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Global existence and energy decay of solutions to a Bresse system with delay terms

Abbes Benaissa, Mostefa Miloudi, Mokhtar Mokhtari

Abstract. We consider the Bresse system in bounded domain with delay terms in the internal feedbacks and prove the global existence of its solutions in Sobolev spaces by means of semigroup theory under a condition between the weight of the delay terms in the feedbacks and the weight of the terms without delay. Furthermore, we study the asymptotic behavior of solutions using multiplier method.

Keywords: Bresse system; delay terms; decay rate; multiplier method

Classification: 35B40, 35L70

1. Introduction

In this paper we investigate the existence and decay properties of solutions for the initial boundary value problem of the linear Bresse system of the type
(P)

$$\begin{cases}\rho_1\varphi_{tt} - Gh(\varphi_x+\psi+l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1\varphi_t + \mu_2\varphi_t(x,t-\tau_1) = 0\\ \rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+\psi+l\omega) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}\psi_t(x,t-\tau_2) = 0\\ \rho_1\omega_{tt} - Eh(\omega_x - l\varphi)_x + lGh(\varphi_x+\psi+l\omega) + \widetilde{\widetilde{\mu_1}}\omega_t + \widetilde{\widetilde{\mu_2}}\omega_t(x,t-\tau_3) = 0\end{cases}$$

where $(x,t)\in(0,L)\times(0,+\infty), \tau_i>0$ $(i=1,2,3)$ is a time delay, μ_1, μ_2, $\widetilde{\mu_1}$, $\widetilde{\mu_2}$, $\widetilde{\widetilde{\mu_1}}$, $\widetilde{\widetilde{\mu_2}}$ are positive real numbers. This system is subject to the Dirichlet boundary conditions

$$\varphi(0,t)=\varphi(L,t)=\psi(0,t)=\psi(L,t)=\omega(0,t)=\omega(L,t)=0,\quad t>0$$

and to the initial conditions

$$\begin{cases}\varphi(x,0)=\varphi_0(x),\quad \varphi_t(x,0)=\varphi_1(x),\quad \psi(x,0)=\psi_0(x),\\ \psi_t(x,0)=\psi_1(x),\quad \omega(x,0)=\omega_0(x),\quad \omega_t(x,0)=\omega_1(x),\quad x\in(0,L)\\ \varphi_t(x,t-\tau_1)=f_0(x,t-\tau_1),\quad \text{in } (0,L)\times[0,\tau_1]\\ \psi_t(x,t-\tau_2)=\widetilde{f}_0(x,t-\tau_2),\quad \text{in } (0,L)\times[0,\tau_2]\\ \omega_t(x,t-\tau_3)=\widetilde{\widetilde{f}}_0(x,t-\tau_3),\quad \text{in } (0,L)\times[0,\tau_3]\end{cases}$$

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where the initial data $(\varphi_0, \varphi_1, \psi_0, \psi_1, \omega_0, \omega_1, f_0, \widetilde{f_0}, \widetilde{\widetilde{f_0}})$ belong to a suitable Sobolev space. By ω, ψ and φ we are denoting the longitudinal, vertical and shear angle displacements. The original Bresse system is given by the following equations (see [1]) :

$$\begin{cases} \rho_1 \varphi_{tt} = Q_x + lN + F_1, \\ \rho_2 \psi_{tt} = M_x - Q + F_2, \\ \rho_1 \omega_{tt} = N_x - lQ + F_3, \end{cases}$$

where we use N, Q and M to denote the axial force, the shear force and the bending moment respectively. These forces are stress-strain relations for elastic behavior and given by

$$N = Eh(\omega_x - l\varphi), \quad Q = Gh(\varphi_x + \psi + l\omega), \quad \text{and } M = EI\psi_x,$$

where G, E, I and h are positive constants. Finally, by the terms F_i we are denoting external forces.

The Bresse system without delay (i.e. $\mu_2 = \widetilde{\mu_2} = \widetilde{\widetilde{\mu_2}} = 0$) is more general than the well-known Timoshenko system where the longitudinal displacement ω is not considered $l = 0$. There are a number of publications concerning the stabilization of Timoshenko system with different kinds of damping (see [2], [3], [4] and [5]). Raposo et al. [6] proved the exponential decay of the solution for the following linear system of Timoshenko-type beam equations with linear frictional dissipative terms:

$$\begin{aligned} &\rho_1 \varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + \mu_1 \varphi_t = 0 \\ &\rho_2 \psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + \widetilde{\mu_1} \psi_t = 0 \end{aligned}$$

Messaoudi and Mustafa [3] (see also [5]) considered the stabilization for the following Timoshenko system with nonlinear internal feedbacks:

$$\begin{aligned} &\rho_1 \varphi_{tt} - Gh(\varphi_x + \psi + l\omega)_x - lEh(\omega_x - l\varphi) + g_1(\psi_t) = 0 \\ &\rho_2 \psi_{tt} - EI\psi_{xx} + Gh(\varphi_x + \psi + l\omega) + g_2(\psi_t) = 0 \end{aligned}$$

Recently, Park and Kang [5] considered the stabilization of the Timoshenko system with weakly nonlinear internal feedbacks.

In [7], Liu and Rao considered a thermoelastic Bresse system that consists of three wave equations and two heat equations coupled in certain way. The two wave equations for the longitudinal displacement and the shear angle displacement are effectively globally damped by the dissipation from the two heat equations. The wave equation about the vertical displacement is subject to a weak thermal damping and indirectly damped through the coupling. They establish exponential energy decay rate when the vertical and the longitudinal waves have the same speed of propagation. Otherwise, a polynomial-type decay is established.

Time delay is the property of a physical system by which the response to an applied force is delayed in its effect (see [8]). Whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. In recent years, the PDEs with time delay effects have become an active area of research and arise in many pratical problems (see for example [9], [10]). The presence of delay may be a source of instability. For example, it was proved in [11] that an arbitrarily small delay may destabilize a system which is uniformly asymptotically stable in the absence of delay. To stabilize a hyperbolic system involving input delay terms, additional control terms will be necessary (see [12] and [13]). For instance, in [12] the authors studied the wave equation with a linear internal damping term with constant delay and determined suitable relations between μ_1 and μ_2, for which the stability or alternatively instability takes place. More precisely, they showed that the energy is exponentially stable if $\mu_2 < \mu_1$ and they found a sequence of delays for which the solution will be instable if $\mu_2 \geq \mu_1$. The main approach used in [12], is an observability inequality obtained with a Carleman estimate. The same results were showed if both the damping and the delay act in the boundary domain. We also recall the result by Xu, Yung and Li [13], where the authors proved the same result as in [12] for the one space dimension by adopting the spectral analysis approach.

Our purpose in this paper is to give a global solvability in Sobolev spaces and energy decay estimates of the solutions to the problem (P) for linear damping and delay terms. To obtain global solutions to the problem (P), we use the argument combining the semigroup theory (see [12] and [14]) with the energy estimate method. To prove decay estimates, we use a multiplier method.

2. Preliminaries and main results

First assume the following hypotheses:

(H1)

$$|\mu_2| < \mu_1,\ |\widetilde{\mu_2}| < \widetilde{\mu_1}, |\widetilde{\widetilde{\mu_2}}| < \widetilde{\widetilde{\mu_1}}. \tag{1}$$

We first state some lemmas which will be needed later.

