Jordan automorphisms of triangular algebras II

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Abstract. We give a sufficient condition under which any Jordan automorphism of a triangular algebra is either an automorphism or an anti-automorphism.

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1. Introduction

Throughout the paper, R denotes a commutative ring such that $\frac{1}{2} \in R$. Let $\mathcal{A}$ and $\mathcal{B}$ be unital algebras over R. Recall that if θ is an R-linear map from $\mathcal{A}$ into $\mathcal{B}$, then:

(i) θ is said to be a *Jordan homomorphism* if $\theta(AB + BA) = \theta(A)\theta(B) + \theta(B)\theta(A)$ for all $A, B \in \mathcal{A}$;

(ii) θ is said to be *a homomorphism* (resp., *an anti-homomorphism*) if $\theta(AB) = \theta(A)\theta(B)$ for all $A, B \in \mathcal{A}$ (resp., $\theta(AB) = \theta(B)\theta(A)$ for all $A, B \in \mathcal{A}$).

Clearly, every homomorphism and every anti-homomorphism is a Jordan homomorphism. It is well-known that the converse is not true in general.

Recall that a left $\mathcal{A}$-module (resp., right $\mathcal{B}$-module) $\mathcal{M}$ is faithful if for any $A \in \mathcal{A}$, $A\mathcal{M} = \{0\}$ (resp., for any $B \in \mathcal{B}$, $\mathcal{M}B = \{0\}$) implies $A = 0$ (resp., $B = 0$).

Let $\mathcal{M}$ be a unital $(A, \mathcal{B})$-bimodule which is faithful as a left $\mathcal{A}$-module and also as a right $\mathcal{B}$-module. The R-algebra

$$\mathcal{U} = Tri(\mathcal{A}, \mathcal{M}, \mathcal{B}) = \left\{ \begin{pmatrix} a & m \\ & b \end{pmatrix} : a \in \mathcal{A}, b \in \mathcal{B}, m \in \mathcal{M} \right\},$$

under the usual matrix operations is called a triangular algebra (see e.g. [2]). Benkovič and Eremita [3] described the three classical examples of triangular rings: upper triangular matrix rings, block upper triangular matrix rings, and nest algebras. In the same manner we can describe upper triangular matrix algebras and block upper triangular matrix algebras.

In [4], I.N. Herstein showed that every Jordan automorphism of a primitive ring of characteristic different from 2 and 3 is either an automorphism or an anti-automorphism. Since then many other results have been shown in a similar vein for different classes of rings and algebras.

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It is shown in [1] that every Jordan automorphism of a triangular algebra is either an automorphism or an anti-automorphism. The authors of [1] proved this result by a method based on calculations using each entry of an element in $\mathcal{U}$. In this paper we will provide a new proof of this result using fundamental properties of Jordan automorphisms of unital algebras obtained by Herstein [4].

2. Main result

Here is a basic lemma which will be used frequently.

Lemma 2.1 (see [4])**.** *Let $\mathcal{A}$ be a unital algebra over R. If θ is a Jordan automorphism of $\mathcal{A}$, then:*

(a) $\theta(A^2) = (\theta(A))^2$ *for every* $A \in \mathcal{A}$,
(b) $\theta(ABA) = \theta(A)\theta(B)\theta(A)$ *for all* $A, B \in \mathcal{A}$,
(c) $\theta(AXB + BXA) = \theta(A)\theta(X)\theta(B) + \theta(B)\theta(X)\theta(A)$ *for all* $A, B, X \in \mathcal{A}$.

Notation 2.2. Let $P = \left(\begin{smallmatrix} 1 & 0 \\ & 0 \end{smallmatrix}\right)$, $Q = \left(\begin{smallmatrix} 0 & 0 \\ & 1 \end{smallmatrix}\right)$, $I = \left(\begin{smallmatrix} 1 & 0 \\ & 1 \end{smallmatrix}\right)$ and if $m \in \mathcal{M}$, we put $E_m = \left(\begin{smallmatrix} 1 & m \\ & 0 \end{smallmatrix}\right)$ and $F_m = \left(\begin{smallmatrix} 0 & m \\ & 1 \end{smallmatrix}\right)$.

Lemma 2.3 (see [5, Proof of Theorem 1])**.** *If both $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, then the set of idempotents of $\mathcal{U}$ is $\Omega = \{E_m, F_m \mid m \in \mathcal{M}\}$.*

Remark 2.4. An easy computation shows that $QXP = 0$ for any $X \in \mathcal{U}$.

