

## A co-ideal based identity-summand graph of a commutative semiring

S. EBRAHIMI ATANI, S. DOLATI PISH HESARI, M. KHORAMDEL

*Abstract.* Let  $I$  be a strong co-ideal of a commutative semiring  $R$  with identity. Let  $\Gamma_I(R)$  be a graph with the set of vertices  $S_I(R) = \{x \in R \setminus I : x + y \in I \text{ for some } y \in R \setminus I\}$ , where two distinct vertices  $x$  and  $y$  are adjacent if and only if  $x + y \in I$ . We look at the diameter and girth of this graph. Also we discuss when  $\Gamma_I(R)$  is bipartite. Moreover, studies are done on the planarity, clique, and chromatic number of this graph. Examples illustrating the results are presented.

*Keywords:* strong co-ideal;  $Q$ -strong co-ideal; identity-summand element; identity-summand graph; co-ideal based

*Classification:* 16Y60, 05C62

### 1. Introduction

Among the most interesting graphs are the zero-divisor graphs, because these involve both ring theory and graph theory. By studying these graphs, we can gain a broader insight into the concepts and properties that involve both graphs and rings. It was Beck (see [3]) who first introduced the notion of a zero-divisor graph for commutative ring. This notion was later redefined by D. F. Anderson and P. S. Livingston in [1]. In [12], Redmond introduced the zero-divisor graph with respect to a proper ideal. Since then, there has been a lot of interest in this subject and various papers were published establishing different properties of these graphs as well as relations between graphs of various extensions (see [2], [11], [12] and [13]). Recently, such graphs are used to study semirings [5], [6] and [9].

Semirings have proven to be useful in theoretical computer science, in particular for studying automata and formal languages, hence, ought to be in the literature [10] and [14]. From now on let  $R$  be a commutative semiring with identity. In [8], the present authors introduced the identity-summand graph, denoted by  $\Gamma(R)$ , such that vertices are all non-identity identity-summands of  $R$  and two distinct vertices are joint by an edge when the sum of them is 1. We use the notation  $S(R)$  to refer to the set of elements of  $R$  that are identity-summands (we use  $S^*(R)$  to denote the set of non-identity identity-summands of  $R$ ), we say that  $r \in R$  is an identity-summand of  $R$ , if there exists  $1 \neq a \in R$  such that  $r + a = 1$ .

In this paper we will generalize this notion by replacing elements whose sum is identity with elements whose sum lies in some strong co-ideal  $I$  of  $R$ . Indeed, we define an undirected graph  $\Gamma_I(R)$  with vertices  $S_I(R) = \{x \in R \setminus I : x + y \in I \text{ for some } y \in R \setminus I\}$ , where distinct vertices  $x$  and  $y$  are adjacent if and only if  $x + y \in I$ . This definition was motivated by [12], [6] and [8]. Here is a brief summary of our paper. We will make an intensive study on identity-summand graph of commutative semirings based on strong co-ideals. In section 2, it is shown that  $\Gamma_I(R)$  is connected with  $\text{diam}(\Gamma_I(R)) \leq 3$ , and if  $I$  is a subtractive co-ideal, then  $\Gamma_I(R)$  is not complete. We show that if  $\Gamma_I(R)$  contains a cycle, then  $\text{gr}(\Gamma_I(R)) \leq 4$  and several characterizations of  $\Gamma_I(R)$  by girth are given. Also it is proved that if  $I$  is a  $Q$ -strong co-ideal and  $\Gamma_I(R)$  and  $\Gamma(R/I)$  has a cycle, then  $\text{gr}(\Gamma_I(R)) = \text{gr}(\Gamma(R/I))$ . In Section 3, it is shown that for a subtractive strong co-ideal  $I$  of  $R$ ,  $\Gamma_I(R)$  is complete bipartite if and only if there exist two distinct prime strong co-ideals  $P_1$  and  $P_2$  of  $R$  such that  $P_1 \cap P_2 = I$ . Section 4 is devoted to study chromatic number, clique number and planar property of  $\Gamma_I(R)$ .

