Existence of nonnegative periodic solutions in neutral integro-differential equations with functional delay

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Abstract. The fixed point theorem of Krasnoselskii and the concept of large contractions are employed to show the existence of a periodic solution of a nonlinear integro-differential equation with variable delay

$$x'(t) = -\int_{t-\tau(t)}^{t} a(t,s)g(x(s))\,ds + \frac{d}{dt}Q(t, x(t-\tau(t))) + G(t, x(t), x(t-\tau(t))).$$

We transform this equation and then invert it to obtain a sum of two mappings one of which is completely continuous and the other is a large contraction. We choose suitable conditions for τ, g, a, Q and G to show that this sum of mappings fits into the framework of a modification of Krasnoselskii's theorem so that existence of nonnegative T-periodic solutions is concluded.

Keywords: Krasnoselskii's fixed points; periodic solution; large contraction

Classification: 34K20, 34K30, 34K40

1. Introduction

The use of ordinary and partial differential equations to model physical or biological systems and processes has a long history, dating to Lotka and Volterra. But all processes take time delays to complete. The delays can represent gestation times, incubation periods, or transport delays. In many cases, time delays can be substantial such as gestation and maturation or can represent little lags such as acceleration and deceleration in physical processes. Therefore, it becomes natural to include time delay terms into the differential equations that model population dynamics. The models that incorporate such delay times are referred to as delay differential equation models.

In the last fifty years, delay models are becoming more common, appearing in many branches of biological, economical and physical modelling (see [1]–[22], [24], [25]). This is due to their advantage of combining a simple, intuitive derivation with a wide variety of possible behavior regimes and to the fact that such models operate on an infinite dimensional space consisting of continuous functions that accommodate high dimensional dynamics (see [14], [20]–[21]).

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More recently investigators have given special attentions to the study of equations in which the delay occurs in the derivative of the state variable as well as in the independent variable, so called neutral differential equations. As known in Hale [20], Hale and Lunel [21] neutral delay differential equations appear as models of electrical networks which contain lossless transmission lines. Such networks arise, for example, in high speed computers where lossless transmission lines are used to interconnect switching circuits.

Existence, uniqueness, stability and positivity of solutions of functional differential equations are of great interest in mathematics and its applications to the modeling of various practical problems (see [1]–[14], [17]–[22], [24]–[25] and references therein). Positivity is one of the most common and most important characteristics of mathematical models. In the problem of economics, the positivity is quite important for processes that model interest rate dynamics on financial markets, because the interest must be positive. Also, in fluid flow problems, densities, pressures, and concentrations are always positive.

In the current paper we study the existence of periodic solutions for the nonlinear neutral integro-differential equation with variable delay

$$\begin{aligned} x'(t) &= -\int_{t-\tau(t)}^{t} a(t,s)g(x(s))\,ds \\ &\qquad + \frac{d}{dt}Q(t,x(t-\tau(t))) + G(t,x(t),x(t-\tau(t))), \\ x(t+T) &= x(t), \end{aligned} \tag{1.1}$$

where $T>0$ be fixed, the nonlinear terms Q and G are an L^1-Carathéodory functions and the function $a \in L^1[0,T]$ is bounded. Equation (1.1) has a long history and the simpler form of it was considered in 1928 by Volterra with a biological application in mind (see [12]). Obviously, the present problem is totally nonlinear with no nontrivial ode linear term and so the variation of parameters cannot be applied directly. Then, we have to transform this equation into a more tractable one suitable for the inversion. After the integration process we derive a fixed point mapping which we express as a sum of two mappings one is a completely continuous function and the other is a kind of contractive function called large contraction. We, prudently, choose hypotheses for the functions τ, g, a, Q and G to show that this sum of mappings fits very nicely into the framework of a modification of Krasnoselskii's theorem so that a theorem of existence of nonnegative T-periodic solutions for (1.1) is proved. For details on Krasnoselskii's theorem we refer the reader to [14] and [23]. In Section 2, we present the inversion of equation (1.1) and the modification of Krasnoselskii's fixed point theorem. We present our main results on existence of periodic solutions in Section 3 and we provide hypotheses for such solutions to be nonnegative in Section 4.

2. Preliminaries

The following definition is essential in our analysis.

Definition 1. A function $F : [0,T] \times \mathbb{R}^n \to \mathbb{R}$ is an L^1-Carathéodory function if it satisfies the following conditions.

(c1) For each $z \in \mathbb{R}^n$, the mapping $t \to F(t,z)$ is Lebesgue measurable.
(c2) For almost all $t \in [0,T]$, the mapping $z \to F(t,z)$ is continuous on $\mathbb{R}^n$.
(c3) For each $r > 0$, there exists $f_r \in L^1([0,T],\mathbb{R})$ such that, for almost all $t \in [0,T]$ and for all z with $|z| < r$, we have $|F(t,z)| \le f_r(t)$.

For $T > 0$ we let P_T to be the space of continuous functions x that are periodic in t, with period T. Then, $(P_T, \|\cdot\|)$ is a Banach space with the supremum norm

$$\|x\| = \sup_{t\in[0,T]} |x(t)| .$$

We will assume that the following conditions hold.

(k1) $a \in L^1_{Loc}(\mathbb{R},\mathbb{R})$ is positive and bounded, satisfies $a(t+T) = a(t)$ for all t and

$$1 - e^{-\int_{t-T}^{t} a(u,u)\,du} \equiv \frac{1}{\eta} \neq 0.$$

(k2) τ twice continuously differentiable and $\tau(t) \ge \tau^* > 0$ and $\tau(t+T) = \tau(t)$.
(k3) Q and G are an L^1-Carathéodory functions, and for all t

$$Q(t+T) = Q(t),\; G(t+T,x,y) = G(t,x,y) .$$

(k4) Function $g(x)$ is locally Lipschitz continuous in x. That is, there exists a positive constant K_1 so that if $|x| \le L$, then

$$|g(x) - g(y)| \le K_1 \|x - y\|. \tag{2.1}$$

Let the mapping F be defined by

$$F(x) = x - g(x), \tag{2.2}$$

where g is the function given in equation (1.1).

Now we are ready to transform equation (1.1) to a more tractable, but equivalent, one having the same properties which we then invert to define a fixed point mapping.