Lemma 2.5. *Let ϕ be a Jordan endomorphism of $\mathcal{U}$ such that $\phi(P) = P$ and $\phi(Q) = Q$. Then for every $A, B, X \in \mathcal{U}$, we have:*

(1) $\phi(PAQ) = P\phi(A)Q$, $\phi(PA) = P\phi(A)$, $\phi(AQ) = \phi(A)Q$, $\phi(AP) = \phi(A)P$ *and* $\phi(QA) = Q\phi(A)$,
(2) $\phi(APXQ) = \phi(A)P\phi(X)Q$,
(3) $\phi(PXQA) = P\phi(X)Q\phi(A)$,
(4) $P\phi(AB)Q = P\phi(A)\phi(B)Q$,
(5) $\phi(ABPXQ) = \phi(A)\phi(B)P\phi(X)Q$,
(6) $\phi(PXQAB) = P\phi(X)Q\phi(A)\phi(B)$.

Proof: (1) Let $A \in \mathcal{U}$. Since $QAP = 0$, we have $\phi(PAQ) = \phi(PAQ + QAP) = P\phi(A)Q + Q\phi(A)P$ by Lemma 2.1(c). But $Q\phi(A)P = 0$. Then,

$$(E_1) \qquad \phi(PAQ) = P\phi(A)Q.$$

Moreover, from Lemma 2.1(b) it follows that

$$(E_2) \qquad \phi(PAP) = P\phi(A)P \text{ and } \phi(QAQ) = Q\phi(A)Q.$$

On account of equations (E_1) and (E_2) and the fact that $P + Q = I$, we have $\phi(PA) = \phi(PAQ) + \phi(PAP) = P\phi(A)Q + P\phi(A)P = P\phi(A)$ and $\phi(AQ) = \phi(QAQ) + \phi(PAQ) = Q\phi(A)Q + P\phi(A)Q = \phi(A)Q$.

In the same manner we can see that $\phi(AP) = \phi(A)P$ and $\phi(QA) = Q\phi(A)$.

(2) Note that $YP = (P+Q)YP = PYP + QYP = PYP$ for all $Y \in \mathcal{U}$. Let A, $X \in \mathcal{U}$. From (1) it follows that

$$\begin{aligned}
\phi(APXQ) &= \phi(PAPXQ) \\
&= \phi((PA)(PXQ) + (PXQ)(PA)) \\
&= \phi(PA)\,\phi(PXQ) + \phi(PXQ)\,\phi(PA) \\
&= P\phi(A)\,P\phi(X)\,Q + P\phi(X)\,QP\phi(A) \\
&= \phi(A)\,P\phi(X)\,Q \text{ since } QP = 0.
\end{aligned}$$

(3) By using the fact that $QY = QY(P+Q) = QYP + QYQ = QYQ$ for all $Y \in \mathcal{U}$, the proof of (3) is similar to that of (2).

(4) Let $A, B \in \mathcal{U}$. We have

$$\begin{aligned}
P\phi(AB)\,Q &= \phi(PABQ) \text{ by (1)} \\
&= \phi(PABQ + BQPA) \text{ since } QP = 0 \\
&= \phi(PA)\,\phi(BQ) + \phi(BQ)\,\phi(PA) \\
&= P\phi(A)\,\phi(B)\,Q + \phi(B)\,QP\phi(A) \text{ by (1)} \\
&= P\phi(A)\,\phi(B)\,Q.
\end{aligned}$$

(5) Let $A, B, X \in \mathcal{U}$. By (1) and (2), we have

$$\begin{aligned}
\phi(ABPXQ) &= \phi(APBPXQ + BPXQAP) \text{ since } BP = PBP \\
&= \phi(AP)\,\phi(BPXQ) + \phi(BPXQ)\,\phi(AP) \\
&= \phi(A)\,P\phi(B)\,P\phi(X)\,Q + \phi(B)\,P\phi(X)\,Q\phi(A)\,P \\
&= \phi(A)\,\phi(B)\,P\phi(X)\,Q \text{ since } \phi(B)P = P\phi(B)P.
\end{aligned}$$

(6) The proof is similar to that of (5) by using the fact that $QA = QAQ$. □

Lemma 2.6. *Let ψ be a Jordan endomorphism of $\mathcal{U}$ such that $\psi(P) = Q$ and $\psi(Q) = P$. Then for every $A, B, X \in \mathcal{U}$, we have:*