In order to make this paper easier to follow, we recall various notions which will be used in the sequel. For a graph  $\Gamma$ , we denote by  $E(\Gamma)$  and  $V(\Gamma)$  the set of all edges and vertices, respectively. We recall that a graph is connected if there exists a path connecting any two distinct vertices. The distance between two distinct vertices  $a$  and  $b$ , denoted by  $d(a, b)$ , is the length of the shortest path connecting them (if such a path does not exist, then  $d(a, b) = \infty$ , also  $d(a, a) = 0$ ). The diameter of graph  $\Gamma$ , denoted by  $\text{diam}(\Gamma)$ , is equal to  $\sup\{d(a, b) : a, b \in V(\Gamma)\}$ . A graph is complete if it is connected with diameter less than or equal to one. We denote the complete graph on  $n$  vertices by  $K_n$ . The girth of a graph  $\Gamma$ , denoted  $\text{gr}(\Gamma)$ , is the length of a shortest cycle in  $\Gamma$ , provided  $\Gamma$  contains a cycle; otherwise  $\text{gr}(\Gamma) = \infty$ . An edge for which the two ends are the same is called a loop at the common vertex. For  $r$  a nonnegative integer, an  $r$ -partite graph is one whose set of vertices can be partitioned into  $r$  subsets so that no edge has both ends in any single subset. A complete  $r$ -partite graph is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite (i.e., 2-partite) graph with parts of size  $m$  and  $n$  is denoted by  $K_{m,n}$ . We will sometimes call  $K_{1,n}$  a star graph. We define a coloring of a graph  $G$  to be an assignment of colors (elements of some set) to vertices of  $G$ , one color to each vertex, so that distinct colors are assigned to adjacent vertices. If  $n$  colors are used, then the coloring is referred to as an  $n$ -coloring. If there exists an  $n$ -coloring of a graph  $G$ , then  $G$  is called  $n$ -colorable. The minimum  $n$  for which a graph  $G$  is  $n$ -colorable is called the chromatic number of  $G$ , and is denoted by  $\chi(G)$ . A clique of a graph is its maximal complete subgraph and the maximal number of vertices in any clique of graph  $G$ , denoted by  $w(G)$ , is called the clique number of  $G$ .

A commutative semiring  $R$  is defined as an algebraic system  $(R, +, \cdot)$  such that  $(R, +)$  and  $(R, \cdot)$  are commutative semigroups, connected by  $a(b + c) = ab + ac$  for all  $a, b, c \in R$ , and there exists  $0, 1 \in R$  such that  $r + 0 = r$  and  $r0 = 0r = 0$

and  $r1 = 1r = r$  for each  $r \in R$ . In this paper all semirings considered will be assumed to be commutative semirings with non-zero identity.

**Definition 1.1.** Let  $R$  be a semiring.

(1) A non-empty subset  $I$  of  $R$  is called *co-ideal*, if it is closed under multiplication and satisfies the condition  $r + a \in I$  for all  $a \in I$  and  $r \in R$  (so  $0 \in I$  if and only if  $I = R$ ). A co-ideal  $I$  of  $R$  is called *strong co-ideal* provided that  $1 \in I$  (in this case,  $1 + x \in I$  for every  $x \in R$ ).

(2) A co-ideal  $I$  of  $R$  is called *subtractive* if  $x, xy \in I$  implies  $y \in I$  (so every subtractive co-ideal is a strong co-ideal).

(3) If  $I$  is a co-ideal of  $R$ , then the co-rad( $I$ ) of  $I$ , is the set of all  $x \in R$  for which  $nx \in I$  for some positive integer  $n$ . This is a co-ideal of  $R$  containing  $I$  [7].

(4) A proper co-ideal  $P$  of  $R$  is called *prime* if  $x + y \in P$  implies  $x \in P$  or  $y \in P$ . The set of all prime co-ideals of  $R$  is denoted by  $\text{co-Spec}(R)$ . A proper co-ideal  $I$  of  $R$  is called *primary* if  $a + b \in I$  implies  $a \in I$  or  $b \in \text{co-rad}(I)$ . If  $I$  is primary, then  $\text{co-rad}(I)$  is a prime co-ideal. We say that  $I$  is  $P$ -primary if  $I$  is primary and  $\text{co-rad}(I) = P$  [7].

(5) If  $D$  is an arbitrary nonempty subset of  $R$ , then the set  $F(D)$  consisting of all elements of  $R$  of the form  $d_1d_2 \dots d_n + r$  (with  $d_i \in D$  for all  $1 \leq i \leq n$  and  $r \in R$ ) is a co-ideal of  $R$  generated by  $D$  [7], [10] and [14].

(6) A semiring  $R$  is called *co-semidomain*, if  $a + b = 1$  ( $a, b \in R$ ) implies either  $a = 1$  or  $b = 1$  [7].

A strong co-ideal  $I$  of a semiring  $R$  is called a *partitioning strong co-ideal* (=  $Q$ -strong co-ideal) if there exists a subset  $Q$  of  $R$  such that the following hold.

- (1)  $R = \bigcup \{qI : q \in Q\}$ , where  $qI = \{qt : t \in I\}$ .
- (2) If  $q_1, q_2 \in Q$ , then  $(q_1I) \cap (q_2I) \neq \emptyset$  if and only if  $q_1 = q_2$ .
- (3) For each  $q_1, q_2 \in Q$ , there exists  $q_3 \in Q$  such that  $q_1I + q_2I \subseteq q_3I$ .