Lemma 1 (Sobolev-Poincaré's inequality)**.** *Let q be a number with $2 \leq q < +\infty$. Then there is a constant $c_* = c_*((0,1), q)$ such that*

$$\|\psi\|_q \leq c_* \|\psi_x\|_2 \quad \textit{for} \quad \psi \in H_0^1((0,1)).$$

Lemma 2 ([15], [16])**.** *Let $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+$ be a non increasing function and assume that there are two constants $\sigma > -1$ and $\omega > 0$ such that*

$$\int_S^{+\infty} \mathcal{E}^{1+\sigma}(t)\, dt \leq \frac{1}{\omega} \mathcal{E}^\sigma(0)\mathcal{E}(S), \quad 0 \leq S < +\infty, \tag{2}$$

then we have

$$\mathcal{E}(t) = 0 \quad \forall t \geq \frac{\mathcal{E}(0)^{\sigma}}{\omega|\sigma|} \qquad \textit{if} \quad -1 < \sigma < 0, \tag{3}$$

$$\mathcal{E}(t) \leq \mathcal{E}(0)\left(\frac{1+\sigma}{1+\omega\sigma t}\right)^{\frac{1}{\sigma}} \quad \forall t \geq 0, \qquad \textit{if} \quad \sigma > 0, \tag{4}$$

$$\mathcal{E}(t) \leq \mathcal{E}(0)e^{1-\omega t} \quad \forall t \geq 0, \qquad \textit{if} \quad \sigma = 0. \tag{5}$$

We introduce, as in [12], the new variables

$$\begin{aligned} z_1(x,\rho,t) &= \phi_t(x, t-\tau_1\rho), \quad x \in (0,L),\ \rho \in (0,1),\ t > 0,\\ z_2(x,\rho,t) &= \psi_t(x, t-\tau_2\rho), \quad x \in (0,L),\ \rho \in (0,1),\ t > 0,\\ z_3(x,\rho,t) &= \omega_t(x, t-\tau_3\rho), \quad x \in (0,L),\ \rho \in (0,1),\ t > 0. \end{aligned} \tag{6}$$

Then, we have

$$\tau_i z_{it}(x,\rho,t) + z_{i\rho}(x,\rho,t) = 0, \quad \text{in } (0,L)\times(0,1)\times(0,+\infty) \text{ for } i = 1,2,3. \tag{7}$$

Therefore, problem (P) takes the form:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - Gh(\varphi_x + \psi + l\omega)_x(x,t) - lEh(\omega_x - l\varphi)(x,t) \\ \qquad\qquad +\mu_1\varphi_t(x,t) + \mu_2 z_1(x,1,t) = 0, \\ \tau_1 z_{1t}(x,\rho,t) + z_{1\rho}(x,\rho,t) = 0, \\ \rho_2\psi_{tt}(x,t) - EI\psi_{xx}(x,t) + Gh(\varphi_x + \psi + l\omega)(x,t) + \widetilde{\mu_1}\psi_t(x,t) \\ \qquad\qquad +\widetilde{\mu_2} z_2(x,1,t) = 0, \\ \tau_2 z_{2t}(x,\rho,t) + z_{2\rho}(x,\rho,t) = 0, \\ \rho_1\omega_{tt}(x,t) - Eh(\omega_x - l\varphi)_x(x,t) + lGh(\varphi_x + \psi + l\omega)(x,t) + \widetilde{\widetilde{\mu_1}}\omega_t(x,t) \\ \qquad\qquad +\widetilde{\widetilde{\mu_2}} z_3(x,1,t) = 0, \\ \tau_3 z_{3t}(x,\rho,t) + z_{3\rho}(x,\rho,t) = 0. \end{cases} \tag{8}$$

The above system subjected to the following initial and boundary conditions

$$\begin{cases} \varphi(0,t) = \varphi(L,t) = \psi(0,t) = \psi(L,t) = \omega(0,t) = \omega(L,t), \qquad t > 0, \\ z_1(x,0,t) = \varphi_t(x,t), z_2(x,0,t) = \psi_t(x,t), z_3(x,0,t) = \omega_t(x,t), \\ \qquad\qquad\qquad\qquad\qquad\qquad x \in (0,L),\ t > 0, \\ \varphi(x,0) = \varphi_0, \varphi_t(x,0) = \varphi_1, \psi(x,0) = \psi_0, \psi_t(x,0) = \psi_1, \\ \omega(x,0) = \omega_0, \omega_t(x,0) = \omega_1, \quad x \in (0,L), \\ z_1(x,1,t) = f_1(x,t-\tau_1) \quad \text{in } (0,L)\times(0,\tau_1), \\ z_2(x,1,t) = f_2(x,t-\tau_2) \quad \text{in } (0,L)\times(0,\tau_2), \\ z_3(x,1,t) = f_3(x,t-\tau_3) \quad \text{in } (0,L)\times(0,\tau_3). \end{cases} \tag{9}$$

Let ξ_1, ξ_2 and ξ_3 be positive constants such that

$$\begin{cases} \tau_1|\mu_2| < \xi_1 < \tau_1(2\mu_1 - |\mu_2|), \\ \tau_2|\widetilde{\mu_2}| < \xi_2 < \tau_2(2\widetilde{\mu_1} - |\widetilde{\mu_2}|), \\ \tau_3|\widetilde{\widetilde{\mu_2}}| < \xi_3 < \tau_3(2\widetilde{\widetilde{\mu_1}} - |\widetilde{\widetilde{\mu_2}}|), \end{cases} \tag{10}$$

thanks to hypothesis $(H1)$. We define the energy associated to the solution of the problem (8) by the following formula:

$$\begin{aligned} \mathcal{E}(t) = {} & \frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x + \psi + l\omega\|_2^2 \\ & + \frac{Eh}{2}\|\omega_x - l\varphi\|_2^2 + \sum_{i=1}^{3}\frac{\xi_i}{2}\int_0^1 \|z_i(x,\rho,t)\|_2^2\,d\rho. \end{aligned} \tag{11}$$

We have the following theorem.

Theorem 1. *Let* $(\varphi_0,\, \varphi_1,\, f_1\,(.,-.\tau_1),\, \psi_0,\, \psi_1,\, f_2\,(.,-.\tau_2),\, \omega_0,\, \omega_1,\, f_3\,(.,-.\tau_3)) \in (H_0^1(0,L) \times L^2(0,L) \times L^2((0,L)\times(0,1)))^3$. *Assume that the hypothesis* $(H1)$ *holds. Then problem* (P) *admits a unique solution*

$$\begin{aligned} \varphi &\in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\ \psi &\in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)) \\ \omega &\in C([0,+\infty); H_0^1(0,L)) \cap C^1([0,+\infty); L^2(0,L)), \\ z_1, z_2, z_3 &\in C([0,+\infty); L^2((0,L)\times(0,1))). \end{aligned}$$

In addition, we have the following decay estimate:

$$\mathcal{E}(t) \le c\mathcal{E}(0)e^{-\omega t}, \qquad \forall\, t \ge 0, \tag{12}$$

where c and ω are positive constants, independent of the initial data.

We finish this section by giving an explicit upper bound for the derivative of the energy.