1. *$\psi(PAQ) = P\psi(A)Q$, $\psi(PA) = \psi(A)\,Q$, $\psi(AQ) = P\psi(A)$, $\psi(AP) = Q\psi(A)$ and $\psi(QA) = \psi(A)P$,*
2. *$\psi(APXQ) = P\psi(X)\,Q\psi(A)$,*
3. *$\psi(PXQA) = \psi(A)\,P\psi(X)\,Q$,*
4. *$P\psi(AB)Q = P\psi(B)\psi(A)Q$,*
5. *$\psi(ABPXQ) = P\psi(X)Q\psi(B)\psi(A)$,*
6. *$\psi(PXQAB) = \psi(B)\psi(A)P\psi(X)Q$.*

Proof: The proof is similar to that of Lemma 2.5. □

Proposition 2.7. (1) *Let ϕ be a Jordan automorphism of $\mathcal{U}$ such that $\phi(P) = P$ and $\phi(Q) = Q$. Then ϕ is an automorphism.*

(2) *Let ψ be a Jordan automorphism of $\mathcal{U}$ such that $\psi(P) = Q$ and $\psi(Q) = P$. Then ψ is an anti-automorphism.*

Proof: (1) Let $A, B, X \in \mathcal{U}$. Lemma 2.5((2), (5)) yields $\phi(AB)P\phi(X)Q = \phi(ABPXQ) = \phi(A)\phi(B)P\phi(X)Q$. So $(\phi(AB) - \phi(A)\phi(B))P\phi(X)Q = 0$. Since ϕ is a Jordan automorphism, we have $(\phi(AB) - \phi(A)\phi(B))P\mathcal{U}Q = 0$. Thus $[P(\phi(AB) - \phi(A)\phi(B))P]P\mathcal{U}Q = 0$ since $P^2 = P$. Note that by hypothesis, $\mathcal{M}$ is a faithful left $\mathcal{A}$-module. Then an easy computation shows that $P(\phi(AB)-\phi(A)\phi(B))P = 0$. In the same manner we can also see that $Q(\phi(AB)-\phi(A)\phi(B))Q = 0$. Moreover, Lemma 2.5(4) gives $P\phi(AB)Q = P\phi(A)\phi(B)Q$. That is, $P(\phi(AB)-\phi(A)\phi(B))Q = 0$. Therefore $(P+Q)(\phi(AB)-\phi(A)\phi(B))(P+Q) = 0$. Consequently, $\phi(AB) = \phi(A)\phi(B)$. This completes the proof.

(2) The proof is similar to that of (1). □

This brings us to the main result of this paper.

Theorem 2.8. *If both $\mathcal{A}$ and $\mathcal{B}$ have only trivial idempotents, then any Jordan automorphism of $\mathcal{U}$ is either an automorphism or an anti-automorphism.*

Proof: Let θ be a Jordan automorphism of $\mathcal{U}$. Since P is an idempotent of $\mathcal{U}$, either $\theta(P) = E_m$ or $\theta(P) = F_m$ for some $m \in \mathcal{M}$. Assume that $\theta(P) = E_m$ for some $m \in \mathcal{M}$. This implies that $\theta(Q) = F_k$ for some $k \in M$. Indeed, if $\theta(Q) = E_x$ for some $x \in M$, we obtain $\theta(PQ+QP) = \theta(P)\theta(Q)+\theta(Q)\theta(P) = E_m+E_x \neq 0$, a contradiction. Therefore $\theta(PQ + QP) = E_mF_k + F_kE_m$. Hence $k + m = 0$. This gives $\theta(Q) = F_{-m}$. It is easy to check that $T = \left(\begin{smallmatrix} 1 & -m \\ & 1 \end{smallmatrix}\right)$ is invertible and its inverse is $T^{-1} = \left(\begin{smallmatrix} 1 & m \\ & 1 \end{smallmatrix}\right)$. Let σ_T be the automorphism of $\mathcal{U}$ defined by $\sigma_T(Y) = TYT^{-1}$ for all $Y \in \mathcal{U}$. It is not difficult to see that $\theta(P) = \sigma_T(P)$ and $\theta(Q) = \sigma_T(Q)$. We thus get $\phi(P) = P$ and $\phi(Q) = Q$, where $\phi = \sigma_{T^{-1}} \circ \theta$ is also a Jordan automorphism of $\mathcal{U}$. By Proposition 2.7, ϕ is an automorphism. Therefore θ is an automorphism.

Similarly, we can prove that if $\theta(P) = F_m$ for some $m \in \mathcal{M}$, then θ is an anti-automorphism. □

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