semi-norms $P_{m,n}$, $m, n = 0, 1, \dots$.
- *$\mathcal{S}_e(\mathbb{R})$ (resp. $\mathcal{S}_o(\mathbb{R})$) the subspace of $\mathcal{S}(\mathbb{R})$ consisting of even (resp. odd) functions.*
- *$\mathcal{S}^2(\mathbb{R})$ the space of $\mathcal{C}^\infty$ functions f on $\mathbb{R}$ such that for all $m, n = 0, 1, \dots,$*

$$Q_{m,n}(f) = \sup_{x\in\mathbb{R}} \left(1+x^2\right)^m \varphi_0(x)^{-1} \left|\frac{d^n}{dx^n} f(x)\right| < \infty.$$

The topology of $\mathcal{S}^2(\mathbb{R})$ is defined by the semi-norms $Q_{m,n}$, $m, n = 0, 1, \dots$.
- *$\mathcal{S}^2_e(\mathbb{R})$ (resp. $\mathcal{S}^2_o(\mathbb{R})$) the subspace of $\mathcal{S}^2(\mathbb{R})$ consisting of even (resp. odd) functions.*
- *$\mathcal{J}$ the map defined by $\mathcal{J}h(x) = \int_{-\infty}^x h(t)\,dt$, $x \in \mathbb{R}$.*

Remark 2.2. (i) By (6) we see that $\mathcal{S}^2(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.

(ii) It is easily checked that $\mathcal{S}^2(\mathbb{R})$ is invariant under the differential-difference operator Λ.

(iii) Due to our assumptions on the function A there is a positive constant k such that

$$A(x) \sim k\, e^{2\rho|x|} \quad \text{as} \quad |x| \to \infty. \tag{10}$$

The following technical lemma will be useful.

Lemma 2.2. *The map $\mathcal{J}$ is a topological isomorphism from $\mathcal{S}^2_o(\mathbb{R})$ onto $\mathcal{S}^2_e(\mathbb{R})$.*

Proof: It is sufficient to show that $\mathcal{J}$ maps continuously $\mathcal{S}^2_o(\mathbb{R})$ into $\mathcal{S}^2_e(\mathbb{R})$. Let $f \in \mathcal{S}^2_o(\mathbb{R})$. Clearly $\mathcal{J}f$ is a C^∞ even function on $\mathbb{R}$. For $n = 1, 2, \dots,$ $Q_{m,n}(\mathcal{J}f) = Q_{m,n-1}(f)$. Moreover, as by (9), φ_0 is decreasing on $[0, \infty[$, we get

$$\begin{aligned}(1+x^2)^m \varphi_0(x)^{-1}|\mathcal{J}f(x)| &\leq (1+x^2)^m\, \varphi_0(x)^{-1} \int_{|x|}^\infty |f(t)|\,dt \\ &\leq \int_{|x|}^\infty (1+t^2)^m\, \varphi_0(t)^{-1}|f(t)|\,dt \\ &\leq Q_{m+1,0}(f) \int_{|x|}^\infty \frac{dt}{(1+t^2)}.\end{aligned}$$

Hence $Q_{m,0}(\mathcal{J}f) \leq \frac{\pi}{2}\, Q_{m+1,0}(f)$. This ends the proof. $\square$

The generalized Fourier transform of a suitable function f on $\mathbb{R}$ is defined by

$$\mathcal{F}_\Lambda(f)(\lambda) = \int_{\mathbb{R}} f(x)\Phi_{-\lambda}(x)A(x)\,dx, \quad \lambda \in \mathbb{R}.$$

Remark 2.3. According to (7), (8) and (10), the generalized Fourier transform $\mathcal{F}_\Lambda$ is well defined on $\mathcal{S}^2(\mathbb{R})$.

Proposition 2.2. *For all $f \in \mathcal{S}^2(\mathbb{R})$,*

$$\mathcal{F}_\Lambda(f)(\lambda) = \mathcal{F}_\Delta(f_e)(\lambda) + (i\lambda - \rho)\,\mathcal{F}_\Delta\mathcal{J}(f_o)(\lambda), \tag{11}$$

where $\mathcal{F}_\Delta$ stands for the Fourier transform related to the differential operator Δ, defined on $\mathcal{S}_e^2(\mathbb{R})$ by

$$\mathcal{F}_\Delta(h)(\lambda) = \int_{\mathbb{R}} h(x)\varphi_\lambda(x)A(x)\,dx, \quad \lambda \in \mathbb{R}.$$

Proof: If $f \in \mathcal{S}_e^2(\mathbb{R})$, identity (11) is obvious. Assume $f \in \mathcal{S}_o^2(\mathbb{R})$. By using (3), (4), (5) and by integrating by parts we obtain

$$\begin{aligned}
\mathcal{F}_\Lambda(f)(\lambda) &= \frac{-1}{i\lambda+\rho}\int_{\mathbb{R}} f(x)\varphi_\lambda'(x)A(x)\,dx \\
&= \frac{1}{i\lambda+\rho}\int_{\mathbb{R}} \mathcal{J}f(x)(A(x)\varphi_\lambda'(x))'\,dx \\
&= \frac{1}{i\lambda+\rho}\int_{\mathbb{R}} \mathcal{J}f(x)\Delta\varphi_\lambda(x)A(x)\,dx \\
&= (i\lambda-\rho)\int_{\mathbb{R}} \mathcal{J}f(x)\varphi_\lambda(x)A(x)\,dx \\
&= (i\lambda-\rho)\,\mathcal{F}_\Delta(\mathcal{J}f)(\lambda),
\end{aligned}$$

which completes the proof. □

Remark 2.4. For $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$, $\alpha \geq \beta > -1/2$, the transform $\mathcal{F}_\Delta$ coincides with the Jacobi transform of order (α,β) (see [10]).