Lemma 3. *Let* $(\varphi, \psi, \omega, z_1, z_2, z_3)$ *be a solution of the problem* (8)*. Then, the energy functional defined by* (11) *satisfies*

$$\begin{aligned} \mathcal{E}'(t) \le {} & -\left(\mu_1 - \frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2 - \left(\widetilde{\mu_1} - \frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|\psi_t\|_2^2 \\ & - \left(\widetilde{\widetilde{\mu_1}} - \frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)\|\omega_t\|_2^2 - \left(\frac{\xi_1}{2\tau_1} - \frac{|\mu_2|}{2}\right)\|z_1(x,1,t)\|_2^2 \\ & - \left(\frac{\xi_2}{2\tau_2} - \frac{|\widetilde{\mu_2}|}{2}\right)\|z_2(x,1,t)\|_2^2 - \left(\frac{\xi_3}{2\tau_3} - \frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)\|z_3(x,1,t)\|_2^2. \end{aligned} \tag{13}$$

Proof: Multiplying the first equation in (8) by φ_t, the third equation by ψ_t, the fifth equation by ω_t, integrating over $(0,L)$ and using integration by parts, we get

$$
\begin{aligned}
&\frac{1}{2}\rho_1\frac{d}{dt}\|\varphi_t\|_2^2 - Gh\int_0^L(\varphi_x+\psi+l\omega)_x\varphi_t dx - lEh\int_0^L(\omega_x-l\varphi)\varphi_t dx + \mu_1\|\varphi_t\|_2^2\\
&+\mu_2\int_0^L z_1(x,1,t)\varphi_t dx = 0,\\
&\frac{1}{2}\rho_2\frac{d}{dt}\|\psi_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + Gh\int_0^L(\varphi_x+\psi+l\omega)\psi_t dx + \widetilde{\mu_1}\|\psi_t\|_2^2\\
&+\widetilde{\mu_2}\int_0^L z_1(x,1,t)\psi_t = 0,\\
&\frac{1}{2}\rho_1\frac{d}{dt}\|\omega_t\|_2^2 - Eh\int_0^L(\omega_x-l\varphi)_x\omega_t dx + lGh\int_0^L(\varphi_x+\psi+l\omega)\omega_t dx + \widetilde{\widetilde{\mu_1}}\|\omega_t\|_2^2\\
&+\widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t dx = 0.
\end{aligned}
$$

Then

(14)
$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\rho_1}{2}\|\varphi_t\|_2^2 + \frac{\rho_2}{2}\|\psi_t\|_2^2 + \frac{\rho_1}{2}\|\omega_t\|_2^2 + \frac{EI}{2}\|\psi_x\|_2^2 + \frac{Gh}{2}\|\varphi_x+\psi+l\omega\|_2^2\right.\\
&\left.+\frac{Eh}{2}\|\omega_x-l\varphi\|_2^2\right) + \mu_1\|\varphi_t\|_2^2 + \widetilde{\mu_1}\|\psi_t\|_2^2 + \widetilde{\widetilde{\mu_1}}\|\omega_t\|_2^2\\
&+\widetilde{\mu_2}\int_0^L z_1(x,1,t)\psi_t dx + \mu_2\int_0^L z_1(x,1,t)\varphi_t dx + \widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t dx = 0.
\end{aligned}
$$

Multiplying the equation in (7) by $\xi_i z_i$ and integrating over $(0,L)\times(0,1)$, obtain:

(15)
$$
\begin{aligned}
\xi_i 2\frac{d}{dt}\int_0^L\int_0^1 z_i^2(x,\rho,t)d\rho\,dx &= -\frac{\xi_i}{\tau_1}\int_0^L\int_0^1 z_i z_{i\rho}d\rho\,dx\\
&= \frac{\xi_i}{2\tau_i}\int_0^L\left(z_i^2(x,0,t)-z_i^2(x,1,t)\right)dx\\
&= \frac{\xi_i}{2\tau_i}\left[\|z_i^2(x,0,t)\|_2^2 - \|z_i(x,1,t)\|_2^2\right],
\end{aligned}
$$

where $z_1(x,0,t)=\varphi_t(x,t)$, $z_2(x,0,t)=\psi_t(x,t)$ and $z_3(x,0,t)=\omega_t(x,t)$. From (11), (14), (15) and using Young inequality we get

$$\begin{aligned}\mathcal{E}'(t)=&-\left(\mu_1-\frac{\xi_1}{2\tau_1}\right)\|\varphi_t\|_2^2-\left(\widetilde{\mu_1}-\frac{\xi_2}{2\tau_2}\right)\|\psi_t\|_2^2-\left(\widetilde{\widetilde{\mu_1}}-\frac{\xi_3}{2\tau_3}\right)\|\omega_t\|_2^2\\ &-\sum_{i=1}^{3}\frac{\xi_i}{2\tau_i}\|z_i(x,1,t)\|_2^2-\mu_2\int_0^L z_1(x,1,t)\varphi_t\,dx\\ &-\widetilde{\mu_2}\int_0^L z_1(x,1,t)\psi_t\,dx-\widetilde{\widetilde{\mu_2}}\int_0^L z_3(x,1,t)\omega_t\,dx.\end{aligned}\tag{16}$$

Due to Young's inequality, we have

$$\begin{aligned}&\int_0^L z_1(x,1,t)\varphi_t(x,t)\,dx\le\frac{1}{2}\|\varphi_t(x,t)\|_2^2+\frac{1}{2}\|z_1(x,1,t)\|_2^2\\ &\int_0^L z_2(x,1,t)\varphi_t(x,t)\,dx\le\frac{1}{2}\|\psi_t(x,t)\|_2^2+\frac{1}{2}\|z_2(x,1,t)\|_2^2\\ &\int_0^L z_3(x,1,t)\omega_t(x,t)\,dx\le\frac{1}{2}\|\omega_t(x,t)\|_2^2+\frac{1}{2}\|z_3(x,1,t)\|_2^2.\end{aligned}\tag{17}$$

Inserting (17) into (16), we obtain

$$\begin{aligned}\mathcal{E}'(t)\le&-\left(\mu_1-\frac{\xi_1}{2\tau_1}-\frac{|\mu_2|}{2}\right)\|\varphi_t\|_2^2-\left(\widetilde{\mu_1}-\frac{\xi_2}{2\tau_2}-\frac{|\widetilde{\mu_2}|}{2}\right)\|\psi_t\|_2^2\\ &-\left(\widetilde{\widetilde{\mu_1}}-\frac{\xi_3}{2\tau_3}-\frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)\|\omega_t\|_2^2-\left(\frac{\xi_1}{2\tau_1}-\frac{|\mu_2|}{2}\right)\|z_1(x,1,t)\|_2^2\\ &-\left(\frac{\xi_2}{2\tau_2}-\frac{|\widetilde{\mu_2}|}{2}\right)\|z_2(x,1,t)\|_2^2-\left(\frac{\xi_3}{2\tau_3}-\frac{|\widetilde{\widetilde{\mu_2}}|}{2}\right)\|z_3(x,1,t)\|_2^2.\end{aligned}$$

