The subspace of weak P-points of $\mathbb{N}^*$

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Abstract. Let W be the subspace of $\mathbb{N}^*$ consisting of all weak P-points. It is not hard to see that W is a pseudocompact space. In this paper we shall prove that this space has stronger pseudocompact properties. Indeed, it is shown that W is a p-pseudocompact space for all $p \in \mathbb{N}^*$.

Keywords: p-pseudocompactness; ultrapseudocompactness; strongly pseudocompactness; strongly p-pseudocompactness; weak P-points; $\mathfrak{c}$-OK points

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Preliminaries and introduction

The Greek letter ω represents the first infinite cardinal number. For an infinite set X, we write $[X]^\omega := \{A \subseteq X : |A| = \omega\}$. Every space in this paper is considered to be Tychonoff and infinite. The *Stone-Čech* compactification $\beta\mathbb{N}$ of the discrete space of natural numbers $\mathbb{N}$ will be identified with the set of all ultrafilters on $\mathbb{N}$ and its remainder $\mathbb{N}^*$ will be identified with the set of all free ultrafilters on $\mathbb{N}$. If $A \subseteq \mathbb{N}$, then $A^* = cl_{\beta\mathbb{N}}(A)\setminus\mathbb{N}$. For a given function $f : \mathbb{N} \to \mathbb{N}$, we represent the extension of f to $\beta(\mathbb{N})$ as $\bar{f}$. Those notions used and not defined in this article have the meaning given to them in [5].

Recall that a space is pseudocompact if every continuous function to the reals is bounded and this is equivalent to the property that every locally finite family of open sets is finite. Following the paper [10], given a space X, $p \in \mathbb{N}^*$ and a sequence $(S_n)_{n\in\mathbb{N}}$ of nonempty subsets of X, we say that $z \in X$ is a *p-limit* of $(S_n)_{n\in\mathbb{N}}$, in symbols $x \in p-\lim S_n$, if $\{n \in \mathbb{N} : S_n \cap W \neq \emptyset\} \in p$ for each neighborhood W of z. In particular, if $(x_n)_{n\in\mathbb{N}}$ is a sequence of points of X and x is a p-limit point of this sequence, then the point x is unique and we simply write $x = p-\lim x_n$ rather than $x \in p-\lim\{x_n\}$. The set $p-\lim S_n$ of p-limits of a sequence $(S_n)_{n\in\mathbb{N}}$ of nonempty subsets of X can have more than one point.

Definition 0.1. Let X be a space.

(1) For $p \in \mathbb{N}^*$, X is called *p-compact* if every sequence of points of X has a p-limit point.

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(2) For $p \in \mathbb{N}^*$, X is called *p-pseudocompact* if every sequence of nonempty open subsets of X has a p-limit point.
(3) X is called *ultrapseudocompact* if X is p-pseudocompact for all $p \in \mathbb{N}^*$.
(4) For $p \in \mathbb{N}^*$, X is called *strongly p-pseudocompact*, if for each sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty open subsets of X there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X and $x \in X$ such that $x = p\text{-}\lim x_n$ and $x_n \in U_n$ for all $n \in \mathbb{N}$.
(5) X is called strongly pseudocompact if for each sequence $(U_n)_{n\in\mathbb{N}}$ of nonempty open subsets of X there is a sequence $(x_n)_{n\in\mathbb{N}}$ of points in X, $p \in \mathbb{N}^*$ and $x \in X$ such that $x = p\text{-}\lim x_n$ and $x_n \in U_n$ for each $n \in \mathbb{N}$.

The concept of p-compactness was introduced by Bernstein [3] and the concept of p-pseudocompactness was introduced by Ginsburg and Saks [10], strong p-pseudocompactness in [2] and strong pseudocompactness in [6]. It is not difficult to verify that a space is pseudocompact if for every sequence of nonempty open subsets of X there is $p \in \mathbb{N}^*$ such that the sequence has a p-limit point. Pseudocompactness and the notions (2)-(5) are equivalent on the hyperspace of compact nonempty subsets of a space (see [1]). In general, all notions introduced in Definition 0.1 are stronger than pseudocompactness and their basic properties are discussed in [2] and [10]. Their connection follow directly from the definition: For each $p \in \mathbb{N}^*$, we have that

p-compactness $\Rightarrow$ strong p-pseudocompactness $\Rightarrow$ p-pseudocompactness.

Also it is evident that

ultrapseudocompactness $\Rightarrow$ p-pseudocompactness, for each $p \in \mathbb{N}^*$, and

strong pseudocompactness $\Rightarrow$ pseudocompactness.

In the context of topological groups, it is shown in [8] that a topological group G is pseudocompact iff it is ultrapseudocompact.

Let us consider the next natural questions.