Lemma 1. *Equation* (1.1) *is equivalent to*

$$\begin{aligned}\frac{d}{dt}\{x(t) - Q(t, x(t-\tau(t)))\} &= B(t, t-\tau(t))(1-\tau'(t))g(x(t-\tau(t)))\\ &\quad + \frac{d}{dt}\int_{t-\tau(t)}^{t} B(t,s)g(x(s))\,ds\\ &\quad + G(t, x(t), x(t-\tau(t))),\end{aligned} \tag{2.3}$$

where

$$B(t,s) := \int_t^s a(u,s)\,du, \quad \textit{with} \;\; B(t,t-\tau(t)) = \int_t^{t-\tau(t)} a(u,t-\tau(t))\,du. \tag{2.4}$$

Furthermore,

$$B(t+T,s+T) = B(t,s) \;\; \textit{and} \;\; B(t+T,t+T-\tau(t+T)) = B(t,t-\tau(t)). \tag{2.5}$$

PROOF: Differentiating the integral term in (2.3), we obtain

$$\begin{aligned}&\frac{d}{dt}\int_{t-\tau(t)}^{t} B(t,s)g(x(s))\,ds\\ &= B(t,t)g\left(x\left(t\right)\right) - B(t,t-\tau(t))(1-\tau'(t))g(x(t-\tau(t)))\\ &\quad + \int_{t-\tau(t)}^{t} \frac{\partial}{\partial t}B(t,s)g(x(s))\,ds.\end{aligned}$$

Substituting this into (2.4), it follows that (2.4) is equivalent to (1.1) provided B satisfies the following conditions

$$B(t,t) = 0 \;\; \text{and} \;\; \frac{\partial}{\partial t}B(t,s) = -a(t,s). \tag{2.6}$$

Now (2.6) implies

$$B(t,s) = -\int_0^t a(u,s)\,du + \phi(s), \tag{2.7}$$

for some function ϕ, and $B(t,s)$ must satisfy

$$B(t,s) = -\int_0^t a(u,t)\,du + \phi(t) = 0.$$

Consequently,

$$\phi(t) = \int_0^t a(u,t)\,du.$$

Substituting this into (2.7), we obtain

$$\begin{aligned} B(t,s) &= -\int_0^t a(u,s)\,du + \int_0^s a(u,s)\,du\\ &= \int_t^s a(u,s)\,du.\end{aligned}$$

This definition of B satisfies (2.6). Consequently, (1.1) is, indeed, equivalent to (2.4).

Now, we must show that

$$B(t+T,s+T)=B(t,s) \quad \text{and} \quad B(t+T,t+T-\tau(t+T))=B(t,t-\tau(t)).$$

We have

$$B(t+T,s+T)=\int_{t+T}^{s+T} a(u,s+T)\,du.$$

By letting $u=y+T$, we see that

$$B(t+T,s+T)=\int_{t}^{s} a(y+T,s+T)\,dy=\int_{t}^{s} a(y,s)\,dy=B(t,s),$$

since $a(t+T,s+T)=a(t,s)$. Also, we have

$$B(t+T,t+T-\tau(t+T))=\int_{t+T}^{t+T-\tau(t+T)} a(u,t+T-\tau(t+T))\,du.$$

Replacing by $u=y+T$, we get

$$\begin{aligned} B(t+T,t+T-\tau(t+T)) &= \int_{t}^{t-\tau(t+T)} a(y+T,t+T-\tau(t+T))\,dy \\ &= \int_{t}^{t-\tau(t)} a(y,t-\tau(t))\,dy=B(t,t-\tau(t)), \end{aligned}$$

since $a(t+T,s+T)=a(t,s)$. □

Lemma 2. *Suppose that conditions* (k1), (k2), (k3) *and* (2.2) *hold. Then,* $x\in P_T$ *is a solution of equation* (1.1) *if and only if* $x\in P_T$ *satisfies*

$$\begin{aligned} x(t) &= Q\left(t,x\left(t-\tau(t)\right)\right)+\int_{t-\tau(t)}^{t}(B(t,u)+a(u,u))g(x(u))\,du \\ &\quad -\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}a(s,s)Q\left(s,x(s-\tau(s))\right)\,ds \\ &\quad -\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}a(s,s)\int_{s-\tau(s)}^{s}(a(u,u)+B(s,u))g(x(u))\,du\,ds \\ &\quad +\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}[a(s-\tau(s),s-\tau(s)) \\ &\quad +B(s,s-\tau(s))](1-\tau'(s))g(x(s-\tau(s)))\,ds \\ &\quad +\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}a(s,s)F(x(s))\,ds \\ &\quad +\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}G(s,x(s),x(s-\tau(s)))\,ds. \end{aligned} \tag{2.8}$$

Proof: Multiplying both sides of (2.4) by the factor $e^{\int_0^t a(u,u)\,du}$ and integrating from $t-T$ to t, we obtain

$$\begin{aligned}
&\int_{t-T}^{t}\left[\{x(s)-Q(s,x(s-\tau(s)))\}\,e^{\int_0^s a(u,u)\,du}\right]'\,ds\\
&\quad=\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}a(s,s)x(s)\,ds\\
&\quad\quad-\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}a(s,s)Q\left(s,x\left(s-\tau\left(s\right)\right)\right)\,ds\\
&\quad\quad+\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}\frac{d}{ds}\int_{s-\tau(s)}^{s}[a(u,u)+B(s,u)]g(x(u))\,du\,ds\\
&\quad\quad-\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}\frac{d}{ds}\int_{s-\tau(s)}^{s}a(u,u)g(x(u))\,du\\
&\quad\quad+\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}B(s,s-\tau(s))(1-\tau'(s))g(x(s-\tau(s)))\,ds\\
&\quad\quad+\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}G(s,x(s),x(s-\tau(s)))\,ds.
\end{aligned}\tag{2.9}$$

An integration by parts gives

$$\begin{aligned}
&\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}\frac{d}{ds}\int_{s-\tau(s)}^{s}[a(u,u)+B(s,u)]g(x(u))\,du\,ds\\
&=\left(e^{\int_0^s a(u,u)\,du}\int_{s-\tau(s)}^{s}[a(u,u)+B(s,u)]g(x(u))\,du\right)_{t-T}^{t}\\
&\quad-\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}a(s,s)\int_{s-\tau(s)}^{s}[a(u,u)+B(s,u)]g(x(u))\,du\,ds\\
&=\left(1-e^{-\int_{t-T}^{t} a(u,u)\,du}\right)e^{\int_0^t a(u,u)\,du}\int_{t-\tau(t)}^{t}[a(u,u)+B(t,u)]g(x(u))\,du\\
&\quad-\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du}a(s,s)\int_{s-\tau(s)}^{s}[a(u,u)+B(s,u)]g(x(u))\,du\,ds.
\end{aligned}$$

Finally, we arrive

(2.10)

$$\begin{aligned}
&x(t)e^{\int_0^t a(u,u)\,du}-x(t-T)e^{\int_0^{t-T} a(u,u)\,du}\\
&=e^{\int_0^t a(u,u)\,du}\left(1-e^{-\int_{t-T}^{t} a(u,u)\,du}\right)Q\left(t,x\left(t-\tau(t)\right)\right)\\
&\quad-\int_{t-T}^{t} e^{-\int_0^s a(u,u)\,du}a(s,s)Q(s,x(s-\tau(s)))\,ds
\end{aligned}$$

$$+\left(1-e^{-\int_{t-T}^{t} a(u,u)\,du}\right) e^{\int_0^t a(u,u)\,du} \int_{t-\tau(t)}^{t} [a(u,u)+B(t,u)]g(x(u))\,du$$
$$-\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du} a(s,s) \int_{s-\tau(s)}^{s} [a(u,u)+B(s,u)]g(x(u))\,du\,ds$$
$$+\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du} [B(s,s-\tau(s))$$
$$+a(s-\tau(s),s-\tau(s))](1-\tau'(s))g(x(s-\tau(s)))\,ds$$
$$+\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du} a(s,s)F(x(s))\,ds$$
$$+\int_{t-T}^{t} e^{\int_0^s a(u,u)\,du} G(s,x(s),x(s-\tau(s)))\,ds.$$

By dividing both sides of the above equation by $(1-e^{-\int_{t-T}^{t} a(u,u)\,du})e^{\int_0^t a(u,u)\,du}$ and using the fact that $x(t)=x(t-T)$, we obtain (2.8). The converse implication is easily obtained by differentiating and using Leibniz rules. The proof is complete. □

Krasnoselskii (see [14, Theorem 1.2.7] or [23]) combined the contraction mapping theorem and Schauder's theorem and formulated the following hybrid, but attractive, result.