This completes the proof of the lemma. □

3. Global existence

In this section we will give well-posedness results for problem (8) and (9) using semigroup theory. Let us introduce the semigroup representation of the Bresse system (8) and (9). Let $U=(\varphi,\varphi_t,z_1,\psi,\psi_t,z_2,\omega,\omega_t,z_3)^T$ and rewrite (8) and (9) as

$$\begin{cases}U'=\mathcal{A}U,\\ U(0)=(\varphi_0,\varphi_1,f_1(.,-.\tau_1),\psi_0,\psi_1,f_2(.,-.\tau_2),\omega_0,\omega_1,f_3(.,-.\tau_3)),\end{cases}\tag{18}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix}\varphi\\ u\\ z_1\\ \psi\\ v\\ z_2\\ \omega\\ \widetilde{\omega}\\ z_3\end{pmatrix}=\begin{pmatrix} u\\ \dfrac{Gh}{\rho_1}(\varphi_x+\psi+l\omega)_x+\dfrac{lEh}{\rho_1}(\omega_x-l\varphi)-\dfrac{\mu_1}{\rho_1}u-\dfrac{\mu_2}{\rho_1}z_1(.,1)\\ -(1/\tau_1)z_{1\rho}\\ v\\ \dfrac{EI}{\rho_2}\psi_{xx}-\dfrac{Gh}{\rho_2}(\varphi_x+\psi+l\omega)-\dfrac{\widetilde{\mu_1}}{\rho_2}v-\dfrac{\widetilde{\mu_2}}{\rho_2}z_2(.,1)\\ -(1/\tau_2)z_{2\rho}\\ \widetilde{\omega}\\ \dfrac{Eh}{\rho_1}(\omega_x-l\varphi)_x-\dfrac{lGh}{\rho_1}(\varphi_x+\psi+l\omega)-\dfrac{\widetilde{\widetilde{\mu_1}}}{\rho_1}\widetilde{\omega}-\dfrac{\widetilde{\widetilde{\mu_2}}}{\rho_1}z_3(.,1)\\ -(1/\tau_3)z_{3\rho}\end{pmatrix}$$

with domain

$$\begin{aligned}D(\mathcal{A})=\{&(\varphi,u,z_1,\psi,v,z_2,\omega,\widetilde{\omega},z_3)^T \text{ in } H: u=z_1(.,0),\\ &v=z_2(.,0),\widetilde{\omega}=z_3(.,0), \text{ in } (0,L)\},\end{aligned}\tag{19}$$

where

$$H=(H^2(0,L)\cap H_0^1(0,L)\times H_0^1(0,L)\times L^2(0,L,H^1(0,1)))^3.$$

Now, the energy space $\mathcal{H}$ is defined as

$$\mathcal{H}=H_0^1(0,L)\times L^2(0,L)\times L^2((0,L)\times(0,1)).$$

For $U=(\varphi,u,z_1,\psi,v,z_2,\omega,\widetilde{\omega},z_3)^T,\overline{U}=(\overline{\varphi},\overline{u},\overline{z}_1,\overline{\psi},\overline{v},\overline{z}_2,\overline{\omega},\overline{\widetilde{\omega}},\overline{z}_3)^T$ and for ξ_i positive constants satisfying (10), we define the following inner product in $\mathcal{H}$

$$\begin{aligned}\langle U,\overline{U}\rangle_{\mathcal{H}}=\int_0^L\Big(&\rho_1u\overline{u}+\rho_2v\overline{v}+\rho_1\widetilde{\omega}\overline{\widetilde{\omega}}+EI\psi_x\overline{\psi}_x\\ &+Gh(\varphi_x+\psi+l\omega)(\overline{\varphi}_x+\overline{\psi}+l\overline{\omega})\\ &+Eh(\omega_x-l\varphi)(\overline{\omega}_x-l\overline{\varphi})+\sum_{i=1}^3\xi_i\int_0^1 z_i\overline{z}_i\,d\rho\Big)\,dx.\end{aligned}$$

We show that the operator $\mathcal{A}$ generates a C_0-semigroup in $\mathcal{H}$. In this step, we prove that the operator $\mathcal{A}$ is dissipative. Let $U=(\varphi,u,z_1,\psi,v,z_2,\omega,\widetilde{\omega},z_3)^T$. Using (18), (13) and the fact that

$$\mathcal{E}(t)=\frac{1}{2}\|U\|^2_{\mathcal{H}},\tag{20}$$

we get

$$
\begin{aligned}
\langle \mathcal{A}U, U\rangle_{\mathcal{H}} &= -\mu_1 \int_0^L u^2\,dx - \widetilde{\mu_1} \int_0^L v^2\,dx - \widetilde{\widetilde{\mu_1}} \int_0^L \widetilde{\omega}^2\,dx \\
&\quad - \mu_2 \int_0^L z_1(x,1)u\,dx - \widetilde{\mu_2} \int_0^L z_2(x,1)v\,dx - \widetilde{\widetilde{\mu_2}} \int_0^L z_3(x,1)\widetilde{\omega}\,dx \\
&\quad - \sum_{i=1}^{3} \frac{\xi_i}{\tau_i} \int_0^L \int_0^1 z_i(x,\rho) z_{i\rho}(x,\rho)\,d\rho\,dx. \\
&\le 0.
\end{aligned} \tag{21}
$$

Consequently, the operator $\mathcal{A}$ is dissipative. Now, we will prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. For this purpose, let $(f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8, f_9)^T \in \mathcal{H}$, we seek $U = (\varphi, u, z_1, \psi, v, z_2, \omega, \widetilde{\omega}, z_3)^T \in D(\mathcal{A})$ solution of the following system of equations

$$
\begin{cases}
\lambda\varphi - u = f_1, \\
\lambda u - \dfrac{Gh}{\rho_1}(\varphi_x + \psi + l\omega)_x - \dfrac{lEh}{\rho_1}(\omega_x - l\varphi) + \dfrac{\mu_1}{\rho_1} u + \dfrac{\mu_2}{\rho_1} z_1(.,1) = f_2, \\
\lambda z_1 + (1/\tau_1) z_{1\rho} = f_3, \\
\lambda\psi - v = f_4, \\
\lambda v - \dfrac{EI}{\rho_2}\psi_{xx} + \dfrac{Gh}{\rho_2}(\varphi_x + \psi + l\omega) + \dfrac{\widetilde{\mu_1}}{\rho_2} v + \dfrac{\widetilde{\mu_2}}{\rho_2} z_2(.,1) = f_5, \\
\lambda z_2 + (1/\tau_2) z_{2\rho} = f_6, \\
\lambda\omega - \widetilde{\omega} = f_7, \\
\lambda\widetilde{\omega} - \dfrac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \dfrac{lGh}{\rho_1}(\varphi_x + \psi + l\omega) + \dfrac{\widetilde{\widetilde{\mu_1}}}{\rho_1}\widetilde{\omega} + \dfrac{\widetilde{\widetilde{\mu_2}}}{\rho_1} z_3(.,1) = f_8, \\
\lambda z_3 + (1/\tau_3) z_{3\rho} = f_9.
\end{cases} \tag{22}
$$

Suppose that we have found φ, ψ and ω. Therefore, the first, the fourth and the seventh equation in (22) give

$$
\begin{cases}
u = \lambda\varphi - f_1, \\
v = \lambda\psi - f_4, \\
\widetilde{\omega} = \lambda\omega - f_7.
\end{cases} \tag{23}
$$

It is clear that $u \in H_0^1(0,L), v \in H_0^1(0,L)$ and $\omega \in H_0^1(0,L)$. Furthermore, by (22) we can find $z_i\,(i = 1,2,3)$ as

$$
z_1(x,0) = u(x), z_2(x,0) = v(x), z_3(x,0) = \widetilde{\omega}(x), \quad \text{for } x \in (0,L). \tag{24}
$$