Question 0.2 ([2, Question 3.11]). *Let $p \in \mathbb{N}^*$. Is it true that every space X is strongly p-pseudocompact if and only if it is p-pseudocompact?*

Question 0.3 ([2, Question 3.11], [12, Question 3.7]). *Let $p \in \mathbb{N}^*$. Is it true that every first countable space X is p-compact if and only if it is p-pseudocompact (strongly p-pseudocompact)?*

Of course Question 0.2 can be made for pseudocompactness and strong pseudocompactness but the reader may find in [9] an example of a pseudocompact group which is not strongly pseudocompact. The first natural idea that arises from this question is to consider the subspaces of P-points and weak P-points of $\mathbb{N}^*$ because these both spaces do not have accumulation points for any countable subset. So, they are not strongly pseudocompact. It is evident that the subspace of all P-points of $\mathbb{N}^*$ is not pseudocompact. Contrary to this, the set of all weak P-points W of $\mathbb{N}^*$ is pseudocompact. Our main goal in this paper is to prove that W is ultrapseudocompact. This answers Question 0.2 in a stronger way. Indeed, in Section 1, we give a necessary and sufficient condition to guarantee that a dense subspace of $\mathbb{N}^*$ is ultrapseudocompact, and take advantage of an strong result of Jan van Mill to prove that the subspaces of $\mathfrak{c}$-OK and weak P-points of $\mathbb{N}^*$ are ultrapseudocompact. In the second section we give an example, under CH, of a first countable strong p-pseudocompact space that it is not countably compact. This answers consistently Question 0.3 in the negative way.

1. The subspace of weak P-points of $\mathbb{N}^*$

For this section let us recall that, if X is a topological space, then x is a weak P-point of X if x is not an accumulation point of any countable subset of X. In [11] K. Kunen provided a technique to construct special weak P-points in $\mathbb{N}^*$ called $\mathfrak{c}$-OK points. As we announced before we will show that these two kind of points of $\mathbb{N}^*$ generate ultrapseudocompact spaces. First we state the following result taken from [14, Theorem 4.5.1].

Theorem 1.1. *There is a finite-to-one function $\pi : \mathbb{N} \to \mathbb{N}$ such that for all $p \in \mathbb{N}^*$ there is a weak P-point q ($\mathfrak{c}$-OK point) such that $\pi(q) = p$.*

This important result is inspired in the construction of Kunen. By using the Van Mill's theorem we will prove that W is ultrapseudocompact. Before that we write two preliminary lemmas which are easy to verify. In what follows, an *ω-partition* $\{P_n : n \in \mathbb{N}\}$ of $\mathbb{N}$ will consist of infinite subsets.

Lemma 1.2. *Let $\{P_n : n \in \mathbb{N}\}$ be an ω-partition of $\mathbb{N}$ and let $f : \mathbb{N} \to \mathbb{N}$ the function defined by $f^{-1}(n) = P_n$ for all $n \in \mathbb{N}$. Then for every $p \in \mathbb{N}^*$*

$$\bar{f}^{-1}(p) = p\text{--}\lim P_n \supseteq p\text{--}\lim P_n^*.$$

The previous lemma implies directly the following.

Lemma 1.3. *Let X be a dense subset of $\mathbb{N}^*$. Then X is ultrapseudocompact iff for every $p \in \mathbb{N}^*$ and every function $f : \mathbb{N} \to \mathbb{N}$ with infinite fibers, there exists $q \in X$ such that $f(q) = p$.*

Now we are ready to prove our main result.

Theorem 1.4. *The set of $\mathfrak{c}$-OK points of $\mathbb{N}^*$ is ultrapseudocompact.*

Proof: Let X be the set of all $\mathfrak{c}$-OK points of $\mathbb{N}^*$. Let $f : \mathbb{N} \to \mathbb{N}$ be a function with infinite fibers and let π be the function defined in Theorem 1.1. It is not hard to show that there is a bijection $g : \mathbb{N} \to \mathbb{N}$ such that for every $m \in \mathbb{N}$, we have that

$$g^{-1}\big(\pi^{-1}(n)\big) \subseteq f^{-1}(m) \text{ for some } m \in \mathbb{N}.$$

Clearly the function $\pi \circ g$ is a copy of π because g is bijective. Let $h : \mathbb{N} \to \mathbb{N}$ be the function defined by

$$h(n) = m \text{ iff } g^{-1}\big(\pi^{-1}(n)\big) \subseteq f^{-1}(m).$$

We claim that $f = h \circ \pi \circ g$. Pick $n \in \mathbb{N}$. Since $n \in g^{-1}\big(\pi^{-1}(\pi \circ g(n))\big)$ we obtain from the definition of g that

$$g^{-1}\big(\pi^{-1}(\pi \circ g(n))\big) \subseteq f^{-1}\big(f(n)\big).$$

Then $h\big(\pi \circ g(n)\big) = f(n)$. Finally, for every given $p \in \mathbb{N}^*$, pick $q \in h^{-1}(p)$. By Theorem 1.1, there is $r \in X \cap \overline{\pi \circ g}^{-1}(q)$. Then $f(r) = p$. Therefore, by Lemma 1.3, we conclude that X is ultrapseudocompact. $\square$

As a direct consequence of the previous corollary we obtain that the subset of weak P-points is an ultrapseudocompact space. Observe that, in the context of Question 0.2, we have obtained two more examples of p-pseudocompact spaces which are not strongly p-pseudocompact (actually these examples are stronger because they are ultrapseudocompact and non-strongly pseudocompact). This leads us to ask:

Question 1.5. *Is the subspace of all Q-points of $\mathbb{N}^*$ ultrapseudocompact, of course in some model of ZFC ?*

At this point, we know that W is an ultrapseudocompact space whose countable subsets are closed and C*-embedded. A connected pseudocompact space all countable subsets of which are closed and C*-embedded was constructed in [13]. Based on this, one may ask the following.