Theorem 1 (Krasnoselskii [14, Theorem 1.2.7])**.** *Let M be a closed convex nonempty subset of a Banach space $(S,\|\cdot\|)$. Suppose that*

(i) *the mapping $A : M \to S$ is completely continuous,*
(ii) *the mapping $B : M \to S$ is a contraction, and*
(iii) *$x, y \in M$, implies $Ax + By \in M$.*

Then the mapping $A + B$ has a fixed point in M.

This is a captivating result and has a number of interesting applications. In recent year much attention has been paid to this theorem. Burton, in ([14, Theorem 1.2.8]), observed that Krasnoselskii's result can be more interesting in application with certain changes. He introduced the concept of large contraction, established a fixed point theorem which concerns this concept and extended Krasnoselskii's theorem as stated below.

Definition 2 (Large contraction)**.** Let (M,d) be a metric space and $B : M \to M$. B is said to be a large contraction if $\varphi,\psi \in M$, with $\varphi \neq \psi$ then $d(B\varphi,B\psi) < d(\varphi,\psi)$ and if for all $\epsilon > 0$ there exists $\delta < 1$ such that

$$[\varphi,\psi \in M,\ d(\varphi,\psi) \geq \epsilon] \Longrightarrow d(B\varphi,B\psi) \leq \delta d(\varphi,\psi).$$

Theorem 2. *Let (M,d) be a complete metric space and B be a large contraction. Suppose there is an $x \in M$ and $L > 0$, such that: $d(x,B^n x) \leq L$ for all $n \geq 1$. Then B has a unique fixed point in M.*

Theorem 3 (Krasnoselskii-Burton [14, Theorem 1.2.8])**.** *Let M be a closed bounded convex nonempty subset of a Banach space $(S, \|\cdot\|)$. Suppose that A, B map M into M and that*

(i) *for all $x, y \in M \Longrightarrow Ax + By \in M$,*
(ii) *A is completely continuous,*
(iii) *B is a large contraction.*

Then there is a $z \in M$ with $z = Az + Bz$.

In what follows, we shall use this theorem to prove the existence of periodic and positive solutions for (1.1).

3. Existence of periodic solutions

To apply Theorem 3, we need to define a Banach space S, a bounded convex subset M of S and construct two mappings, one of which is a large contraction and the other is completely continuous. So, let $(S, \|\cdot\|)$ be the space P_T endowed with the supremum norm and define the operator H by

$$\varphi(t) = (\mathfrak{B}\varphi)(t) + (A\varphi)(t) := (H\varphi)(t), \tag{3.1}$$

where A, $\mathfrak{B}$ are defined on P_T as follows

$$(\mathfrak{B}\varphi)(t) = \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\left[\varphi(s) - g(\varphi(s))\right] ds, \tag{3.2}$$

and

$$\begin{aligned}(A\varphi)(t) = {} & Q\left(t, \varphi\left(t - \tau(t)\right)\right) + \int_{t-\tau(t)}^{t} (B(t,u) + a(u,u)) g(\varphi(u))\,du \\ & - \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) Q\left(s, \varphi\left(s - \tau\left(s\right)\right)\right) ds \\ & - \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) \int_{s-\tau(s)}^{s} (a(u,u) + B(s,u)) g(\varphi(u))\,du\,ds \\ & + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} [a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))] \\ & \times (1 - \tau'(s)) g(\varphi(s-\tau(s)))\,ds \\ & + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} G\left(s, \varphi(s), \varphi(s-\tau(s))\right) ds.\end{aligned} \tag{3.3}$$

We need the following restrictions on the nonlinear term Q.

(k5) The function $Q(t,x)$ is continuous in t and there exist bounded positive periodic functions $q_1, q_2 \in L^1[0,T]$, with period T, such that

$$\left|Q\left(t,x\right)\right| \leq q_1\left(t\right)\left|x\right| + q_2\left(t\right), \tag{3.4}$$

for all $x \in \mathbb{R}$.

Our first lemma in this section shows that A maps P_T into itself and is completely continuous.

Lemma 3. *Let A be given in* (3.3). *Suppose that conditions* (k1)–(k5) *hold. Then $A : P_T \to P_T$ is completely continuous.*

Proof: Let A be defined by (3.2). Clearly, $A\varphi$ is continuous if φ is such. Having in mind conditions (k1)–(k3) and using a change of variables, it can be seen that $(A\varphi)(t+T) = (A\varphi)(t)$. To see that A is continuous, let $\{\varphi_n\} \subset P_T$ be an arbitrary sequence such that $\varphi_n \to \varphi$ as $n \to \infty$. By the Dominated Convergence theorem we obtain

$$
\begin{aligned}
&\lim_{n\to\infty} |(A\varphi_n)(t) - (A\varphi)(t)| \\
&\le \lim_{n\to\infty} |Q(t, \varphi_n(t-\tau(t))) - Q(t, \varphi(t-\tau(t)))| \\
&\quad + \lim_{n\to\infty} \int_{t-\tau(t)}^{t} |B(t,u) + a(u,u)|\, |g(\varphi_n(u)) - g(\varphi(u))|\, du \\
&\quad + \lim_{n\to\infty} \eta \Big\{ \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) \int_{s-\tau(s)}^{s} |a(u,u) + B(s,u)| \\
&\quad \times |g(\varphi_n(u)) - g(\varphi(u))|\, du\, ds \\
&\quad + \lim_{n\to\infty} \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\, |Q(s, \varphi_n(s-\tau(s))) - Q(s, \varphi(s-\tau(s)))|\, ds \\
&\quad + \lim_{n\to\infty} \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}\, |a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))|\, |1-\tau'(s)| \\
&\quad \times |g(\varphi_n(s-\tau(s))) - g(\varphi(s-\tau(s)))|\, ds \\
&\quad + \lim_{n\to\infty} \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} |G(s, \varphi_n(s), \varphi_n(s-\tau(s))) \\
&\quad - G(s, \varphi(s), \varphi(s-\tau(s)))|\, ds \Big\} \\
&\le \lim_{n\to\infty} |Q(t, \varphi_n(t-\tau(t))) - Q(t, \varphi(t-\tau(t)))| \\
&\quad + K_1 \int_{t-\tau(t)}^{t} \lim_{n\to\infty} |B(t,u) + a(u,u)|\, |\varphi_n(u) - \varphi(u)|\, du \\
&\quad + \eta \Big\{ K_1 \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) \lim_{n\to\infty} \int_{s-\tau(s)}^{s} |a(u,u) + B(s,u)| \\
&\quad \times |\varphi_n(u) - \varphi(u)|\, du\, ds \\
&\quad + \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) \lim_{n\to\infty} |Q(s, \varphi_n(s-\tau(s))) - Q(s, \varphi(s-\tau(s)))|\, ds \\
&\quad + K_1 \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} \lim_{n\to\infty} |a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))|\, |1-\tau'(s)|
\end{aligned}
$$