Following the same approach as in [12], we obtain, by using equations for z_i in (22),

$$\begin{aligned} z_1(x,\rho) &= u(x)e^{-\lambda\tau_1\rho} + \tau_1 e^{-\lambda\tau_1\rho}\int_0^\rho f_3(x,s)e^{\lambda\tau_1 s}\,ds,\\ z_2(x,\rho) &= v(x)e^{-\lambda\tau_2\rho} + \tau_2 e^{-\lambda\tau_2\rho}\int_0^\rho f_6(x,s)e^{\lambda\tau_2 s}\,ds,\\ z_3(x,\rho) &= \widetilde{\omega}(x)e^{-\lambda\tau_3\rho} + \tau_3 e^{-\lambda\tau_3\rho}\int_0^\rho f_9(x,s)e^{\lambda\tau_3 s}\,ds. \end{aligned}$$

From (23), we obtain

$$(25)\qquad \begin{cases} z_1(x,\rho) &= \lambda\varphi(x)e^{-\lambda\tau_1\rho} - f_1 e^{-\lambda\tau_1\rho} + \tau_1 e^{-\lambda\tau_1\rho}\int_0^\rho f_3(x,s)e^{\lambda\tau_1 s}\,ds,\\ z_2(x,\rho) &= \lambda\psi(x)e^{-\lambda\tau_2\rho} - f_4 e^{-\lambda\tau_2\rho} + \tau_2 e^{-\lambda\tau_2\rho}\int_0^\rho f_6(x,s)e^{\lambda\tau_2 s}\,ds,\\ z_3(x,\rho) &= \lambda\omega(x)e^{-\lambda\tau_3\rho} - f_7 e^{-\lambda\tau_3\rho} + \tau_3 e^{-\lambda\tau_3\rho}\int_0^\rho f_9(x,s)e^{\lambda\tau_3 s}\,ds. \end{cases}$$

By using (23) and (22) the functions φ, ψ and ω satisfying the following system

$$(26)\qquad \begin{cases} \lambda^2\varphi - \frac{Gh}{\rho_1}(\varphi_x+\psi+l\omega)_x - \frac{lEh}{\rho_1}(\omega_x - l\varphi) + \frac{\mu_1}{\rho_1}u + \frac{\mu_2}{\rho_1}z_1(.,1) = f_2 + \lambda f_1,\\ \lambda^2\psi - \frac{EI}{\rho_2}\psi_{xx} + \frac{Gh}{\rho_2}(\varphi_x+\psi+l\omega) + \frac{\widetilde{\mu_1}}{\rho_2}v + \frac{\widetilde{\mu_2}}{\rho_2}z_2(.,1) = f_5 + \lambda f_4,\\ \lambda^2\omega - \frac{Eh}{\rho_1}(\omega_x - l\varphi)_x + \frac{lGh}{\rho_1}(\varphi_x+\psi+l\omega) + \frac{\widetilde{\widetilde{\mu_1}}}{\rho_1}\widetilde{\omega} + \frac{\widetilde{\widetilde{\mu_2}}}{\rho_1}z_3(.,1) = f_8 + \lambda f_7. \end{cases}$$

Solving system (26) is equivalent to finding $(\varphi,\psi,\omega) \in (H^2 \cap H_0^1(0,L))^3$ such that

$$(27)\qquad \begin{cases} \int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)w_x - lEh(\omega_x - l\varphi)w + \mu_1 uw + \mu_2 z_1(.,1)w)\,dx\\ = \int_0^L \rho_1(f_2+\lambda f_1)w\,dx,\\ \int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x + Gh(\varphi_x+\psi+l\omega)\chi + \widetilde{\mu_1}v\chi + \widetilde{\mu_2}z_2(.,1)\chi)\,dx\\ = \int_0^L \rho_2(f_5+\lambda f_4)\chi\,dx,\\ \int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)\zeta_x + lGh(\varphi_x+\psi+l\omega)\zeta + \widetilde{\widetilde{\mu_1}}\widetilde{\omega}\zeta + \widetilde{\widetilde{\mu_2}}z_3(.,1)\zeta)\,dx\\ = \int_0^L \rho_1(f_8+\lambda f_7)\zeta\,dx \end{cases}$$

for all $(w,\chi,\zeta) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$. From (25), we have

$$\begin{cases} z_1(x,1) = \lambda\varphi(x)e^{-\lambda\tau_1} - f_1 e^{-\lambda\tau_1} + \tau_1 e^{-\lambda\tau_1}\displaystyle\int_0^1 f_3(x,s)e^{\lambda\tau_1 s}\,ds,\\ z_2(x,1) = \lambda\psi(x)e^{-\lambda\tau_2} - f_4 e^{-\lambda\tau_2} + \tau_2 e^{-\lambda\tau_2}\displaystyle\int_0^1 f_6(x,s)e^{\lambda\tau_2 s}\,ds,\\ z_3(x,1) = \lambda\omega(x)e^{-\lambda\tau_3} - f_7 e^{-\lambda\tau_3} + \tau_3 e^{-\lambda\tau_3}\displaystyle\int_0^1 f_9(x,s)e^{\lambda\tau_3 s}\,ds. \end{cases}$$

Consequently, problem (27) is equivalent to the problem

$$a((\varphi,\psi,\omega),(w,\chi,\zeta)) = L(w,\chi,\zeta) \tag{28}$$

where the bilinear form $a : [H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)]^2 \to \mathbb{R}$ and the linear form $L : H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L) \to \mathbb{R}$ are defined by

$$\begin{aligned} a((\varphi,\psi,\omega),(w,\chi,\zeta)) &= \int_0^L (\rho_1\lambda^2\varphi w + Gh(\varphi_x+\psi+l\omega)_x(w_x+\chi+l\zeta))\,dx\\ &+ \int_0^L (\rho_2\lambda^2\psi\chi + EI\psi_x\chi_x)\,dx + \int_0^L (\rho_1\lambda^2\omega\zeta + Eh(\omega_x - l\varphi)(\zeta_x - lw)\,dx\\ &+ \int_0^L \lambda\varphi(\mu_1+\mu_2 e^{-\lambda\tau_1})w\,dx\\ &+ \int_0^L \lambda\varphi(\widetilde{\mu_1}+\widetilde{\mu_2}e^{-\lambda\tau_2})w\,dx + \int_0^L \lambda\varphi(\widetilde{\widetilde{\mu_1}}+\widetilde{\widetilde{\mu_2}}e^{-\lambda\tau_3})w\,dx \end{aligned}$$

and

$$\begin{aligned} L(w,\chi,\zeta) &= \int_0^L (\mu_1 f_1 - \mu_2 M_1)w\,dx + \int_0^L (\widetilde{\mu_1}f_4 - \widetilde{\mu_2}M_2)\chi\,dx\\ &+ \int_0^L (\widetilde{\widetilde{\mu_1}}f_7 - \widetilde{\widetilde{\mu_2}}M_3)\zeta\,dx + \int_0^L \rho_1(f_2+\lambda f_1)w\,dx\\ &+ \int_0^L \rho_2(f_5+\lambda f_4)\chi\,dx + \int_0^L \rho_1(f_8+\lambda f_7)\zeta\,dx. \end{aligned}$$

It is easy to verify that a is continuous and coercive, and L is continuous. So applying the Lax-Milgram theorem, we deduce that for all $(w,\chi,\zeta) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$ problem (28) admits a unique solution $(\varphi,\psi,\omega) \in H_0^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$. Applying the classical elliptic regularity, it follows from (27) that $(\varphi,\psi,\omega) \in H^2(0,L) \times H^2(0,L) \times H^2(0,L)$. Therefore, the operator $\lambda I - A$ is surjective for any $\lambda > 0$. Consequently, the existence result of Theorem 1 follows from the Hille-Yosida theorem. $\square$

4. Asymptotic behavior

First we state and prove a lemma that will be needed to establish the asymptotic behavior.