Let  $I$  be a  $Q$ -strong co-ideal of a semiring  $R$  and let  $R/I = \{qI : q \in Q\}$ . Then  $R/I$  forms a semiring under the binary operations  $\oplus$  and  $\odot$  defined as follows:  $(q_1I) \oplus (q_2I) = q_3I$ , where  $q_3$  is the unique element in  $Q$  such that  $(q_1I + q_2I) \subseteq q_3I$ , and  $(q_1I) \odot (q_2I) = q_3I$ , where  $q_3$  is the unique element in  $Q$  such that  $(q_1q_2)I \subseteq q_3I$  [7]. If  $q_e$  is the unique element in  $Q$  such that  $1 \in q_eI$ , then  $q_eI = I$  is the identity of  $R/I$ . Note that every  $Q$ -strong co-ideal is subtractive [7]. Throughout this paper we shall assume unless otherwise stated, that  $q_0I$  (resp.  $q_eI$ ) is the zero element (resp. the identity element) of  $R/I$ . In the following, we give an example of a  $Q$ -strong co-ideal. One can see another example of  $Q$ -strong co-ideal in [7].

**Example 1.2.** Let  $R$  be the set of all non-negative integers. Define  $a + b = \text{gcd}(a, b)$  and  $a \times b = \text{lcm}(a, b)$  (take  $0 + 0 = 0$  and  $0 \times 0 = 0$ ). Then  $(R, +, \times)$  is easily checked to be a commutative semiring. Let  $I$  be the set of all non-negative

odd integers. Then  $I$  is a strong co-ideal of  $R$ . Set  $Q = \{0, 1, 2, 4, 8, 16, 32, 64, \dots\}$ . It is clear that  $I$  is a  $Q$ -strong co-ideal.

**2. Examples and basic properties of  $\Gamma_I(R)$**

In this section we study the diameter, girth and cut-point of  $\Gamma_I(R)$ , when  $I$  is a strong co-ideal of the semiring  $R$ .

**Proposition 2.1.** *Let  $I$  be a subtractive co-ideal of a semiring  $R$ . Then the following hold:*

- (1) if  $xy \in I$ , then  $x, y \in I$  for all  $x, y \in R$ ;
- (2)  $I = \text{co-rad}(I)$ ;
- (3)  $(I : a) = \{r \in R : r + a \in I\}$  is a subtractive co-ideal of  $R$  for all  $a \in R$ ;
- (4) if  $I$  is a  $Q$ -strong co-ideal of  $R$  and  $q_e I$  is the identity element in  $R/I$ , then  $q_e I \oplus qI = q_e I$  and  $qI \oplus qI = qI$  for all  $qI \in R/I$ .

PROOF: (1) Observe that  $1 + x \in I$  for each  $x \in R$ . If  $xy \in I$ , then  $y(1 + x) = xy + y \in I$  gives  $y \in I$ , since  $I$  is subtractive. Similarly,  $x \in I$ .

(2) It suffices to show that  $\text{co-rad}(I) \subseteq I$ . Let  $x \in \text{co-rad}(I)$ , so  $nx \in I$  for some positive integer  $n \in \mathbb{N}$ . Thus  $nx = \underbrace{x(1 + 1 + \dots + 1)}_{n \text{ times}} \in I$  gives  $x \in I$ .

(3) Clearly,  $1 \in (I : a)$ . If  $x, y \in (I : a)$ , then  $x + a \in I$  and  $y + a \in I$ , implying  $a^2 + ax + ay + xy \in I$ . Since  $(xy + a)(1 + a)(1 + y)(1 + x) \in I$ ,  $xy + a \in I$  by (1). Thus  $xy \in (I : a)$ . As  $I$  is a co-ideal,  $r + x + a \in I$  for each  $r \in R$  and so  $x + r \in (I : a)$  for each  $r \in R$ . This shows that  $(I : a)$  is a co-ideal of  $R$ . Now let  $xy, x \in (I : a)$ . Then  $xy + a + y + xa = (x + 1)(y + a) \in I$ , which gives  $y + a \in I$ , and so  $y \in (I : a)$ , as desired.

(4) Let  $q_e I \oplus qI = q'I$ , where  $q'$  is the unique element in  $Q$  such that  $q_e I + qI \subseteq q'I$ . Since  $I$  is co-ideal,  $qI + q_e I \subseteq q_e I \cap q'I$ , which gives  $q_e I = q'I$ . Finally,  $qI \oplus qI = qI \odot (q_e I \oplus q_e I) = qI \odot q_e I = qI$ . □

**Proposition 2.2.** *Let  $I$  be a strong co-ideal of a semiring  $R$ . Then  $S_I(R) = \emptyset$  if and only if  $I$  is a prime strong co-ideal of  $R$ .*

PROOF: This follows directly from the definitions. □

**Theorem 2.3.** *Let  $I$  be a  $Q$ -strong co-ideal of  $R$ . Then the following are equivalent:*

- (1)  $S_I(R) = \emptyset$ ;
- (2)  $I$  is a prime co-ideal of  $R$ ;
- (3)  $S^*(R/I) = \emptyset$ ;
- (4)  $I$  is  $P$ -primary.