$$
\begin{aligned}
&\quad \times \left|\varphi_n(s-\tau(s)) - \varphi\left(s-\tau(s)\right)\right| \, ds \\
&\quad + \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} \lim_{n\to\infty} \left| G\left(s, \varphi_n(s), \varphi_n(s-\tau(s))\right) \right. \\
&\quad \left. - G(s, \varphi(s), \varphi(s-\tau(s))) \right| ds \Big\} \\
&= 0.
\end{aligned}
$$

Thus, A maps P_T into P_T and is continuous.

It remains to show that A is a compact function. Toward this, let $\mathbb{S} \subset P_T$ be a closed bounded subset and let R be a constant such that $\|\varphi\| \leq R$ for all $\varphi \in \mathbb{S}$. Let

$$
\begin{aligned}
q_1^* &= \max_{u\in[0,T]} q_1(u), \qquad q_2^* = \max_{u\in[0,T]} q_2\left(u\right), \\
\sigma &= \max_{s\in[t-T,t]} \{a(s,s)\}, \quad \theta = \max_{t\in[t-T,t]} e^{-\int_s^t a(u,u)\,du}.
\end{aligned}
$$

Then,

$$
\begin{aligned}
\left|\left(A\varphi\right)(t)\right| \leq q_1^* R + q_2^* + \left(K_1 L + |g\left(0\right)|\right) \int_{t-\tau(t)}^{t} \left|B(t,u) + a(u,u)\right| du \\
+ \eta\theta \Big\{ \sigma\left(K_1 L + |g(0)|\right) \int_{t-T}^{t} \int_{s-\tau(s)}^{s} \left|a(u,u) + B(s,u)\right| du\, ds \\
+ \left(K_1 L + |g(0)|\right) \int_{t-T}^{t} \left|1 - \tau'(s)\right| \big| a(s-\tau(s), s-\tau(s)) \\
+ B(s, s-\tau(s)) \big| \, ds + \sigma \int_{t-T}^{t} q_R(u)\, du + \int_{t-T}^{t} g_R(u)\, du \Big\} \\
= D,
\end{aligned}
$$

for some constant D. So, the family of functions $A\varphi$ is uniformly bounded.

Again, let $\varphi \in \mathbb{S}$. Without loss of generality, pick $t_1 < t_2$ such that $t_2 - t_1 < T$, then

$$
\begin{aligned}
&\left|\left(A\varphi\right)(t_2) - \left(A\varphi\right)(t_1)\right| \\
=&\Big| \, Q\left(t_2, \varphi\left(t_2 - \tau\left(t_2\right)\right)\right) + \int_{t_2-\tau(t_2)}^{t_2} \left[a(u,u) + B(t_2,u)\right] g(\phi(u))\, du \\
&-\eta \int_{t_2-T}^{t_2} e^{-\int_s^{t_2} a(u,u)\,du} a(s,s) Q\left(s, x\left(s-\tau\left(s\right)\right)\right) ds \\
&-\eta \int_{t_2-T}^{t_2} e^{-\int_s^{t_2} a(u,u)\,du} a(s,s) \int_{s-\tau(s)}^{s} (a(u,u) + B(s,u)) g(\varphi(u))\, du\, ds \\
&+\eta \int_{t_2-T}^{t_2} e^{-\int_s^{t_2} a(u,u)\,du} e^{-\int_s^{t} a(u,u)\,du} \left[a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))\right]
\end{aligned}
$$

$$
\begin{aligned}
&\times(1-\tau'(s))g(\varphi(s-\tau(s)))\,ds\\
&+\eta\int_{t_2-T}^{t_2} e^{-\int_s^{t_2} a(u,u)\,du} G\left(s,x(s),x(s-\tau(s))\right)ds\\
&-Q\left(t_1,\varphi\left(t_1-\tau\left(t_1\right)\right)\right)-\int_{t_1-\tau(t_1)}^{t_1}\left[a(u,u)+B(t_2,u)\right]g(\phi(u))\,du\\
&-\eta\int_{t_1-T}^{t_1}\{e^{-\int_s^{t_1} a(u,u)\,du}[a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))]\\
&\times(1-\tau'(s))g(\varphi(s-\tau(s)))\,ds\\
&-\eta\int_{t_1-T}^{t_1} e^{-\int_s^{t_1} a(u,u)\,du} G\left(s,x(s),x(s-\tau(s))\right)ds\\
&+\eta\int_{t_1-T}^{t_1} e^{-\int_s^{t_1} a(u,u)\,du} a(s,s)Q\left(s,x\left(s-\tau\left(s\right)\right)\right)\,ds\\
&+\eta\int_{t_1-T}^{t_1} e^{-\int_s^{t_1} a(u,u)\,du} a(s,s)\int_{s-\tau(s)}^{s}(a(u,u)+B(s,u))g(\varphi(u))\,du\,ds\}\ | \ .
\end{aligned}
$$