Theorem 2.1. *The generalized Fourier transform $\mathcal{F}_\Lambda$ is a topological isomorphism between $\mathcal{S}^2(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$. Moreover,*

$$\mathcal{F}_\Lambda^{-1}(g)(x) = \mathcal{F}_\Delta^{-1}(g_e)(x) + \left(\rho\,\mathrm{I} + \frac{d}{dx}\right)\mathcal{F}_\Delta^{-1}\left(\frac{g_o}{i\lambda}\right)(x)$$

for all $g \in \mathcal{S}(\mathbb{R})$.

Proof: By [14] we know that the transform $\mathcal{F}_\Delta$ is a topological isomorphism from $\mathcal{S}_e^2(\mathbb{R})$ onto $\mathcal{S}_e(\mathbb{R})$. Then the result follows from (11), Lemma 2.2 and the fact that the map $f \to \lambda f$ is a topological isomorphism from $\mathcal{S}_e(\mathbb{R})$ onto $\mathcal{S}_o(\mathbb{R})$. The identity above follows easily from (11). □

Note 2.2. *We denote by*

- *$\mathcal{D}_a(\mathbb{R})$, $a > 0$, the space of C^∞ functions on $\mathbb{R}$ supported in $[-a,a]$, provided with the topology of compact convergence for all derivatives.*
- *$\mathcal{D}(\mathbb{R}) = \bigcup_{a>0}\mathcal{D}_a(\mathbb{R})$ endowed with the inductive limit topology.*

- $\mathcal{D}_e(\mathbb{R})$ *(resp.* $\mathcal{D}_o(\mathbb{R})$*) the subspace of* $\mathcal{D}(\mathbb{R})$ *consisting of even (resp. odd) functions.*

- $\mathbf{H}_a$, $a > 0$, *the space of entire, rapidly decreasing functions of exponential type* a*; that is,* $f \in \mathbf{H}_a$ *if and only if* f *is entire on* $\mathbb{C}$ *and for all* $m = 0, 1, \ldots,$

$$p_m(f) = \sup_{\lambda \in \mathbb{C}} \left| (1+\lambda)^m f(\lambda) \mathrm{e}^{-a|\mathrm{Im}\lambda|} \right| < \infty.$$

  $\mathbf{H}_a$ *is equipped with the topology defined by the semi-norms* p_m, $m = 0, 1, \ldots$.
- $\mathbf{H} = \bigcup_{a>0} \mathbf{H}_a$*, equipped with the inductive limit topology.*
- $\mathcal{H}_a$, $a > 0$, *the space of entire, slowly increasing functions of exponential type* a*; that is,* $f \in \mathcal{H}_a$ *if and only if* f *is entire on* $\mathbb{C}$ *and there is* $m = 0, 1, \ldots$ *such that*

$$\sup_{\lambda \in \mathbb{C}} \left| (1+|\lambda|)^{-m} f(\lambda) \mathrm{e}^{-a|\mathrm{Im}\lambda|} \right| < \infty.$$

- $\mathcal{H} = \bigcup_{a>0} \mathcal{H}_a$.

Another standard result for the generalized Fourier transform $\mathcal{F}_\Lambda$ is as follows.

Theorem 2.2 (Paley-Wiener). (i) *The generalized Fourier transform* $\mathcal{F}_\Lambda$ *is a bijection from* $\mathcal{E}'(\mathbb{R})$ *onto* $\mathcal{H}$*. More precisely,* T *has its support in* $[-a, a]$ *if and only if* $\mathcal{F}_\Lambda(T) \in \mathcal{H}_a$.

(ii) *The generalized Fourier transform* $\mathcal{F}_\Lambda$ *is a topological isomorphism from* $\mathcal{D}(\mathbb{R})$ *onto* $\mathbf{H}$*. More precisely,* $f \in \mathcal{D}_a(\mathbb{R})$ *if and only if* $\mathcal{F}_\Lambda(f) \in \mathbf{H}_a$.

According to [11] the inverse generalized Fourier transform $\mathcal{F}_\Lambda^{-1}$ may also be expressed as follows.