PROOF: (1)  $\Leftrightarrow$  (2) follows from Proposition 2.2.

(2)  $\Leftrightarrow$  (3) By [[7], Theorem 3.8],  $I$  is prime if and only if  $R/I$  is co-semidomain. Therefore  $I$  is prime if and only if  $S^*(R/I) = \emptyset$ .

(2)  $\Rightarrow$  (4) is clear.

(4)  $\Rightarrow$  (2) If  $I$  is a  $P$ -primary strong co-ideal of  $R$ , then  $I = \text{co-rad}(I) = P$  by Proposition 2.1(2) and [7, Proposition 2.2]; hence  $I$  is prime.  $\square$

Redmond [12] explored the relationship between  $\Gamma_I(R)$  and  $\Gamma(R/I)$ . He gave an example of rings  $R, T$  and ideals  $I \trianglelefteq R, J \trianglelefteq T$ , where  $\Gamma(R/I) \cong \Gamma(T/J)$  but  $\Gamma_I(R) \not\cong \Gamma_J(T)$ . Here we generalize this concept to the case of semirings.

**Example 2.4.** Let  $X = \{a, b, c\}$  and  $R = (P(X), \cup, \cap)$  a semiring with  $1_R = X$ , where  $P(X)$  is the set of all subsets of  $X$ . If  $I = \{X, \{a, b\}\}$ , then  $I$  is a  $Q$ -strong co-ideal, where  $Q = \{q_1 = \{c\}, q_2 = \{a, c\}, q_3 = \{b, c\}, q_e = X\}$ . An inspection will show that  $q_2I \oplus q_3I = q_eI$  and  $S^*(R/I) = \{q_2I, q_3I\}$ . Also  $S_I(R) = \{\{a\}, \{b\}, \{a, c\}, \{b, c\}\}$ . Let  $T = \{0, 1, 2, 3, 4, 6, 12\}$ . Then  $(T, \text{gcd}, \text{lcm})$  (take  $\text{gcd}(0, 0) = 0$  and  $\text{lcm}(0, 0) = 0$ ) is a commutative semiring. If  $J = \{1, 2\}$ , then it easily can be checked that  $J$  is a  $Q$ -strong co-ideal with  $Q = \{0, 1, 3, 4, 12\}$ ,  $T/J = \{0J, 1J, 3J, 4J, 12J\}$ ,  $S^*(T/J) = \{3J, 4J\}$  and  $S_J(T) = \{3, 4, 6\}$ . Thus  $\Gamma(R/I) \cong \Gamma(T/J)$ , however  $\Gamma_I(R) \not\cong \Gamma_J(T)$ .

The next several results investigate the relationship between  $\Gamma(R/I)$  and  $\Gamma_I(R)$ .

**Proposition 2.5.** *Let  $I$  be a  $Q$ -strong co-ideal of a semiring  $R$  and let  $x, y \in S_I(R)$  such that  $x \in q_1I$  and  $y \in q_2I$ , for some  $q_1, q_2 \in Q$ . Then:*

- (1)  $x$  is adjacent to  $y$  in  $\Gamma_I(R)$  if and only if  $q_1I$  and  $q_2I$  are adjacent in  $\Gamma(R/I)$  and  $q_1 \neq q_2$ . In particular, each elements of  $q_1I$  are adjacent to each elements of  $q_2I$  in  $\Gamma_I(R)$ .
- (2) If  $q_1I \in S^*(R/I)$ , then all the distinct elements of  $q_1I$  are not adjacent to each other in  $\Gamma_I(R)$ .