Lemma 4. *There exists a positive constant C such that the following inequality holds for every $(\varphi, \psi, \omega) \in (H_0^1(0,L))^3$*

$$\begin{aligned}(29)\quad &\int_0^L (|\varphi_x|^2 + |\psi_x|^2 + |\omega_x|^2)\,dx \\ &\qquad \leq C \int_0^L (EI|\psi_x|^2 + Gh|\varphi_x + \psi + l\omega|^2 + Eh|\omega_x - l\varphi|^2)\,dx \leq \mathcal{E}(t).\end{aligned}$$

Proof: We will argue by contradiction. Indeed, let us suppose that (29) is not true. So, we can find a sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ in $(H_0^1(0,L))^3$ satisfying

$$(30)\qquad \int_0^L (EI|\psi_{\nu x}|^2 + Gh|\varphi_{\nu x} + \psi + l\omega_\nu|^2 + Eh|\omega_{\nu x} - l\varphi_\nu|^2)\,dx \leq \frac{1}{\nu}$$

and

$$(31)\qquad \int_0^L (|\varphi_{\nu x}|^2 + |\psi_{\nu x}|^2 + |\omega_{\nu x}|^2)\,dx = 1.$$

From (31), the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ is bounded in $(H_0^1(0,L))^3$. Since the embedding $H_0^1(0,L) \hookrightarrow L^2(0,L)$ is compact, then the sequence $\{(\varphi_\nu, \psi_\nu, \omega_\nu)\}_{\nu\in\mathbb{N}}$ converges strongly in $(L^2(0,L))^3$.
From (30)

$$(32)\qquad \psi_{\nu x} \to 0 \text{ strongly in } L^2(0,L).$$

Using Poincaré's inequality we can conclude that

$$(33)\qquad \psi_\nu \to 0 \text{ strongly in } L^2(0,L).$$

Now, setting $\varphi_\nu \to \varphi$ and $\omega_\nu \to \omega$ strongly in $L^2(0,L)$.
From (30), we have

$$(34)\qquad \varphi_{\nu x} + \psi_\nu + l\omega_\nu \to 0 \text{ strongly in } L^2(0,L).$$

Then

$$(35)\qquad \varphi_{\nu x} + \psi_\nu + l\omega_\nu = \varphi_{\nu x} + \psi_\nu + l(\omega_\nu - \omega) + l\omega \to 0 \text{ strongly in } L^2(0,L)$$

which implies that

$$(36)\qquad \varphi_{\nu x} \to -l\omega \text{ strongly in } L^2(0,L).$$

Then, $\{\varphi_\nu\}_n$ is a Cauchy sequence in $H^1(0,L)$. Therefore $\{\varphi_\nu\}_n$ converges to a function φ_1 in $H^1(0,L)$. Consequently $\{\varphi_\nu\}_n$ converges to φ_1 in $L^2(0,L)$. Thus by the uniqueness of the limit $\varphi_1=\varphi$. Moreover $\varphi\in H_0^1(0,L)$.
From (36) we deduce that

$$\varphi_x + l\omega = 0 \text{ a.e } x\in(0,L). \tag{37}$$

Similarly, we have

$$\omega_x - l\varphi = 0 \text{ a.e } x\in(0,L) \tag{38}$$

and $\omega\in H_0^1(0,L)$. (37) and (38) provides us $\varphi=\omega=0$, contradicting (31). $\square$

From now on, we denote by c various positive constants which may be different at different occurrences. Multiplying the first equation in (8) by $\mathcal{E}^q\varphi$, the third equation by $\mathcal{E}^q\psi$ and the fifth equation by $\mathcal{E}^q\omega$ we obtain

$$\begin{aligned}0 = \int_S^T \mathcal{E}^q \int_0^L \varphi\,(\rho_1\varphi_{tt} - Gh(\varphi_x+\psi+l\omega)_x - lEh(\omega_x-l\varphi)\\ +\mu_1\varphi_t + \mu_2 z_1(x,1,t))\;dx\,dt,\end{aligned}$$

$$\begin{aligned}0 \;=\;& \left[\mathcal{E}^q\rho_1\int_0^L \varphi\varphi_t dx\right]_S^T - \int_S^T \rho_1 q\mathcal{E}^{'}\mathcal{E}^{q-1}\int_0^L \varphi\varphi_t\,dx\,dt - \rho_1\int_S^T \mathcal{E}^q\|\varphi_t\|_2^2\,dt\\ &-\int_S^T \mathcal{E}^q\int_0^L \varphi_x Gh(\varphi_x+\psi+l\omega)\,dx\,dt - \int_S^T \mathcal{E}^q\int_0^L \varphi(lEh)(\omega_x-l\varphi)\,dx\,dt\\ &+\mu_1\int_S^T \mathcal{E}^q\int_0^L \varphi_t\varphi\,dx\,dt + \mu_2\int_S^T\mathcal{E}^q\int_0^L \varphi z_1(x,1,t)\,dx\,dt,\end{aligned}$$

$$0 = \int_S^T \mathcal{E}^q\int_0^L \psi\,(\rho_2\psi_{tt} - EI\psi_{xx} + Gh(\varphi_x+\psi+l\omega) + \widetilde{\mu_1}\psi_t + \widetilde{\mu_2}z_2(x,1,t))\;dx\,dt,$$

$$\begin{aligned}0 \;=\;& \left[\mathcal{E}^q\rho_2\int_0^L \psi\psi_t dx\right]_S^T - \int_S^T \rho_2 q\mathcal{E}^{'}\mathcal{E}^{q-1}\int_0^L \psi\psi_t\,dx\,dt - \rho_2\int_S^T \mathcal{E}^q\|\psi_t\|_2^2\,dt\\ &+\int_S^T \mathcal{E}^q EI\|\psi_x\|_2^2\,dt + \int_S^T\mathcal{E}^q\int_0^L \psi Gh(\varphi_x+\psi+l\omega))\,dx\,dt\\ &+\widetilde{\mu_1}\int_S^T \mathcal{E}^q\int_0^L \psi\psi_t\,dx\,dt + \widetilde{\mu_2}\int_S^T\mathcal{E}^q\int_0^L \psi z_2(x,1,t)\,dx\,dt,\end{aligned}$$

$$\begin{aligned}0 = \int_S^T \mathcal{E}^q\int_0^L \omega\Big(\rho_1\omega_{tt} - Eh(\omega_x-l\varphi)_x\\ + lGh(\varphi_x+\psi+l\omega) + \widetilde{\widetilde{\mu_1}}\omega_t + \widetilde{\widetilde{\mu_2}}z_3(x,1,t)\Big)\,dx\,dt,\end{aligned}$$

$$\begin{aligned} 0 \;=\;& \left[\mathcal{E}^q\rho_1\int_0^L \omega\omega_t dx\right]_S^T - \int_S^T \rho_1 q\mathcal{E}^{'}\mathcal{E}^{q-1}\int_0^L \omega\omega_t\,dx\,dt \\ & -\rho_1\int_S^T \mathcal{E}^q\|\omega_t\|_2^2\,dt + \int_S^T \mathcal{E}^q\int_0^L Eh\omega_x(\omega_x - l\varphi)\,dx\,dt \\ & + \int_S^T \mathcal{E}^q\int_0^L \omega(lGh)(\varphi_x+\psi+l\omega)\,dx\,dt \\ & +\widetilde{\widetilde{\mu_1}}\int_S^T \mathcal{E}^q\int_0^L \omega_t\omega\,dx\,dt + \widetilde{\widetilde{\mu_2}}\int_S^T \mathcal{E}^q\int_0^L \omega z_3(x,1,t)\,dx\,dt. \end{aligned}$$