Theorem 2.3. *For all* $g \in \mathcal{S}(\mathbb{R})$,

$$\mathcal{F}_\Lambda^{-1}(g)(x) = \int_{\mathbb{R}} g(\lambda) \Phi_{-\lambda}(-x)\, d\sigma(\lambda),$$

with

$$d\sigma(\lambda) = \left(\frac{\lambda - i\rho}{\lambda} \right) \frac{d\lambda}{2\pi |c(|\lambda|)|^2}\,, \tag{12}$$

where $c(s)$ *is a continuous function on* $]0, \infty[$ *such that*

$$\begin{aligned} c(s)^{-1} &\sim k_1\ s^{\alpha+\frac{1}{2}} \quad \text{as} \quad s \to \infty, \\ c(s)^{-1} &\sim k_2\ s, \quad \text{as} \quad s \to 0, \end{aligned} \tag{13}$$

for some $k_1, k_2 \in \mathbb{C}$.

Remark 2.5. (i) The tempered measure σ is called the spectral measure associated with the differential-difference operator Λ.

(ii) Let $g \in \mathcal{S}_e(\mathbb{R})$. By (3) and (12),

$$\begin{aligned}\int_{\mathbb{R}} g(\lambda)\Phi_{-\lambda}(-x)\,d\sigma(\lambda) &= \int_{\mathbb{R}} g(\lambda)\varphi_\lambda(x)\left(1-\frac{i\rho}{\lambda}\right)\frac{d\lambda}{2\pi|c(|\lambda|)|^2} \\ &\quad - i\int_{\mathbb{R}} g(\lambda)\frac{\varphi'_\lambda(x)}{\lambda}\frac{d\lambda}{2\pi|c(|\lambda|)|^2} \\ &= \int_{\mathbb{R}} g(\lambda)\varphi_\lambda(x)\frac{d\lambda}{2\pi|c(|\lambda|)|^2}\end{aligned}$$

By comparing Theorems 2.1 and 2.3 we deduce that

$$\mathcal{F}_\Lambda^{-1}(g)(x) = \int_{\mathbb{R}} g(\lambda)\varphi_\lambda(x)\frac{d\lambda}{2\pi|c(|\lambda|)|^2} = \mathcal{F}_\Delta^{-1}(g)(x).$$

This further shows that $\frac{d\lambda}{2\pi|c(|\lambda|)|^2}$ is the spectral measure tied to the differential operator Δ.

(iii) For $A(x) = (\sinh|x|)^{2\alpha+1}(\cosh x)^{2\beta+1}$, $\alpha \geq \beta > -1/2$, we have

$$c(s) = \frac{2^{\alpha+\beta+2-is}\,\Gamma(is)\,\Gamma(\alpha+1)}{\Gamma\big[(\alpha+\beta+1+is)/2\big]\,\Gamma\big[(\alpha-\beta+1+is)/2\big]}, \quad s>0.$$

The next statement provides a Parseval type formula for the generalized Fourier transform $\mathcal{F}_\Lambda$.

Theorem 2.4. *For all $f, g \in \mathcal{D}(\mathbb{R})$,*

$$\int_{\mathbb{R}} f(x)g(-x)A(x)\,dx = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)\,d\sigma(\lambda).$$

To prove Theorem 2.4 we need some facts about the transform $\mathcal{F}_\Delta$.

Lemma 2.3. (i) *For all $f \in \mathcal{D}_e(\mathbb{R})$,*

$$\mathcal{F}_\Delta(\Delta f)(\lambda) = -\left(\lambda^2+\rho^2\right)\mathcal{F}_\Delta(f)(\lambda).$$

(ii) *For all $f, g \in \mathcal{D}_e(\mathbb{R})$,*

$$\int_{\mathbb{R}} f(x)g(x)A(x)\,dx = \int_{\mathbb{R}} \mathcal{F}_\Delta(f)(\lambda)\mathcal{F}_\Delta(g)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2}.$$

Proof: (i) Using (4), (5) together with an integration by parts we have

$$\begin{aligned}\mathcal{F}_\Delta(\Delta f)(\lambda) &= \int_{\mathbb{R}} \Delta f(x)\varphi_\lambda(x)A(x)\,dx \\ &= \int_{\mathbb{R}} (A(x)f'(x))'\varphi_\lambda(x)\,dx\end{aligned}$$

$$\begin{aligned}
&= -\int_{\mathbb{R}} f'(x)\varphi'_\lambda(x)A(x)\,dx \\
&= \int_{\mathbb{R}} f(x)(A(x)\varphi'_\lambda(x))'\,dx \\
&= \int_{\mathbb{R}} f(x)\Delta\varphi_\lambda(x)A(x)\,dx \\
&= -\big(\lambda^2+\rho^2\big)\mathcal{F}_\Delta(f)(\lambda).
\end{aligned}$$

(ii) Notice that φ_λ is real whenever λ is real. So $\overline{\mathcal{F}_\Delta(g)(\lambda)} = \mathcal{F}_\Delta(\overline{g})(\lambda)$ for all $\lambda \in \mathbb{R}$. This when combined with a Parseval formula for the transform $\mathcal{F}_\Delta$ (see [14, Theorem II.4]) yields