Question 1.6. *Is there a connected ultrapseudocompact space all countable subsets of which are closed and C*-embedded ?*

2. A first countable, strongly p-pseudocompact and non-countably compact space

This article finishes with a consistent answer to Question 0.3. To do that we need to introduce some notation and terminology, and state one lemma.

An infinite family $\mathcal{A} \subseteq [\mathbb{N}]^\omega$ is said to be an *almost disjoint family* (AD family) if $A \neq B$ iff $|A \cap B| < \omega$ for all $A, B \in \mathcal{A}$. A maximal AD family is called *maximal almost disjoint family* (MAD family). For an AD family $\mathcal{A}$, the *Mrówka-Isbell* space associated to the family $\mathcal{A}$ is denoted by $\Psi(\mathcal{A})$. It is not hard to prove that the space $\Psi(\mathcal{A})$ is always first countable and it is pseudocompact iff $\mathcal{A}$ is a MAD family. For an AD family $\mathcal{A}$, we write $\mathcal{A}^* = \bigcup_{A \in \mathcal{A}} A^*$ and $\widehat{\mathcal{A}} = \mathbb{N} \cup \mathcal{A}^*$.

Lemma 2.1. *Let $p \in \mathbb{N}^*$.*

(1) ([4]) *If $\mathcal{A}$ is an AD family and p is not a P-point*[1], *then there is a permutation ψ of $\mathbb{N}$ such that $\bar{\psi}(p) \notin \mathcal{A}^*$.*

(2) ([7, Theorem 2.6]) *Assume the Continuum Hypothesis. If p is a P-point, then there is a MAD family $\mathcal{A}$ such that $f(p) \in \widehat{\mathcal{A}}$ for all $f : \mathbb{N} \to \mathbb{N}$.*

[1] An ultrafilter $p \in \mathbb{N}^*$ is called a P-point if the intersection of countably many neighborhoods of p is again a neighborhood of it. It is well known that the existence of these points is independent from the axioms of ZFC.

We omit the proof of the following easy lemma.

Lemma 2.2. *Let $\mathcal{A}$ be a MAD family and $p \in \mathbb{N}^*$. Then $\widehat{\mathcal{A}}$ is strongly p-pseudocompact (p-pseudocompact) iff $\Psi(\mathcal{A})$ is so.*

Theorem 2.3. **[CH]** *The following statements are equivalent for $p \in \mathbb{N}^*$.*

(1) *There exists a MAD family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is strongly p-pseudocompact.*
(2) *p is a P-point of $\mathbb{N}^*$.*

Proof: (1) $\Rightarrow$ (2). If p is not a P-point and $\mathcal{A}$ is a MAD family, then there is a permutation ψ of $\mathbb{N}$ such that $\bar{\psi}(p) \notin \widehat{\mathcal{A}}$. As $\bar{\psi}(p)$ is the only point in $\lim\{\psi(n)\}$, this shows that $\widehat{\mathcal{A}}$ is not strongly p-pseudocompact. Hence $\Psi(\mathcal{A})$ is not strongly p-pseudocompact by Lemma 2.2.

(2) $\Rightarrow$ (1). Assume that p is a P-point of $\mathbb{N}^*$. According to Lemma 2.1.2, we can find a MAD family $\mathcal{A}$ so that $f(p) \in \widehat{\mathcal{A}}$ for all $f : \mathbb{N} \to \mathbb{N}$. We will obtain that $\widehat{\mathcal{A}}$ is strongly p-pseudocompact. Indeed, given a sequence $(U_n)_{n\in\mathbb{N}}$ of non-empty open sets, let $f : \mathbb{N} \to \mathbb{N}$ be such that $f(n) \in U_n$ for all $n \in \mathbb{N}$. Then $f(p) = p-\lim f(n)$ is in $\widehat{\mathcal{A}}$ as required by strong p-pseudocompactness. Therefore, by Lemma 2.2, $\Psi(\mathcal{A})$ is strongly p-pseudocompact. $\square$

Corollary 2.4. **[CH]** *For each P-point $p \in \mathbb{N}^*$ there exists a MAD family $\mathcal{A}$ such that $\Psi(\mathcal{A})$ is first countable, strongly p-pseudocompact and not countably compact.*

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