Taking their sum, we obtain

$$\begin{aligned} 0 =& \left[\mathcal{E}^q\rho_1\int_0^L \varphi\varphi_t dx\right]_S^T + \left[\mathcal{E}^q\rho_2\int_0^L \psi\psi_t dx\right]_S^T + \left[\mathcal{E}^q\rho_1\int_0^L \omega\omega_t dx\right] \\ & - \int_S^T \rho_1 q\mathcal{E}^{'}\mathcal{E}^{q-1}\int_0^L \left(\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1\omega\omega_t\right)\,dx\,dt \\ & - 2\rho_1\int_S^T \mathcal{E}^q\|\varphi_t\|_2^2\,dt - 2\rho_2\int_S^T \mathcal{E}^q\|\psi_t\|_2^2\,dt - 2\rho_1\int_S^T \mathcal{E}^q\|\omega_t\|_2^2\,dt \\ & + \int_S^T \mathcal{E}^q\left(\rho_1\|\varphi_t\|_2^2 + \rho_2\|\psi_t\|_2^2 + \rho_1\|\omega_t\|_2^2 + Gh\|\varphi_x+\psi+l\omega\|_2^2\right. \\ & \left.+EI\|\psi_x\|_2^2 + Eh\|\omega_x - l\psi\|_2^2\right) \\ & + \mu_1\int_S^T \mathcal{E}^q\int_0^L \varphi_t\varphi\,dx\,dt + \mu_2\int_S^T \mathcal{E}^q\int_0^L \varphi z_1(x,1,t)\,dx\,dt \\ & + \widetilde{\mu_1}\int_S^T \mathcal{E}^q\int_0^L \psi\psi_t\,dx\,dt + \widetilde{\mu_2}\int_S^T \mathcal{E}^q\int_0^L \psi z_2(x,1,t)\,dx\,dt \\ & + \widetilde{\widetilde{\mu_1}}\int_S^T \mathcal{E}^q\int_0^L \omega_t\omega\,dx\,dt + \widetilde{\widetilde{\mu_2}}\int_S^T \mathcal{E}^q\int_0^L \omega z_3(x,1,t)\,dx\,dt. \end{aligned} \tag{39}$$

Similarly, we multiply the equation of (7) by $\mathcal{E}^q\xi_i e^{-2\tau_i\rho}z_i(x,\rho,t)$ and get

$$\begin{aligned} 0 =& \int_S^T \mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}\xi_i z_i(\tau_i z_{it} + z_{i\rho})\,d\rho\,dx\,dt \\ =& \left[\frac{1}{2}\xi_i\tau_i\mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho dx\right]_S^T \\ & - \frac{\tau_i\xi_i}{2}\int_S^T q\mathcal{E}^{q-1}\mathcal{E}^{'}\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\,dt \\ & + \int_S^T \mathcal{E}^q\xi_i\int_0^L\int_0^1 \frac{e^{-2\tau_i\rho}}{2}\frac{d}{d\rho}(z_i^2)\,d\rho\,dx\,dt, \end{aligned} \tag{40}$$

$$
\begin{aligned}
0 = &\left[\frac{1}{2}\xi_i\tau_i\mathcal{E}^q \int_0^L \int_0^1 e^{-2\tau_i\rho} z_i^2 \, d\rho dx\right]_S^T \\
&- \frac{\tau_i\xi_i}{2} \int_S^T q\mathcal{E}^{q-1}E^{'} \int_0^L \int_0^1 e^{-2\tau_i\rho} z_i^2 \, d\rho \, dx \, dt \\
&+ \frac{\xi_i}{2} \int_S^T \mathcal{E}^q \int_0^L \int_0^1 \left[\frac{d}{d\rho}\left(e^{-2\tau_i\rho} z_i^2\right) + 2\tau_i e^{-2\tau_i\rho} z_i^2\right] d\rho \, dx \, dt, \\
0 = &\left[\frac{1}{2}\xi_i\tau_i\mathcal{E}^q \int_0^L \int_0^1 e^{-2\tau_i\rho} z_i^2 \, d\rho dx\right]_S^T \\
&- \frac{\tau_i\xi_i}{2} \int_S^T q\mathcal{E}^{q-1}\mathcal{E}^{'} \int_0^L \int_0^1 e^{-2\tau_i\rho} z_i^2 \, d\rho \, dx \, dt \\
&+ \frac{\xi_i}{2} \int_S^T \mathcal{E}^q \int_0^L \left[e^{-2\tau_i} z_i^2(x,1,t) - z_i^2(x,0,t)\right] dx \, dt \\
&+ \xi_i\tau_i \int_S^T \int_0^L \int_0^1 e^{-2\tau_i\rho} z_i^2 \, d\rho \, dx \, dt.
\end{aligned}
$$

Recalling the definition of $\mathcal{E}$ and from (39), (40), we get

(41)

$$
\begin{aligned}
A\int_S^T \mathcal{E}^{q+1} \, dt \leq &- \left[\rho_1\mathcal{E}^q \int_0^L \varphi\varphi_t \, dx\right]_S^T - \left[\rho_2\mathcal{E}^q \int_0^L \psi\psi_t dx\right]_S^T \\
&- \left[\rho_1\mathcal{E}^q \int_0^L \omega\omega_t \, dx\right]_S^T \\
&+ \int_S^T q\mathcal{E}^{'}\mathcal{E}^{q-1} \int_0^L \left(\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1\omega\omega_t\right) dx \, dt \\
&+ 2\int_S^T \mathcal{E}^q \left(\rho_1\|\varphi_t\|_2^2 + \rho_2\|\psi_t\|_2^2 + \rho_1\|\omega_t\|_2^2\right) dt \\
&- \mu_1 \int_S^T \mathcal{E}^q \int_0^L \varphi_t\varphi \, dx \, dt - \mu_2 \int_S^T \mathcal{E}^q \int_0^L \varphi z_1(x,1,t) \, dx \, dt \\
&- \widetilde{\mu_1} \int_S^T \mathcal{E}^q \int_0^L \psi\psi_t \, dx \, dt - \widetilde{\mu_2} \int_S^T \mathcal{E}^q \int_0^L \psi z_2(x,1,t) \, dx \, dt \\
&- \widetilde{\widetilde{\mu_1}} \int_S^T \mathcal{E}^q \int_0^L \omega_t\omega \, dx \, dt - \widetilde{\widetilde{\mu_2}} \int_S^T \mathcal{E}^q \int_0^L \omega z_3(x,1,t) \, dx \, dt \\
&- \sum_{i=1}^{3} \left[\frac{1}{2}\xi_i\tau_i\mathcal{E}^q \int_0^L \int_0^1 e^{-2\tau_i\rho} z_i^2 \, d\rho dx\right]_S^T
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{i=1}^{3}\frac{\tau_i\xi_i}{2}\int_S^T q\mathcal{E}^{q-1}E^{'}\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,d\rho\,dx\,dt\\
&-\sum_{i=1}^{3}\frac{\xi_i}{2}\int_S^T \mathcal{E}^{q}e^{-2\tau_i}\int_0^L z_i^2(x,1,t)\,dx\,dt\\
&+\sum_{i=1}^{3}\frac{\xi_i}{2}\int_S^T \mathcal{E}^{q}\|z_i(x,0,t)\|_2^2\,dt
\end{aligned}
$$