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{F}_\Delta(f)(\lambda)\mathcal{F}_\Delta(g)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2} &= \int_{\mathbb{R}} \mathcal{F}_\Delta(f)(\lambda)\overline{\mathcal{F}_\Delta(\overline{g})(\lambda)}\frac{d\lambda}{2\pi|c(|\lambda|)|^2} \\
&= \int_{\mathbb{R}} f(x)g(x)A(x)\,dx,
\end{aligned}$$

which achieves the proof. □

Proof of Theorem 2.4: By (11),

$$\begin{aligned}
\int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)d\sigma(\lambda) = &\int_{\mathbb{R}} \mathcal{F}_\Delta(f_e)(\lambda)\mathcal{F}_\Delta(g_e)(\lambda)\,d\sigma(\lambda) \\
&+\int_{\mathbb{R}} (i\lambda-\rho)\mathcal{F}_\Delta(f_e)(\lambda)\mathcal{F}_\Delta\mathcal{J}(g_o)(\lambda)\,d\sigma(\lambda) \\
&+\int_{\mathbb{R}} (i\lambda-\rho)\mathcal{F}_\Delta\mathcal{J}(f_o)(\lambda)\mathcal{F}_\Delta(g_e)(\lambda)\,d\sigma(\lambda) \\
&+\int_{\mathbb{R}} (i\lambda-\rho)^2\,\mathcal{F}_\Delta\mathcal{J}(f_o)(\lambda)\mathcal{F}_\Delta\mathcal{J}(g_o)(\lambda)\,d\sigma(\lambda) \\
= &\,\kappa_1+\kappa_2+\kappa_3+\kappa_4.
\end{aligned}$$

By (12), we have

$$\begin{aligned}
\kappa_2 &= i\int_{\mathbb{R}} \frac{\lambda^2+\rho^2}{\lambda}\mathcal{F}_\Delta(f_e)(\lambda)\mathcal{F}_\Delta\mathcal{J}(g_o)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2} = 0; \\
\kappa_3 &= i\int_{\mathbb{R}} \frac{\lambda^2+\rho^2}{\lambda}\mathcal{F}_\Delta\mathcal{J}(f_o)(\lambda)\mathcal{F}_\Delta(g_e)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2} = 0.
\end{aligned}$$

Again by (12) and Lemma 2.3,

$$\begin{aligned}
\kappa_1 &= \int_{\mathbb{R}} \left(1-\frac{i\rho}{\lambda}\right)\mathcal{F}_\Delta(f_e)(\lambda)\mathcal{F}_\Delta(g_e)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2} \\
&= \int_{\mathbb{R}} \mathcal{F}_\Delta(f_e)(\lambda)\mathcal{F}_\Delta(g_e)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2}
\end{aligned}$$

$$= \int_{\mathbb{R}} f_e(x)g_e(x)A(x)\,dx;$$

$$\begin{aligned}
\kappa_4 &= -\int_{\mathbb{R}} \left(1+\frac{i\rho}{\lambda}\right)\left(\lambda^2+\rho^2\right)\mathcal{F}_\Delta\mathcal{J}(f_o)(\lambda)\mathcal{F}_\Delta\mathcal{J}(g_o)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2}\\
&= -\int_{\mathbb{R}} \left(\lambda^2+\rho^2\right)\mathcal{F}_\Delta\mathcal{J}(f_o)(\lambda)\mathcal{F}_\Delta\mathcal{J}(g_o)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2}\\
&= \int_{\mathbb{R}} \mathcal{F}_\Delta(\Delta\mathcal{J}f_o)(\lambda)\mathcal{F}_\Delta(\mathcal{J}g_o)(\lambda)\frac{d\lambda}{2\pi|c(|\lambda|)|^2}\\
&= \int_{\mathbb{R}} \Delta\mathcal{J}(f_o)(x)\mathcal{J}(g_o)(x)A(x)\,dx\\
&= \int_{\mathbb{R}} (Af_o)'(x)\mathcal{J}(g_o)(x)\,dx\\
&= -\int_{\mathbb{R}} f_o(x)g_o(x)A(x)\,dx.
\end{aligned}$$

Hence

$$\kappa_1+\kappa_4 = \int_{\mathbb{R}}\left[f_e(x)g_e(x)-f_o(x)g_o(x)\right]A(x)\,dx = \int_{\mathbb{R}} f(x)g(-x)A(x)\,dx.$$

This concludes the proof. □

Note 2.3. *We denote by*

- $L^p(\mathbb{R}, A(x)dx)$, $1 \le p \le \infty$, *the class of measurable functions* f *on* $\mathbb{R}$ *for which* $\|f\|_{p,A} < \infty$, *where*

$$\|f\|_{p,A} = \left(\int_{\mathbb{R}} |f(x)|^p A(x)\,dx\right)^{1/p}, \quad \textit{if} \;\; p<\infty,$$

  and $\|f\|_{\infty,A} = \|f\|_\infty$.
- $L^p(\mathbb{R}, |\sigma|)$, $1 \le p \le \infty$, *be the class of measurable functions* f *on* $\mathbb{R}$ *for which* $\|f\|_{p,|\sigma|} < \infty$, *where*

$$\|f\|_{p,|\sigma|} = \left(\int_{\mathbb{R}} |f(\lambda)|^p \,d|\sigma|(\lambda)\right)^{1/p}, \quad \textit{if} \;\; p<\infty,$$

  and $\|f\|_{\infty,|\sigma|} = \|f\|_\infty$.