PROOF: (1) Let  $x$  be adjacent to  $y$  in  $\Gamma_I(R)$ , so  $x + y \in q_eI = I$ . Let  $q_1I \oplus q_2I = q_3I$ , where  $q_3$  is the unique element in  $Q$  such that  $q_1I + q_2I \subseteq q_3I$ . Since  $x + y \in q_3I \cap q_eI$ ,  $q_3 = q_e$ . Thus  $q_1I$  is adjacent to  $q_2I$  in  $\Gamma(R/I)$ . We show  $q_1 \neq q_2$ . Suppose, on the contrary,  $q_1 = q_2$ . Since  $q_1I$  and  $q_2I$  are adjacent, we have  $I = q_eI = q_1I \oplus q_2I = q_1I \oplus q_1I = q_1I$  by Proposition 2.1(4), a contradiction. Thus  $q_1 \neq q_2$ . Conversely, let  $q_1I$  be adjacent to  $q_2I$  in  $\Gamma(R/I)$ , so  $q_1I \oplus q_2I = q_eI$ , where  $(q_1I + q_2I) \subseteq q_eI$ . Then  $x + y \in q_1I + q_2I \subseteq q_eI = I$ ; hence  $x$  is adjacent to  $y$  in  $\Gamma_I(R)$ . Now, from above discussion, it is clear that each elements of  $q_1I$  are adjacent to each elements of  $q_2I$  in  $\Gamma_I(R)$ .

(2) It is similar to the proof of (1).  $\square$

An edge for which the two ends are the same is called a loop at the common vertex.

**Theorem 2.6.** *Let  $I$  be a strong co-ideal of a semiring  $R$ .*

- (1) *If  $I$  is subtractive, then  $\Gamma_I(R)$  has no loop.*
- (2) *If  $I$  is a  $Q$ -strong co-ideal and  $\Gamma(R/I) \neq \emptyset$ , then  $\Gamma(R/I)$  has at least two vertices and has no loop.*
- (3) *If  $I$  is subtractive and  $a \in R$  is a vertex of  $\Gamma_I(R)$  which is adjacent to every other vertex, then  $a + a = a$  and  $(I : a)$  is a maximal element of the*

set  $\Delta = \{(I : x) : x \in R \setminus I\}$  with respect to inclusion. Moreover,  $(I : a)$  is a prime co-ideal of  $R$ .

PROOF: (1) Suppose that  $a \in R \setminus I$  with  $a + a = a(1 + 1) \in I$ . Since  $I$  is subtractive  $a \in I$ , which is a contradiction. So  $\Gamma_I(R)$  has no loop.

(2) By Proposition 2.1(4),  $\Gamma(R/I)$  has no loop, so it has more than one vertex.

(3) Let  $a + a \neq a$ . As  $I$  is subtractive and  $a \notin I$ ,  $a + a \notin I$ . Since  $a$  is adjacent to every other vertex in  $\Gamma_I(R)$ ,  $a + a + x \in I$  for each  $x \in S_I(R)$ . Thus  $a + a \in S_I(R)$ . Hence  $a + a + a = a(1 + 1 + 1) \in I$  gives  $a \in I$ , a contradiction. So  $a + a = a$ . Suppose, on the contrary,  $(I : a)$  is not maximal. So there is  $x \in R \setminus I$  such that  $(I : a) \subset (I : x)$ . Since  $a$  is adjacent to every other vertex in  $\Gamma_I(R)$ ,  $x + a \in I$ , which gives  $x \in (I : a) \subset (I : x)$ . So  $x + x \in I$ , a contradiction by (1).

Let  $x + y \in (I : a)$  be such that  $x \notin (I : a)$ . So  $x + a \notin I$ . As  $(I : a) \subseteq (I : x + a)$  and  $(I : a)$  is maximal in  $\Delta$ , we have  $(I : a) = (I : x + a)$ . Since  $x + y \in (I : a)$ , we get  $y \in (I : a + x) = (I : a)$ . Thus  $(I : a)$  is prime.  $\square$

Note that the condition that  $I$  is subtractive is necessary in Proposition 2.6 (1) as the following example shows.

**Example 2.7.** Let  $R = (\{0, 1, 2, 3\}, +, \times)$ , where

$$a + b = \begin{cases} 3 & \text{if } a, b \neq 0, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0. \end{cases}$$

and  $1 \times 1 = 1, 2 \times 1 = 1 \times 2 = 2, 3 \times 1 = 1 \times 3 = 3, 2 \times 2 = 1, 2 \times 3 = 3 \times 2 = 3, 3 \times 3 = 3$ , moreover  $r \times 0 = 0 \times r = 0$  for all  $r \in R$ . Then  $I = \{1, 3\}$  is a strong co-ideal of  $R$  which is not subtractive because  $3, 3 \times 2 \in I$  but  $2 \notin I$ . It is easy to see that  $S_I(R) = \{2\}$  and  $\Gamma_I(R)$  has loop.

**Theorem 2.8.** Let  $I$  be a strong co-ideal of a semiring  $R$ . Then the following statements hold.

- (1)  $\Gamma_I(R)$  is connected with  $\text{diam}(\Gamma_I(R)) \leq 3$ .
- (2) If  $I$  is a subtractive co-ideal of  $R$  with  $|S_I(R)| \geq 3$  then  $\Gamma_I(R)$  is not a complete graph. In particular,  $\text{diam}(\Gamma_I(R)) = 2$  or  $3$ .