where $A = 2\min\{1, 2\tau_1e^{-2\tau_1}, 2\tau_2e^{-2\tau_2}, 2\tau_3e^{-2\tau_3}\}$. Using the Young and Sobolev-Poincaré inequalities and Lemma (4), we find that

$$
\begin{aligned}
-\left[\mathcal{E}^q\int_0^L \varphi\varphi_t\,dx\right]_S^T &= \mathcal{E}^q(S)(S)\int_0^L \varphi(S)\varphi_t(S)\,dx - \mathcal{E}^q(T)\int_0^L \varphi(T)\varphi_t(T)\,dx\\
&\leq C\mathcal{E}^{q+1}(S)
\end{aligned}
$$

$$
\left|\int_S^T (q\mathcal{E}'\mathcal{E}^{q-1})\int_0^L(\rho_1\varphi\varphi_t+\rho_2\psi\psi_t+\rho_1\omega\omega_t)\,dx\,dt\right| \leq c\int_S^T(-\mathcal{E}')\mathcal{E}^q\,dt \leq c\mathcal{E}^{q+1}(S),
$$

$$
\left|\frac{1}{2}\xi_i\tau_i\mathcal{E}^q\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2\,dx\,d\rho\right| \leq c\mathcal{E}(S)^{q+1} \quad \forall t\geq S,
$$

$$
\int_S^T \mathcal{E}^q\int_0^L u_t^2\,dx\,dt \leq c\int_S^T \mathcal{E}^q(-\mathcal{E}')\,dt \leq c\mathcal{E}^{q+1}(S),
$$

$$
\int_S^T \mathcal{E}^q\xi_i\int_0^L e^{-2\tau_i}z_i^2(x,1,t)\,dx\,dt \leq c\int_S^T \mathcal{E}^q(-\mathcal{E}')\,dt \leq c\mathcal{E}^{q+1}(S),
$$

$$
\begin{aligned}
\frac{1}{2}\int_S^T \mathcal{E}^q\xi_i\int_0^L z_i^2(x,0,t)\,dx\,dt &= \frac{1}{2}\int_S^T \mathcal{E}^q\xi_i\int_0^L \varphi'^2\,dx\,dt\\
&\leq c\mathcal{E}^{q+1}(S),
\end{aligned}
$$

$$
\left|\frac{\tau_i\xi_i}{2}\int_S^T q\mathcal{E}^{q-1}\mathcal{E}'\int_0^L\int_0^1 e^{-2\tau_i\rho}z_i^2dx\,d\rho\,dt\right| \leq c\int_S^T(-\mathcal{E}')\mathcal{E}^q\,dt \leq c\mathcal{E}^{q+1}(S),
$$

$$
\begin{aligned}
\left|\int_S^T \mathcal{E}^q \int_0^L \varphi\varphi_t\,dx\,dt\right| &\le \varepsilon \int_S^T \mathcal{E}^q \int_0^L \varphi^2\,dx\,dt + c(\varepsilon)\int_S^T \mathcal{E}^q \int_0^L \varphi_t^2\,dx\,dt \\
&\le \varepsilon c \int_0^L \mathcal{E}^{q+1}\,dt + c(\varepsilon)\int_S^T \mathcal{E}^q \int_0^L \varphi_t^2\,dx\,dt \\
&\le \varepsilon c \int_0^L \mathcal{E}^{q+1}\,dt + c(\varepsilon)\int_S^T \mathcal{E}^q(-\mathcal{E}')\,dt \\
&\le \varepsilon c \int_0^L \mathcal{E}^{q+1}\,dt + c(\varepsilon)\mathcal{E}(S)^{q+1}
\end{aligned} \tag{42}
$$

and

$$
\begin{aligned}
\left|\int_S^T \mathcal{E}^q \int_0^L \varphi z_1(x,1,t)\,dx\,dt\right| &\le \varepsilon_1 \int_S^T \mathcal{E}^q \int_0^L \varphi^2\,dx\,dt + c(\varepsilon_1)\int_S^T \mathcal{E}^q \int_0^L z_1(x,1,t)^2\,dx\,dt \\
&\le \varepsilon_1 c \int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1)\int_S^T \mathcal{E}^q \int_0^L z_1(x,1,t)^2\,dx\,dt \\
&\le \varepsilon_1 c \int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1)\int_S^T \mathcal{E}^q(-\mathcal{E}')\,dt \\
&\le \varepsilon_1 c \int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1)\mathcal{E}^{q+1}(S).
\end{aligned} \tag{43}
$$

$$
\left|\int_S^T \mathcal{E}^q \int_0^L \psi\psi_t\,dx\,dt\right| \le \varepsilon' c \int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon')\mathcal{E}(S)^{q+1}, \tag{44}
$$

$$
\left|\int_S^T \mathcal{E}^q \int_0^L \psi z_2(x,1,t)\,dx\,dt\right| \le \varepsilon_1' c \int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1')\mathcal{E}(S)^{q+1}, \tag{45}
$$

$$
\left|\int_S^T \mathcal{E}^q \int_0^L \omega\omega_t\,dx\,dt\right| \le \varepsilon'' c \int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon'')\mathcal{E}(S)^{q+1}, \tag{46}
$$

$$
\left|\int_S^T \mathcal{E}^q \int_0^L \omega z_3(x,1,t)\,dx\,dt\right| \le \varepsilon_1'' c \int_S^T \mathcal{E}^{q+1}\,dt + c(\varepsilon_1'')\mathcal{E}(S)^{q+1}. \tag{47}
$$

Choosing $\varepsilon, \varepsilon_1, \varepsilon', \varepsilon_1', \varepsilon''$ and ε_1'' small enough, we deduce from (41), (42), (43), (44), (45), (46) and (47) that

$$
\int_S^T \mathcal{E}^{q+1}\,dt \le c\mathcal{E}^{q+1}(S),
$$

where c is a positive constant independent of $E(0)$. We choose $q = 0$. Hence, we deduce from Lemma (2) that

$$\mathcal{E}(t) \leq c\mathcal{E}(0)e^{-\omega t}, \qquad t \geq 0.$$

This ends the proof of Theorem 1. $\square$

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Laboratory of Analysis and Control of Partial Differential Equations, Djillali Liabes University, P.O.Box 89, Sidi Bel Abbes 22000, Algeria

E-mail: : benaissa_abbes@yahoo.com

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