Remark 2.6. By (8) there is a positive constant $k>0$ such that

$$|\mathcal{F}_\Lambda(f)(\lambda)| \le k\,(1+|\lambda|)\,\|f\|_{1,A}$$

for all $f \in L^1(\mathbb{R}, A(x)dx)$.

Lemma 2.4. *For all* $f \in L^1(\mathbb{R}, A(x)dx)$ *and* $g \in \mathcal{D}(\mathbb{R})$,

$$\int_{\mathbb{R}} f(x)g(-x)A(x)\,dx = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)\,d\sigma(\lambda).$$

Proof: Fix $g \in \mathcal{D}(\mathbb{R})$. For $f \in L^1(\mathbb{R}, A(x)dx)$ put

$$l_1(f) = \int_{\mathbb{R}} f(x)g(-x)A(x)\,dx$$

and

$$l_2(f) = \int_{\mathbb{R}} \mathcal{F}_\Lambda(f)(\lambda)\mathcal{F}_\Lambda(g)(\lambda)\,d\sigma(\lambda).$$

In view of Theorem 2.4, $l_1(f) = l_2(f)$ for each $f \in \mathcal{D}(\mathbb{R})$. Moreover,

$$|l_1(f)| \leq \|g\|_\infty \, \|f\|_{1,A}$$

and

$$|l_2(f)| \leq k \, \|f\|_{1,A} \int_{\mathbb{R}} |\mathcal{F}_\Lambda(g)(\lambda)|\,(1+|\lambda|)\,d|\sigma|(\lambda)$$

by virtue of Remark 2.6. This shows that the linear functionals l_1 and l_2 are bounded on $L^1(\mathbb{R}, A(x)dx)$. Therefore $l_1 = l_2$, and the lemma is proved. □

An immediate consequence of the lemma above is

Corollary 2.1. *The generalized Fourier transform* $\mathcal{F}_\Lambda$ *is injective on* $L^1(\mathbb{R}, A(x)dx)$.

For $t > 0$, the Gaussian kernel E_t associated with the differential-difference operator Λ is defined by

$$E_t(x) = \int_{\mathbb{R}} e^{-t(\lambda^2+\rho^2)}\Phi_{-\lambda}(-x)\,d\sigma(\lambda), \quad x \in \mathbb{R}. \tag{14}$$

This kernel enjoys the following properties.

Proposition 2.3. (i) $E_t \in \mathcal{S}^2(\mathbb{R})$ *and*

$$\mathcal{F}_\Lambda(E_t)(\lambda) = e^{-t\left(\lambda^2+\rho^2\right)}, \quad \text{for all } \lambda \in \mathbb{R}. \tag{15}$$

(ii) E_t *is even, positive and* $\int_{\mathbb{R}} E_t(x)A(x)\,dx = 1$.

(iii) *The function* $u(x,t) = E_t(x)$ *is* C^∞ *on* $\mathbb{R}\times\,]0,\infty[$ *and solves the partial differential equation*

$$\Delta_x u(x,t) = \frac{\partial}{\partial t}u(x,t),$$

where Δ *is given by* (5).

(iv) *There are two positive constants* $C_1(t)$ *and* $C_2(t)$ *such that*

$$C_1(t)\frac{e^{-\frac{x^2}{4t}}}{\sqrt{B(x)}} \leq E_t(x) \leq C_2(t)\frac{e^{-\frac{x^2}{4t}}}{\sqrt{B(x)}}. \tag{16}$$

(v) *Let* $p \in [0, \infty[$. *Then there exists a positive constant* $M(p,t)$ *such that*

$$(E_t(x))^p \leq M(p,t)E_{t/p}(x). \tag{17}$$

Proof: Assertion (i) follows directly from Theorems 2.1 and 2.3. A combination of (14) and Remark 2.5(ii) yields

$$E_t(x) = \int_0^\infty e^{-t\left(\lambda^2+\rho^2\right)}\,\varphi_\lambda(x)\frac{d\lambda}{\pi|c(\lambda)|^2}. \tag{18}$$

But according to [6], the right hand side of (18) satisfies (ii), (iii) and (iv). According to our assumptions on the function A, there is a constant $k > 0$ such that $B(x) \geq k$ for all $x \in \mathbb{R}$. The majorization (17) is then an easy consequence of (16). □

3. Hardy and Cowling-Price theorems

The following technical lemmas will greatly simplify the proofs of our main theorems.

Lemma 3.1 ([1]). *Let* g *be an entire function on* $\mathbb{C}$. *Suppose that*

$$|g(z)| \leq M(1+|z|)^m\, e^{a(Rez)^2} \quad \textit{for all} \quad z \in \mathbb{C}$$

and

$$|g(x)| \leq M \quad \textit{for all} \quad x \in \mathbb{R},$$

for some $a, M > 0$ *and* $m \in \mathbb{N}$. *Then* g *is constant on* $\mathbb{C}$.