PROOF: (1) Let  $x, y \in S_I(R)$ . If  $x + y \in I$ , then  $x, y$  are adjacent and  $d(x, y) = 1$ . Thus suppose that  $x + y \notin I$ . By Theorem 2.6(1),  $x + x \notin I, y + y \notin I$ . As  $x, y \in S_I(R)$ ,  $x + a \in I, y + b \in I$  for some  $a, b \in R \setminus (I \cup \{x, y\})$ . If  $a = b$ , then  $x - a - y$  is a path. If  $a \neq b$  and  $a + b \in I$ , then  $x - a - b - y$  is a path. If  $a \neq b$  and  $a + b \notin I$ , then  $x - a + b - y$  is a path. Thus  $\Gamma_I(R)$  is connected with  $\text{diam}\Gamma_I(R) \leq 3$ .

(2) Assume that  $\Gamma_I(R)$  is complete and let  $a, b, c \in S_I(R)$  be distinct elements. Then  $a + c, a + b \in I$ , so  $bc \in (I : a)$ , since  $(I : a)$  is a strong co-ideal of  $R$  by Proposition 2.1(3). If  $bc \in I$ , then Proposition 2.1(1) gives  $b, c \in I$  that is a contradiction. So  $bc \notin I$ . If  $bc = c$ , then  $c + b = bc + b = b(1 + c) \in I$ , implying  $b \in I$  by Proposition 2.1, a contradiction. So  $bc \neq c$ . Since  $\Gamma_I(R)$  is complete,

$c(b + 1) = bc + c \in I$ ; hence  $c \in I$  which is a final contradiction. Thus  $\Gamma_I(R)$  is not complete (so  $\text{diam}(\Gamma_I(R)) \neq 1$ ). Finally, by (1) and Proposition 2.6(1),  $\text{diam}(\Gamma_I(R)) = 2$  or  $3$ .  $\square$

Note that the condition that  $I$  is subtractive is necessary in Theorem 2.8(2), as the following example shows.

**Example 2.9.** Assume that  $R = \{0, 1, 2, 3, 4, 5\}$ . Define

$$a + b = \begin{cases} 5 & \text{if } a \neq 0, b \neq 0, a \neq b, \\ a & \text{if } a = b, \\ b & \text{if } a = 0, \\ a & \text{if } b = 0. \end{cases}$$

and

$$a * b = \begin{cases} 0 & \text{if } a = 0 \text{ or } b = 0, \\ 3 & \text{if } a = b = 2, \\ b & \text{if } a = 1, \\ a & \text{if } b = 1, \\ 5 & \text{otherwise.} \end{cases}$$

Then  $(R, +, *)$  is easily checked to be a commutative semiring. An inspection will show that  $I = \{1, 5\}$  is a co-ideal of  $R$  which is not subtractive because  $5 * 2 \in I$ ,  $5 \in I$  but  $2 \notin I$ . Also  $S_I(R) = \{2, 3, 4\}$  and  $\Gamma_I(R)$  is a complete graph.

A vertex  $x$  of a connected graph  $G$  is a cut-point of  $G$  if there are vertices  $y$  and  $z$  of  $G$  such that  $x$  is in every path from  $y$  to  $z$  (and  $x \neq y, x \neq z$ ). Equivalently, for a connected graph  $G$ ,  $x$  is a cut-point of  $G$  if  $G - \{x\}$  is not connected.

**Example 2.10.** Let  $R = (\{0, 1, 2, 4, 5, 10, 20, 25, 50, 100\}, \text{gcd}, \text{lcm})$  (take  $\text{gcd}(0, 0) = 0$  and  $\text{lcm}(0, 0) = 0$ ) and  $I = \{1, 2\}$  be a strong co-ideal of  $R$ . Observe that  $S_I(R) = \{4, 5, 10, 25, 50\}$ . It can be easily seen that 4 is a cut-point of  $\Gamma_I(R)$ .

In the next theorems, we completely characterize the girth of the graph  $\Gamma_I(R)$ . A cycle graph or a circular graph is a graph that consists of a single cycle, or in other words, some number of vertices connected in a closed chain.