The calculus shows that

$$
\begin{aligned}
&\left|\left(A\varphi\right)\left(t_2\right)-\left(A\varphi\right)\left(t_1\right)\right|\\
&\leq\left|Q\left(t_2,\varphi\left(t_2-\tau\left(t_2\right)\right)\right)-Q\left(t_1,\varphi\left(t_1-\tau\left(t_1\right)\right)\right)\right|\\
&\quad+\ |\int_{t_2-\tau(t_2)}^{t_2}\left[a(u,u)+B(t_2,u)\right]g(\phi(u))\,du\\
&\quad-\int_{t_1-\tau(t_1)}^{t_1}\left[a(u,u)+B(t_2,u)\right]g(\phi(u))\,du\ |\\
&\quad+\eta\int_{t_1}^{t_2}[a(s,s)\int_{s-\tau(s)}^{s}\left|a(u,u)+B(s,u)\right|\left|g(\phi(u))\right|\,du\\
&\quad+\left|a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'\left(s\right)\right|\left|g(\varphi(s-\tau(s)))\right|\\
&\quad+a\left(s,s\right)\left|Q\left(s,x\left(s-\tau(s)\right)\right)\right|\,ds+\left|G(s,x(s),x(s-\tau(s)))\right|]e^{-\int_s^{t_2} a(u,u)\,du}\,ds\\
&\quad+\eta\int_{t_2-T}^{t_1}[a(s,s)\int_{s-\tau(s)}^{s}\left|a(u,u)+B(s,u)\right|\left|g(\phi(u))\right|\,du\\
&\quad+\left|a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'\left(s\right)\right|\left|g(\varphi(s-\tau(s)))\right|\\
&\quad+a\left(s,s\right)\left|Q\left(s,x\left(s-\tau(s)\right)\right)\right|+\left|G(s,x(s),x(s-\tau(s)))\right|]\\
&\quad\times\left|e^{-\int_s^{t_2} a(u,u)\,du}-e^{-\int_s^{t_1} a(u,u)\,du}\right|\,ds\\
&\quad+\eta\int_{t_1-T}^{t_2-T}[a(s,s)\int_{s-\tau(s)}^{s}\left|a(u,u)+B(s,u)\right|\left|g(\phi(u))\right|\,du\\
&\quad+\left|a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'\left(s\right)\right|\left|g(\varphi(s-\tau(s)))\right|
\end{aligned}
$$

$$
\begin{aligned}
&+ a(s,s)\left|Q\left(s,x(s-\tau(s))\right)\right| + \left|G\left(s,x(s),x(s-\tau(s))\right)\right|]e^{-\int_s^{t_1} a(u,u)\,du}\,ds \\
&\le \left|Q\left(t_2,\varphi\left(t_2-\tau\left(t_2\right)\right)\right) - Q\left(t_1,\varphi\left(t_1-\tau\left(t_1\right)\right)\right)\right| \\
&+ \Big| \int_{t_2-\tau(t_2)}^{t_2} \left[a(u,u)+B(t_2,u)\right] g(\phi(u))\,du \\
&- \int_{t_1-\tau(t_1)}^{t_1} \left[a(u,u)+B(t_2,u)\right] g(\phi(u))\,du \Big| \\
&+ 2\eta\theta\Big\{\int_{t_1}^{t_2} [\sigma\left(K_1L+|g(0)|\right) \int_{s-\tau(s)}^{s} \left|a(u,u)+B(s,u)\right|\,du \\
&+ \left(K_1L+|g(0)|\right)\left|a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'(s)\right| \\
&+ \sigma q_R(s) + g_R(s)]\,ds\Big\} \\
&+ \eta \int_{t_2-T}^{t_1} [\sigma\left(K_1L+|g(0)|\right) \int_{s-\tau(s)}^{s} \left|a(u,u)+B(s,u)\right|\,du \\
&+ \left(K_1L+|g(0)|\right)\left|a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'(s)\right| \\
&+ \sigma q_R(s) + g_R(s)]\left|e^{-\int_s^{t_2} a(u,u)\,du} - e^{-\int_s^{t_1} a(u,u)\,du}\right|\,ds.
\end{aligned}
$$

Now, we observe that

$$
\begin{aligned}
&\left|Q\left(t_2,\varphi\left(t_2-\tau\left(t_2\right)\right)\right) - Q\left(t_1,\varphi\left(t_1-\tau\left(t_1\right)\right)\right)\right| \to 0, \\
&\Big| \int_{t_2-\tau(t_2)}^{t_2} \left[a(u,u)+B(t_2,u)\right] g(\phi(u))\,du \\
&\quad - \int_{t_1-\tau(t_1)}^{t_1} \left[a(u,u)+B(t_2,u)\right] g(\phi(u))\,du \Big| \to 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_{t_1}^{t_2} [\sigma\left(K_1L+|g(0)|\right) \int_{s-\tau(s)}^{s} \left|a(u,u)+B(s,u)\right|\,du \\
&\quad + \left(K_1L+|g(0)|\right)\left|a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'(s)\right| \\
&\quad + \sigma q_R(s) + g_R(s)]\,ds \to 0
\end{aligned}
$$

as $(t_2-t_1)\to 0$. Also,

$$
\begin{aligned}
&\int_{t_2-T}^{t_1} [\sigma\left(K_1L+|g(0)|\right) \int_{s-\tau(s)}^{s} \left|a(u,u)+B(s,u)\right|\,du \\
&\quad + \left(K_1L+|g(0)|\right)\left|a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))\right|\left|1-\tau'(s)\right| \\
&\quad + \sigma q_R(s) + g_R(s)]\left|e^{-\int_s^{t_2} a(u,u)\,du} - e^{-\int_s^{t_1} a(u,u)\,du}\right|\,ds \\
&\le \int_0^T [\left(K_1L+|g(0)|\right) \int_{s-\tau(s)}^{s} \left|a(u,u)+B(s,u)\right|\,du
\end{aligned}
$$

$$+ (K_1L + |g(0)|)\, |a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))|\, |1-\tau'(s)|$$
$$+ \sigma q_R(s) + g_R(s)] \left| e^{-\int_s^{t_2} a(u,u)\, du} - e^{-\int_s^{t_1} a(u,u)\, du} \right| ds,$$

and $|e^{-\int_s^{t_2} a(u,u)\, du} - e^{-\int_s^{t_1} a(u,u)\, du}| \to 0$ as $(t_2 - t_1) \to 0$. Consequently, by the Dominated Convergence Theorem, one can have

$$\int_{t_2-T}^{t_1} [\sigma (K_1L + |g(0)|) \int_{s-\tau(s)}^{s} |a(u,u) + B(s,u)|\, du$$
$$+ (K_1L + |g(0)|)\, |a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))|\, |1-\tau'(s)|$$
$$+ \sigma q_R(s) + g_R(s)] \left| e^{-\int_s^{t_2} a(u,u)\, du} - e^{-\int_s^{t_1} a(u,u)\, du} \right| ds \to 0,$$

as $(t_2 - t_1) \to 0$. Thus, $|(A\varphi)(t_2) - (A\varphi)(t_1)| \to 0$ as $(t_2 - t_1) \to 0$ and the limit does not depend on $\varphi \in \mathbb{S}$. We conclude that the family of functions $A\varphi$ is equicontinuous on P_T. By the Arzelà-Ascoli theorem, A is a compact function and the proof is complete. □

Now, we state an important result implying that the mapping F given by (2.2) is a large contraction. This result was already obtained in [1, Theorem 3.4] and for convenience we present below its proof.

Proposition 1. *Let $\|\cdot\|$ be the supremum norm,*

$$M := \{\varphi \in S,\ \|\varphi\| \le L\},$$

and F defined by (2.2). *Suppose that g has the following properties.*

H1. *$g : \mathbb{R} \to \mathbb{R}$ is continuous on $[-L, L]$ and differentiable on $(-L, L)$,*
H2. *The function g is strictly increasing on $[-L, L]$.*
H3. $\sup_{t\in(-L,L)} g'(t) \le 1$.