Lemma 3.2 ([1]). *Let* $q \in [1, \infty[$ *and* g *be an entire function on* $\mathbb{C}$. *Suppose that*

$$\int_{\mathbb{R}} |g(x)|^q\, dx < \infty$$

and

$$|g(z)| \leq M(1+|z|)^m\, e^{a(Rez)^2} \quad \textit{for all} \quad z \in \mathbb{C},$$

for some $a, M > 0$ *and* $m \in \mathbb{N}$. *Then* $g = 0$ *on* $\mathbb{C}$.

Lemma 3.3. *Let* $q \in [1, \infty[$ *and* g *be an entire function on* $\mathbb{C}$. *Suppose that*

$$\|g\|_{q,|\sigma|} < \infty$$

and

$$|g(z)| \leq M(1+|z|)^m\, e^{a(Rez)^2} \quad \textit{for all} \quad z \in \mathbb{C},$$

for some $a, M > 0$ *and* $m \in \mathbb{N}$. *Then* $g = 0$ *on* $\mathbb{C}$.

Proof: By (12),

$$\begin{aligned}
\|g\|_{q,|\sigma|}^q &\geq \int_{|\lambda|\geq 1} |g(\lambda)|^q \, d|\sigma|(\lambda) \\
&= \int_{|\lambda|\geq 1} |g(\lambda)|^q \left| \frac{\lambda - i\rho}{\lambda} \right| \frac{d\lambda}{2\pi\,|c(|\lambda|)|^2} \\
&\geq \int_{|\lambda|\geq 1} |g(\lambda)|^q \frac{d\lambda}{2\pi\,|c(|\lambda|)|^2}.
\end{aligned}$$

According to (13), there is a constant $k > 0$ such that $|c(|\lambda|)|^{-2} \geq k|\lambda|^{2\alpha+1}$ for all $\lambda \in \mathbb{R}$ with $|\lambda| \geq 1$. Therefore

$$\|g\|_{q,|\sigma|}^q \geq \frac{k}{2\pi} \int_{|\lambda|\geq 1} |g(\lambda)|^q |\lambda|^{2\alpha+1} \, d\lambda \geq \frac{k}{2\pi} \int_{|\lambda|\geq 1} |g(\lambda)|^q \, d\lambda,$$

which shows that $\|g\|_q < \infty$. The result is now a direct consequence of Lemma 3.2.

□

Lemma 3.4. *Let $a, b > 0$, $d \geq 1$, $\gamma \in \mathbb{R}$ and*

$$g(y) = \int_0^\infty e^{-a(x-by)^2} (1+x)^d e^{\gamma x} \, dx, \quad y \geq 0.$$

Then there is a positive constant C such that

$$g(y) \leq C\,(1+y)^d\, e^{\gamma by} \quad \text{for all} \quad y \geq 0.$$

Proof: By the convexity of x^d we have

$$\begin{aligned}
g(y) &= e^{\gamma by} \int_{-by}^\infty e^{-az^2+\gamma z} (1+z+by)^d \, dz \\
&\leq e^{\gamma by} \int_{-by}^\infty e^{-az^2+|\gamma||z|} (1+|z|+by)^d \, dz \\
&\leq e^{\gamma by} \int_{-\infty}^\infty e^{-az^2+|\gamma||z|} (1+|z|+by)^d \, dz \\
&= 2e^{\gamma by} \int_0^\infty e^{-az^2+|\gamma|z} (1+z+by)^d \, dz \\
&\leq \text{const. } e^{\gamma by} \int_0^\infty e^{-az^2+|\gamma|z} \left(1+z^d+(by)^d\right) \, dz \\
&= \text{const. } e^{\gamma by} \left(\int_0^\infty e^{-az^2+|\gamma|z} \left(1+z^d\right) \, dz + (by)^d \int_0^\infty e^{-az^2+|\gamma|z} \, dz \right) \\
&\leq \text{const. } \left(1+y^d\right) e^{\gamma by} \\
&\leq \text{const. } (1+y)^d\, e^{\gamma by}
\end{aligned}$$

which ends the proof.

□

Lemma 3.5. *Let $1 \le q \le \infty$ and $a > 0$. Then there is a positive constant C such that for all $\lambda = \xi + i\eta \in \mathbb{R} + i\,\mathbb{R}$:*

(i) $\|E_{\frac{1}{4a}}\Phi_{-\lambda}\|_\infty \le C(1+|\lambda|)\, e^{\frac{\eta^2}{4a}}$;

(ii) $\|E_{\frac{1}{4a}}\Phi_{-\lambda}\|_{q,A} \le C(1+|\lambda|)^3\, e^{\frac{\eta^2}{4a}+\frac{(2-q)\rho|\eta|}{2aq}}$, *if $q < \infty$.*