**Theorem 2.11.** *Let  $I$  be a strong co-ideal of a semiring  $R$ .*

- (1) *If  $\Gamma_I(R)$  contains a cycle, then  $\text{gr}(\Gamma_I(R)) \leq 4$ .*
- (2) *If  $I$  is a  $Q$ -strong co-ideal such that  $\Gamma(R/I)$  and  $\Gamma_I(R)$  contain a cycle, then  $\text{gr}(\Gamma_I(R)) = \text{gr}(\Gamma(R/I))$ . Moreover, If  $\Gamma(R/I)$  has only two vertices  $q_1I$  and  $q_2I$  with  $|q_iI| \geq 2$  ( $i = 1, 2$ ), then  $\text{gr}(\Gamma_I(R)) = 4$ .*
- (3) *If  $I$  is a subtractive co-ideal, then the only cycle graph with respect to  $I$  is  $K_{2,2}$ .*

PROOF: (1) It is well-known that for any connected graph  $G$ , if  $G$  contains a cycle, then  $\text{gr}(G) \leq 2\text{diam}(G) + 1$ . Suppose that  $\Gamma_I(R)$  contains a cycle. Hence  $\text{gr}(\Gamma_I(R)) \leq 7$ . Suppose that  $\text{gr}(\Gamma_I(R)) = n$ , where  $n \in \{5, 6, 7\}$  and let  $x_1 - x_2 - \dots - x_n - x_1$  be a cycle of minimum length. Since  $x_1$  is not adjacent to  $x_3$ ,

$x_1 + x_3 \notin I$ . If  $x_1 + x_3 \neq x_i$  for each  $1 \leq i \leq n$ , then  $x_2 - x_3 - x_4 - x_1 + x_3 - x_2$  is a 4-cycle, that is, a contradiction. Therefore  $x_1 + x_3 = x_i$  for some  $1 \leq i \leq n$ . We split the proof into three cases.

**Case 1:** If  $x_1 + x_3 = x_1$  (resp.  $x_1 + x_3 = x_3$ ), then  $x_1 - x_2 - x_3 - x_4 - x_1$  (resp.  $x_1 - x_2 - x_3 - x_n - x_1$ ) is a 4-cycle, a contradiction.

**Case 2:** If  $x_1 + x_3 = x_2$  (resp.  $x_1 + x_3 = x_4$ ), then  $x_2 - x_3 - x_4 - x_2$  (resp.  $x_2 - x_3 - x_4 - x_2$ ) is a 3-cycle, that is, a contradiction.

**Case 3:** If  $x_1 + x_3 = x_n$ , then  $x_2 - x_3 - x_4 - x_n - x_2$  is a 4-cycle, which is a contradiction. Thus, every case leads to a contradiction; hence  $\text{gr}(\Gamma_I(R)) \leq 4$ .

(2) Assume that  $\text{gr}(\Gamma_I(R)) = n$  and let  $x_1 - x_2 - \dots - x_n - x_1$  be a cycle in  $\Gamma_I(R)$ . Since  $I$  is a  $Q$ -strong co-ideal, there exist unique elements  $q_i \in Q$  ( $1 \leq i \leq n$ ) such that  $x_i \in q_i I$ . By Proposition 2.5,  $q_1 I - q_2 I - \dots - q_n I - q_1 I$  is a cycle in  $\Gamma(R/I)$ ; thus  $\text{gr}(\Gamma(R/I)) \leq \text{gr}(\Gamma_I(R))$ . Now suppose that  $\text{gr}(\Gamma(R/I)) = m$  and let  $q_1 I - q_2 I - \dots - q_m I - q_1 I$  be a cycle of length  $m$  in  $\Gamma(R/I)$ . Then  $q_1 - q_2 - \dots - q_m - q_1$  is a cycle of length  $m$  in  $\Gamma_I(R)$  by Proposition 2.5, so  $\text{gr}(\Gamma_I(R)) \leq \text{gr}(\Gamma(R/I))$ . Thus  $\text{gr}(\Gamma_I(R)) = \text{gr}(\Gamma(R/I))$ . Let  $\Gamma(R/I)$  have only two vertices  $q_1 I$  and  $q_2 I$ ; we show that  $\text{gr}(\Gamma_I(R)) = 4$ . Let  $x, y \in S_I(R)$ . If  $x, y$  are adjacent, then  $x \in q_i I$  and  $y \in q_j I$ , where  $i \neq j \in \{1, 2\}$ , and if  $x, y$  are not adjacent, then either  $x, y \in q_1 I$  or  $x, y \in q_2 I$  by Proposition 2.5. Also, as  $q_1 I$  and  $q_2 I$  are adjacent in  $\Gamma(R/I)$ , every element of  $q_1 I$  and  $q_2 I$  are adjacent in  $\Gamma_I(R)$  by Proposition 2.5. Hence  $\Gamma_I(R)$  is complete bipartite with two parts  $q_1 I$  and  $q_2 I$ . Since  $|q_i I| \geq 2$  for  $i = 1, 2$ ,  $\text{gr}(\Gamma_I(R)) = 4$ .

(3) By Theorem 2.8(2), there is no 3-cycle graph. By (1), there are no cycle graph with five or more vertices. So the only cycle graph is  $K_{2,2}$ .  $\square$

Note that the condition that  $\Gamma_I(R)$  and  $\Gamma(R/I)$  contain cycle in Theorem 2.11(2) is necessary as the following example shows.