Then the mapping F in (2.2) *is a large contraction on the set M.*

Proof: Let $\phi, \varphi \in M$ with $\phi \ne \varphi$. Then $\varphi(t) \ne \phi(t)$ for some $t \in \mathbb{R}$. Let us denote the set of all such t by $D(\phi, \varphi)$. That is

$$D(\phi, \varphi) := \{t \in \mathbb{R} : \phi(t) \ne \varphi(t)\}.$$

For all $t \in D(\phi, \varphi)$ we have

$$\begin{aligned} |F\phi(t) - F\varphi(t)| &= |\phi(t) - g(\phi(t)) - \varphi(t) + g(\varphi(t))| \\ &= |\phi(t) - \varphi(t)| \left| 1 - \left(\frac{g(\phi(t)) - g(\varphi(t))}{\phi(t) - \varphi(t)} \right) \right|. \end{aligned} \tag{3.5}$$

Since g is strictly increasing, we have

$$\frac{g(\phi(t)) - g(\varphi(t))}{\phi(t) - \varphi(t)} > 0, \quad \text{for all } t \in D(\phi, \varphi). \tag{3.6}$$

For each fixed $t \in D(\phi, \varphi)$, define the interval $U_t \subset [-L, L]$ by

$$U_t = \begin{cases} (\varphi(t), \phi(t)) & \text{if } \phi(t) > \varphi(t), \\ (\phi(t), \varphi(t)) & \text{if } \phi(t) < \varphi(t). \end{cases}$$

The Mean Value Theorem implies that for each fixed $t \in D(\phi, \varphi)$ there exists a real number $c_t \in U_t$ such that

$$\frac{g(\phi(t)) - g(\varphi(t))}{\phi(t) - \varphi(t)} = g'(c_t).$$

By (H2)–(H3) we have

$$(3.7)\quad 0 \leq \inf_{u \in (-L,L)} g'(u) \leq \inf_{u \in U_t} g'(u) \leq g'(c_t) \leq \sup_{u \in U_t} g'(u) \leq \sup_{u \in (-L,L)} g'(u) \leq 1.$$

Hence, by (3.5)–(3.7) we obtain

$$|F\phi(t) - F\varphi(t)| \leq \left|1 - \inf_{u \in (-L,L)} g'(u)\right| |\phi(t) - \varphi(t)|, \tag{3.8}$$

for all $t \in D(\phi, \varphi)$. This implies that F is a large contraction in the supremum norm. To see this, choose a fixed $\epsilon \in (0, 1)$ and assume that ϕ and φ are two functions in M satisfying

$$\epsilon \leq \sup_{t \in D(\phi,\varphi)} |\phi(t) - \varphi(t)| = \|\phi - \varphi\|.$$

If $|\phi(t) - \varphi(t)| \leq \frac{\epsilon}{2}$ for some $t \in D(\phi, \varphi)$, then by (3.7) and (3.8) we get

$$|F\phi(t) - F\varphi(t)| \leq |\phi(t) - \varphi(t)| \leq \frac{1}{2} \|\phi - \varphi\|. \tag{3.9}$$

Since g is continuous and strictly increasing the function $g(u + \frac{\epsilon}{2}) - g(u)$ attains its minimum on the closed and bounded interval $[-L, L]$.

Consequently, if $\frac{\epsilon}{2} < |\phi(t) - \varphi(t)|$ for some $t \in D(\phi, \varphi)$, then by (H2) and (H3) we conclude that

$$1 \geq \frac{g(\phi(t)) - g(\varphi(t))}{\phi(t) - \varphi(t)} > \upsilon,$$

where

$$\upsilon := \frac{1}{2L} \min\left\{g\left(u + \frac{\epsilon}{2}\right) - g(u) : u \in [-L, L]\right\} > 0.$$

Hence (3.5) implies

$$|F\phi(t) - F\varphi(t)| \leq (1 - \upsilon) \|\phi(t) - \varphi(t)\|. \tag{3.10}$$

Therefore, combining (3.9) and (3.10) we obtain

$$|F\phi(t) - F\varphi(t)| \leq \varsigma \|\phi - \varphi\|,$$

where $\varsigma = \max\{\frac{1}{2}, 1-\upsilon\} < 1$. The proof is complete. $\square$

We shall prove that the mapping H given by (3.1) has a fixed point which solves (1.1) whenever its derivative exists. For that proof we need further conditions on the nonlinear term G.

(k6) There exist positive periodic functions $g_1, g_2, g_3 \in L^1[0,T]$ with period T such that

$$|G(t,x,y)| \leq g_1(t)\,|x| + g_2(t)|y| + g_3(t),$$

for all $x, y \in \mathbb{R}$.

Lemma 4. *Suppose that conditions (k5) and (k6) hold. Suppose further that*

$$(k_1 L + |g(0)|) \int_{t-\tau(t)}^{t} |B(t,u) + a(u,u)|\, du \leq \frac{R_1}{2} L, \tag{3.11}$$

$$|1-\tau'(t)|\,|a(t-\tau(t), t-\tau(t)) + B(t, t-\tau(t))| \leq \zeta a(t,t), \tag{3.12}$$

$$\zeta\,(k_1 L + |g(0)|) \leq R_2 L, \tag{3.13}$$

$$q_1(t)\,L + q_2(t) \leq \delta L, \tag{3.14}$$

$$[g_1(t) + g_2(t)]\,L + g_3(t) \leq \beta L a(t,t), \tag{3.15}$$

$$J\,(R_1 + R_2 + 2\delta + \beta) \leq 1, \tag{3.16}$$

where β, δ, R_1, R_2 and J are constants with $J > 3$.

For A defined by (3.3), if $\varphi \in M$, then $|(A\varphi)(t)| \leq L/J < L$ for all t.

Proof: Let $\varphi \in M$, then $\|\varphi\| \leq L$. Let A be defined by (3.2), we have

$$\begin{aligned}
&|(A\varphi)(t)| \\
&\leq |Q(t, \varphi(t-\tau(t)))| + \int_{t-\tau(t)}^{t} |B(t,u) + a(u,u)|\,|g(\varphi(u))|\, du \\
&\quad + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\,|Q(s, x(s-\tau(s)))|\, ds \\
&\quad + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) \int_{s-\tau(s)}^{s} |a(u,u) + B(s,u)|\,|g(\varphi(u))|\, du\, ds \\
&\quad + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}\,|a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))| \\
&\quad \times |1-\tau'(s)|\,|g(\varphi(s-\tau(s)))|\, ds \\
&\quad + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}\,|G(s, \varphi(s), \varphi(s-\tau(s)))|\, ds \\
&\leq q_1(t)\,L + q_2(t) + \frac{R_1}{2} L + \frac{R_1}{2} L\eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\, ds
\end{aligned}$$

$$
\begin{aligned}
&+\left[q_1(t)L+q_2(t)\right]\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\,ds \\
&+\left[\zeta\left(k_1L+|g(0)|\right)+\beta L\right]\left(1-e^{-\int_{t-T}^{t} a(u,u)\,du}\right)^{-1}\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\,ds \\
\le{}& \left(R_1+2\delta+R_2+\beta\right)L\le\frac{L}{J}<L.
\end{aligned}
$$

Therefore, A maps M into itself. □

We need $\mathfrak{B}\varphi$, and $A\varphi+\mathfrak{B}\psi$ to reside in M, whenever $\varphi,\psi\in M$. For that purpose, we require that

$$
\max\left(|F(L)|,|F(-L)|\right)\le\frac{(J-1)L}{J}, \tag{3.17}
$$

where $F(x)=x-g(x)$ is the one of (2.2).