Proof: As the function $1/\sqrt{B(x)}$ is bounded, it follows from (8) and (16) that

$$\begin{aligned}\left|E_{\frac{1}{4a}}(x)\Phi_{-\lambda}(x)\right| &\le \text{const.}\,(1+|\lambda|)(1+|x|)^2\, e^{-ax^2+(|\eta|-\rho)|x|}\\ &= \text{const.}\,(1+|\lambda|)(1+|x|)^2\, e^{\frac{\eta^2}{4a}}\, e^{-a\left(|x|-\frac{|\eta|}{2a}\right)^2-\rho|x|},\end{aligned}$$

which proves (i). For $q < \infty$ we have

$$\begin{aligned}\left\|E_{\frac{1}{4a}}\Phi_{-\lambda}\right\|_{q,A} &\le \text{const.}\,(1+|\lambda|)\, e^{\frac{\eta^2}{4a}}\left(\int_0^\infty e^{-aq\left(x-\frac{|\eta|}{2a}\right)^2}(1+x)^{2q}e^{(2-q)\rho x}\,dx\right)^{1/q}\\ &\le \text{const.}\,(1+|\lambda|)\,(1+|\eta|)^2\, e^{\frac{\eta^2}{4a}+\frac{(2-q)\rho|\eta|}{2aq}}\\ &\le \text{const.}\,(1+|\lambda|)^3\, e^{\frac{\eta^2}{4a}+\frac{(2-q)\rho|\eta|}{2aq}}\end{aligned}$$

by virtue of (10) and Lemma 3.4. □

Lemma 3.6. *Let $1 \le p, p' \le \infty$ such that $1/p + 1/p' = 1$. Let f be a measurable function on $\mathbb{R}$ such that $\|E_{\frac{1}{4a}}^{-1} f\|_{p,A} < \infty$ for some $a > 0$. Then the generalized Fourier transform of f is well defined and entire on $\mathbb{C}$. Moreover, there is a positive constant C such that for all $\lambda = \xi + i\eta \in \mathbb{R} + i\,\mathbb{R}$:*

(i) $|\mathcal{F}_\Lambda(f)(\lambda)| \le C(1+|\lambda|)\, e^{\frac{\eta^2}{4a}}$, *if $p = 1$;*

(ii) $|\mathcal{F}_\Lambda(f)(\lambda)| \le C(1+|\lambda|)^3\, e^{\frac{\eta^2}{4a}+\frac{(2-p')\rho|\eta|}{2ap'}}$, *if $p > 1$.*

Proof: The result follows easily by using Lemma 3.5, Hölder's inequality and the derivation theorem under the integral sign. □

We can now state our main results.

Theorem 3.1. *Let $1 \le p, q \le \infty$. Let f be a measurable function on $\mathbb{R}$ such that*

$$E_{\frac{1}{4a}}^{-1} f \in L^p(\mathbb{R}, A(x)dx) \tag{19}$$

and

$$e^{b\lambda^2}\mathcal{F}_\Lambda(f) \in L^q(\mathbb{R}, |\sigma|), \tag{20}$$

for some positive constants a and b. Then

- *if $ab > 1/4$, we have $f = 0$ almost everywhere;*
- *if $ab < 1/4$, for all $t \in]b, 1/(4a)[$, E_t satisfies (19)–(20).*

Proof: We divide the proof in two steps.

Step 1. $ab > 1/4$.

Let $t \in]1/(4a), b[$ and

$$g(\lambda) = e^{t\lambda^2}\,\mathcal{F}_\Lambda(f)(\lambda), \quad \lambda \in \mathbb{C}.$$

By Lemma 3.6, g is entire in $\mathbb{C}$, and there is $C > 0$ such that

$$|g(\lambda)| \leq C(1+|\lambda|)^3\, e^{t(Re\lambda)^2}$$

for all $\lambda \in \mathbb{C}$. Furthermore,

$$\|g\|_{q,|\sigma|} = \left\| e^{b\lambda^2}\mathcal{F}_\Lambda(f)\, e^{(t-b)\lambda^2} \right\|_{q,|\sigma|} \leq \left\| e^{b\lambda^2}\mathcal{F}_\Lambda(f) \right\|_{q,|\sigma|} < \infty.$$

(i) If $q < \infty$, it follows from Lemma 3.3 that $g(\lambda) = 0$ for all $\lambda \in \mathbb{C}$. That is, $\mathcal{F}_\Lambda(f)(\lambda) = 0$ for all $\lambda \in \mathbb{R}$. Therefore, $f = 0$ a.e. on $\mathbb{R}$, by virtue of Corollary 2.1.

(ii) If $q = \infty$, then by Lemma 3.1 there is a constant $K \in \mathbb{C}$ such that $g(\lambda) = K$ for all $\lambda \in \mathbb{C}$. That is, $\mathcal{F}_\Lambda(f)(\lambda) = Ke^{-t\lambda^2}$ for all $\lambda \in \mathbb{R}$. Hence, $f = Ke^{t\rho^2}E_t$ a.e. on $\mathbb{R}$. But due to assumption (19), this is impossible unless $K = 0$. Thus $f = 0$ a.e. on $\mathbb{R}$.

Step 2. $ab < 1/4$.

Let $t \in]b, 1/(4a)[$. By (16), there are two positive constants $C_1(a,t)$ and $C_2(a,t)$ such that

$$C_1(a,t)e^{-(\frac{1}{4t}-a)x^2} \leq E^{-1}_{\frac{1}{4a}}(x)E_t(x) \leq C_2(a,t)e^{-(\frac{1}{4t}-a)x^2},$$

for all $x \in \mathbb{R}$. This shows that $E^{-1}_{\frac{1}{4a}}E_t \in L^p(\mathbb{R}, A(x)dx)$. Moreover,

$$\left\| e^{b\lambda^2}\mathcal{F}_\Lambda(E_t) \right\|_{q,|\sigma|} = e^{-t\rho^2}\left\| e^{-(t-b)\lambda^2} \right\|_{q,|\sigma|} < \infty,$$

by virtue of (15) and the fact that σ is tempered. This completes the proof of Theorem 3.1. □

Theorem 3.2. *Let $1 \leq p \leq 2$ and $1 \leq q \leq \infty$. Let f be a measurable function on $\mathbb{R}$ satisfying (19) and (20) for some positive constants a and b. If $ab = 1/4$ then $f = 0$ almost everywhere.*

Proof: Let

$$g(\lambda) = e^{b\lambda^2}\,\mathcal{F}_\Lambda(f)(\lambda), \quad \lambda \in \mathbb{C}.$$

Let p' be the conjugate exponent of p. As by hypothesis $p' \geq 2$, we deduce from Lemma 3.6 that g is entire on $\mathbb{C}$, and there is $C > 0$ such that

$$|g(\lambda)| \leq C(1+|\lambda|)^3\, e^{b(Re\lambda)^2}$$

for all $\lambda \in \mathbb{C}$. The rest of the proof is now analogous to Step 1 in the proof of Theorem 3.1. □

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References

[1] Ben Farah S., Mokni K., *Uncertainty principle and $L^p - L^q$ sufficient pairs on noncompact real symmetric spaces*, C.R. Acad. Sci. Paris **336** (2003), 889–892.

[2] Ben Farah S., Mokni K., Trimèche K., *An $L^p - L^q$-version of Hardy's theorem for spherical Fourier transform on semi-simple Lie groups*, Int. J. Math. Math. Sci. **33** (2004), 1757–1769.

[3] Cherednik I., *A unification of Knizhnik-Zamolodchikov equations and Dunkl operators via affine Hecke algebras*, Invent. Math. **106** (1991), 411–432.

[4] Cowling M.G., Price J.F., *Generalisations of Heisenberg's inequality*, Lecture Notes in Mathematics, 992, Springer, Berlin, 1983, pp. 443–449.

[5] Eguchi M., Korzumi S., Kumahara K., *An L^p version of the Hardy theorem for the motion group*, J. Austral. Math. Soc. Ser. A **68** (2000), 55–67.

[6] Fitouhi A., *Heat polynomials for a singular differential operator on* $(0, \infty)$, J. Constr. Approx. **5** (1989), 241–270.

[7] Gallardo L., Trimèche K., *Positivity of the Jacobi-Cherednik intertwining operator and its dual*, Adv. Pure Appl. Math. **1** (2010), no. 2, 163–194.

[8] Hardy G.H., *A theorem concerning Fourier transform*, J. London Math. Soc. **8** (1933), 227–231.

[9] Heckman G.J., Schlichtkrull H., *Harmonic Analysis and Special Functions on Symmetric Spaces*, Academic Press, San Diego, CA, 1994.

[10] Koornwinder T.H., *A new proof of a Paley-Wiener type theorem for the Jacobi transform*, Ark. Mat. **13** (1975), 145–159.

[11] Mourou M.A., *Transmutation operators and Paley-Wiener theorem associated with a Cherednik type operator on the real line*, Anal. Appl. (Singap.) **8** (2010), no. 4, 387–408.

[12] Opdam E., *Dunkl Operators for Real and Complex Reflection Groups*, MSJ Memoirs, 8, Mathematical Society of Japan, Tokyo, 2000.

[13] Schapira B., *Contributions to the hypergeometric function theory of Heckman and Opdam: sharp estimates, Schwartz spaces, heat kernel*, Geom. Funct. Anal. **18** (2008), 222–250.

[14] Trimèche K., *Inversion of the Lions transmutation operators using generalized wavelets*, Appl. Comput. Harmon. Anal. **4** (1997), 97–112.

[15] Trimèche K., *Cowling-Price and Hardy theorems on Chébli-Trimèche hypergroups*, Glob. J. Pure Appl. Math. **1** (2005), no. 3, 286–305.

[16] Trimèche K., *The trigonometric Dunkl intertwining operator and its dual associated with the Cherednik operators and the Heckman-Opdam theory*, Adv. Pure Appl. Math. **1** (2010), no. 3, 293–323.

[17] Trimèche K., *Harmonic analysis associated with the Cherednik operators and the Heckman-Opdam theory*, Adv. Pure Appl. Math. **2** (2011), no. 1, 23–46.

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