Lemma 5. *Suppose* (k1)–(k6), (3.11)–(3.17) *and all the conditions of Proposition* 1 *hold and let* A, $\mathfrak{B}$ *be the functions defined in* (3.3) *and* (3.2). *If* $\varphi,\psi\in M$ *are arbitrary, then*

$$
A\varphi+\mathfrak{B}\psi: M\to M.
$$

Moreover, $\mathfrak{B}$ is a large contraction on M with a unique fixed point in M.

Proof: Let $\mathfrak{B}$ be defined by (3.2). Obviously, $\mathfrak{B}\varphi$ is continuous if φ is such. A change of variables shows that $(\mathfrak{B}\varphi)(t+T)=(\mathfrak{B}\varphi)(t)$. From Proposition 1 we know that $\varphi-g(\varphi)$ is a large contraction in the supremum norm. Let $\varphi,\psi\in M$ be arbitrary functions. For any ϵ, from the proof of that proposition, we have found a $\varsigma<1$, such that

$$
\begin{aligned}
\|\mathfrak{B}\varphi-\mathfrak{B}\psi\| &\le \left(1-e^{-\int_{t-T}^{t} a(u,u)\,du}\right)^{-1}\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\varsigma\|\varphi-\psi\|\,ds \\
&\le \varsigma\|\varphi-\psi\|.
\end{aligned}
$$

Furthermore, using the definition of $\mathfrak{B}$, the condition (3.17) and the monotonicity of F of (2.2), we see that

$$
\begin{aligned}
|(\mathfrak{B}\psi)(t)| &\le \left(1-e^{-\int_{t-T}^{t} a(u,u)\,du}\right)^{-1}\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)|F(s)|\,ds \\
&\le \frac{(J-1)L}{J}\left(1-e^{-\int_{t-T}^{t} a(u,u)\,du}\right)^{-1}\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\,ds \\
&\le \frac{(J-1)L}{J}\le L.
\end{aligned}
$$

Thus, $\mathfrak{B}$ maps M into M. Next, remembering the majoration of $|(A\varphi)(t)|$ given in the proof of Lemma 4 and using the last one on $|(B\psi)(t)|$ we obtain

$$\begin{aligned} |(A\varphi)(t) + (B\psi)(t)| &\leq |(A\varphi)(t)| + |(B\psi)(t)| \\ &\leq \frac{L}{J} + \frac{(J-1)L}{J} = L. \end{aligned}$$

Hence, $A\varphi + \mathfrak{B}\psi$ resides in M whenever $\varphi, \psi \in M$. This completes the proof. □

Theorem 4. *Let* $(S, \|\cdot\|)$ *be the Banach space of continuous T-periodic real functions and*

$$M = \{\varphi \in S,\ \|\varphi\| \leq L\},$$

where L is positive constant. Suppose (k1)–(k6), (3.11)–(3.17) *and all conditions of the Proposition* 1 *hold. Then equation* (1.1) *processes a T- periodic solution in the subset M.*

Proof: By Lemma 2, φ is a solution of (1.1) if

$$\varphi = A\varphi + \mathfrak{B}\varphi,$$

where A and $\mathfrak{B}$ are given by (3.3), (3.2) respectively. By Lemma (4), $A : M \to M$ is completely continuous. By Lemma (5), $A\varphi + \mathfrak{B}\psi \in M$, whenever $\varphi, \psi \in M$. Moreover, $\mathfrak{B} : M \to M$ is a large contraction. Clearly, all hypotheses of Theorem 3 of Krasnoselskii-Burton are satisfied. Thus, there exists a fixed point $\varphi \in M$ such that $\varphi = A\varphi + \mathfrak{B}\varphi$. Consequently, equation (1.1) has a T- periodic solution in M. □

4. Existence of nonnegative periodic solutions

Now, we turn our attention to the positivity of solutions of (1.1). But positive solutions need some careful adjustments. So, we begin by defining new sufficient quantities to reach our goal. Let

$$\begin{aligned} &\theta^* = \min_{t\in[t-T,t]} e^{-\int_s^t a(u,u)\,du},\ \gamma^* = \max_{t\in[t-T,t]} e^{-\int_s^t a(u,u)\,du}, \\ &f(t,x) = \int_{t-\tau(t)}^{t} (B(t,u) + a(u,u))g(x(u))\,du, \end{aligned} \tag{4.1}$$

where $\theta^*$and γ^* are positive constants. Given a constant $0 < K$, define the set

$$\mathbb{M}_2 := \{\varphi \in P_T : 0 \leq \varphi \leq K,\ t \in [0,T]\}. \tag{4.2}$$

We ask that the following conditions hold.

(k7) There exist constants $L^* > 0$ such that $0 \leq Q(t,\varphi) \leq L^*\varphi$ for all $\varphi \in \mathbb{M}_2$.

In the case $f(t,x) \geq 0$, we assume that there exist a positive constant E_1 such that

$$0 \leq f(t,x) \leq E_1 x, \quad \text{for all} \quad x \in \mathbb{M}_2, \tag{4.3}$$

$$L^* + E_1 < 1, \tag{4.4}$$

$$\begin{aligned} 0 \leq & -a(s,s) \int_{s-\tau(s)}^{s} (a(u,u) + B(s,u)) g(\varphi)\, du \\ & + [a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))](1-\tau'(s)) g(\varphi) \\ & - a(s,s) Q\left(s,\varphi\right) + G(s,\varphi,\varphi) \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} & [a(s-\tau(s), s-\tau(s)) + B(s, s-\tau(s))](1-\tau'(s)) g(\varphi) \\ & \quad - a(s,s) \int_{s-\tau(s)}^{s} (a(u,u) + B(s,u)) g(\varphi)\, du - a(s,s) Q\left(s,\varphi\right) \\ & \quad + G(s,\varphi,\varphi) + a(s,s)\left[\psi - g(\psi)\right] \\ & \leq \frac{\left(1 - L^* - E_1\right) K}{\eta \gamma^* T}, \end{aligned} \tag{4.6}$$

for all $\varphi, \psi \in \mathbb{M}_2$ and $s \in [t-T, t]$.

Lemma 6. *Suppose that conditions* (k1)–(k5), (k7), (4.1)–(4.6) *hold. Then A and $\mathfrak{B}$ maps* $\mathbb{M}_2 \to \mathbb{M}_2$.

Proof: Let A be defined by (3.3). A similar argumentation as in the proof of Lemma 3 shows that $(A\varphi)(t+T) = (A\varphi)(t)$. Furthermore, for any $\varphi \in \mathbb{M}_2$, by (4.1)-(4.6), we have

$$\begin{aligned} 0 & \leq (A\varphi)(t) \\ & \leq L^* K + E_1 K + \frac{\left(1 - L^* - E_1\right) K}{\eta \gamma^* T} \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\, du}\, ds \\ & \leq K. \end{aligned}$$

That is $A\varphi \in \mathbb{M}_2$.

Likewise, let $\mathfrak{B}$ be defined by (3.2), then $(\mathfrak{B}\varphi)(t+T) = (\mathfrak{B}\varphi)(t)$ is obtained as in the proof of Lemma 5. For any $\varphi \in \mathbb{M}_2$, we have

$$\begin{aligned} 0 \leq \left(\mathfrak{B}\varphi\right)(t) & \leq \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\, du} a(s,s) \left[\varphi(s) - g(\varphi(s))\right] ds \\ & \leq \frac{\left(1 - L^* - E_1\right) K}{\eta \gamma^* T} \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\, du}\, ds \\ & \leq K, \end{aligned}$$

since (4.4) hold. Thus, $\mathfrak{B}\varphi \in \mathbb{M}_2$. □

Theorem 5. *Let* $(S, \|\cdot\|)$ *be the Banach space of continuous T-periodic real functions and* $\mathbb{M}_2 = \{\varphi \in S,\ 0 \le \varphi \le K,\ t \in [0,T]\}$, *where* K *is a positive constant. Suppose that conditions* (k1)–(k5), (k7), *and* (4.1)–(4.6) *hold. Then there exists a nonnegative T-periodic solution of* (1.1).

Proof: By Lemma 2, φ is a solution of (1.1) if

$$\varphi = A\varphi + \mathfrak{B}\varphi,$$

where A and $\mathfrak{B}$ are given by (3.3), (3.2) respectively. By Lemma 6, $A, \mathfrak{B} : \mathbb{M}_2 \to \mathbb{M}_2$. By Lemma 3, A is completely continuous. Moreover, $\mathfrak{B}$ is a large contraction. We just need to show that condition (iii) of Theorem 3 is satisfied. Toward this, let $\varphi, \psi \in \mathbb{M}_2$, then

$$A\varphi\,(t) + \mathfrak{B}\psi\,(t) \ge 0,$$

$$\begin{aligned}
&A\varphi\,(t) + \mathfrak{B}\psi\,(t)\\
&= Q(t, \varphi\,(t - \tau(t))) + f(t, \varphi)\\
&\quad - \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) \int_{s-\tau(s)}^{s} (a(u,u) + B(s,u)) g(\varphi(u))\,du\,ds\\
&\quad + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} [a(s-\tau(s), s-\tau(s))\\
&\quad + B(s, s-\tau(s))](1-\tau'(s)) g(\varphi(s-\tau(s)))\,ds\\
&\quad - \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) Q\,(s, x\,(s-\tau(s)))\,ds\\
&\quad + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s) F\,(\psi\,(s))\,ds\\
&\quad + \eta \int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} G\,(s, \varphi(s), \varphi(s-\tau(s)))\,ds\\
&\le L^* K + E_1 K + \eta\gamma^* T \left(\frac{(1 - L^* - E_1) K}{\eta\gamma^* T}\right) = K.
\end{aligned}$$

By Theorem 3 the operator H has a fixed point in $\mathbb{M}_2$. This fixed point is a solution of (1.1) and the proof is complete. □

In the case $f(t,x) \le 0$, we substitute conditions (4.3)–(4.6) with the following conditions respectively. We assume that there exists a negative constant E_2 such that

$$E_2 x \le f(t,x) \le 0, \quad \text{for all} \quad x \in \mathbb{M}_2, \tag{4.7}$$

$$-E_2 + L^* < 1, \tag{4.8}$$

$$\begin{aligned} \frac{-E_2}{\eta\theta^* T} \leq -a(s,s) &\int_{s-\tau(s)}^{s} (a(u,u)+B(s,u))g(\varphi)\,du \\ &+ [a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))](1-\tau'(s))g(\varphi) \\ &- a(s,s)Q\,(s,\varphi)+G(s,\varphi,\varphi)\,ds \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} &[a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))](1-\tau'(s))g(\varphi) \\ &\quad - a(s,s)\int_{s-\tau(s)}^{s}(a(u,u)+B(s,u))g(\varphi)\,du \\ &\quad - \mu(s)\varphi + G(s,\varphi,\varphi) + a(s,s)[\psi - g(\psi)] \\ &\leq \frac{(1-L^*)\,K}{\eta\gamma^* T}, \end{aligned} \tag{4.10}$$

for all $\varphi,\psi \in \mathbb{M}_2$ and $s \in [t-T,t]$.

Theorem 6. *Suppose* (k1)–(k5), (k7), (4.1), (4.7)–(4.10) *hold. Then, equation* (1.1) *has a T-periodic nonnegative solution x in the subset* $\mathbb{M}_2$.

PROOF: As in the proof of Theorem 5, we can show easily that $A,\mathfrak{B}$ maps: $\mathbb{M}_2 \to \mathbb{M}_2$. We just need to show that condition (iii) of Theorem 3 is satisfied. Let $\varphi,\psi \in \mathbb{M}_2$. Then

$$\begin{aligned} &A\varphi(t)+\mathfrak{B}\psi\,(t) \\ &= Q(t,\varphi(t-\tau(t))) + f(t,\varphi) \\ &\quad -\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)\int_{s-\tau(s)}^{s}(a(u,u)+B(s,u))g(\varphi(u))\,du\,ds \\ &\quad +\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du}[a(s-\tau(s),s-\tau(s))+B(s,s-\tau(s))] \\ &\quad \times(1-\tau'(s))g(\varphi(s-\tau(s)))\,ds \\ &\quad -\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)Q\,(s,x\,(s-\tau(s)))\,\,ds \\ &\quad +\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} a(s,s)F(\psi(s))\,ds \\ &\quad +\eta\int_{t-T}^{t} e^{-\int_s^t a(u,u)\,du} G(s,\varphi(s),\varphi(s-\tau(s)))\,ds \\ &\leq L^*K + \eta\gamma^* T\left(\frac{(1-L^*)\,K}{\eta\gamma^* T}\right) = K. \end{aligned}$$

Likewise

$$A\varphi(t)+\mathfrak{B}\psi(t) \geq h(t,\varphi)+\eta\theta^* T\left(\frac{-E_2}{\eta\theta^* T}\right)$$

$$\geq \frac{E_2}{\eta\theta^* T} + \eta\theta^* T\left(\frac{-E_2}{\eta\theta^* T}\right)$$
$$\geq 0.$$

By Theorem 3 the operator H has a fixed point in $\mathbb{M}_2$. This fixed point is a solution of (1.1) and the proof is complete. □

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