**Example 2.12.** Let  $R$  and  $I$  be as stated in Example 2.4. As we see  $\text{gr}(\Gamma(R/I)) = \infty$  and  $\text{gr}(\Gamma_I(R)) = 4$ .

For a graph  $G$  and vertex  $x \in V(G)$ , the degree of  $x$ , denoted  $\text{deg}(x)$ , is the number of edges of  $G$  incident with  $x$ .

**Theorem 2.13.** *Let  $I$  be a subtractive co-ideal of a semiring  $R$ . Then the following assertions hold:*

- (1)  $\text{gr}(\Gamma_I(R)) = \infty$  if and only if  $\Gamma_I(R)$  is a star graph,
- (2)  $\text{gr}(\Gamma_I(R)) = 4$  if and only if  $\Gamma_I(R)$  is bipartite but not a star graph,
- (3)  $\text{gr}(\Gamma_I(R)) = 3$  if and only if  $\Gamma_I(R)$  contains an odd cycle,
- (4) if  $\text{gr}(\Gamma_I(R)) = 4$ , then there is no end vertex (i.e, vertex with degree 1) in  $\Gamma_I(R)$ .

PROOF: (1) First suppose that  $\text{gr}(\Gamma_I(R)) = \infty$  and  $\Gamma_I(R)$  is not a star graph. So  $|S_I(R)| \geq 4$ , because  $\Gamma_I(R)$  is not complete by Theorem 2.8(2). Since  $\Gamma_I(R)$  is connected, there exists a vertex  $x \in S_I(R)$  such that  $\text{deg}(x) \geq 2$ . As  $\Gamma_I(R)$  is not a star graph, there exists a path of the form  $a - x - b - c$  in  $\Gamma_I(R)$  for some

$a, b, c \in S_I(R)$ . If  $a$  is adjacent to  $c$ , then  $a - x - b - c - a$  is a cycle in  $\Gamma_I(R)$ , a contradiction. If  $a$  is not adjacent to  $c$ , then  $a + c \notin I$ . Since  $a + c + x \in I$ ,  $a + c \in S_I(R)$  and  $x - a + c - b - x$  is a cycle which is a contradiction. Thus  $\Gamma_I(R)$  is a star graph. The other implication is clear.

(2) Let  $\text{gr}(\Gamma_I(R)) = 4$ . So  $\Gamma_I(R)$  is not a star graph by (1). It is known that a graph is bipartite if and only if it contains no odd cycle [[4], Theorem 4.7]. Thus it suffices to show that  $\Gamma_I(R)$  has no odd cycle. Assume that  $x_1 - x_2 - \dots - x_n - x_1$  is an odd cycle of minimal length  $n$  in  $\Gamma_I(R)$ . Since  $\text{gr}(\Gamma_I(R)) = 4$ ,  $n \geq 5$ . As  $\text{gr}(\Gamma_I(R)) \neq 3$ ,  $x_2$  is not adjacent to  $x_4$ , and so  $x_2 + x_4 \notin I$ . Since  $x_2 + x_4 + x_1 \in I$ ,  $x_2 + x_4 \in S_I(R)$ . It follows that  $x_1 - x_2 + x_4 - x_5 - \dots - x_n - x_1$  is an odd cycle of length  $n - 2$  in  $\Gamma_I(R)$ , a contradiction. Hence  $\Gamma_I(R)$  is a bipartite graph. Conversely, let  $\Gamma_I(R)$  be bipartite which is not a star graph. Therefore  $\Gamma_I(R)$  has no odd cycle, and so  $\text{gr}(\Gamma_I(R)) \neq 3$ . By (1),  $\text{gr}(\Gamma_I(R)) \neq \infty$ . Therefore  $\text{gr}(\Gamma_I(R)) = 4$  by Theorem 2.11(1).

(3) If  $\text{gr}(\Gamma_I(R)) = 3$ , then we are done. Conversely, assume that  $\Gamma_I(R)$  has an odd cycle. Let  $\text{gr}(\Gamma_I(R)) \neq 3$ . If  $\text{gr}(\Gamma_I(R)) = 4$ , then (2) implies that  $\Gamma_I(R)$  is a bipartite graph which is not a star graph. Therefore, by [4, Theorem 4.7],  $\Gamma_I(R)$  contains no odd cycle, a contradiction. If  $\text{gr}(\Gamma_I(R)) = \infty$ , then  $\Gamma_I(R)$  is a star graph which contradicts our assumption. Therefore  $\text{gr}(\Gamma_I(R)) = 3$ .

(4) First we show that if  $a - b - c - d$  is a path in  $\Gamma_I(R)$  such that the edge  $b - c$  is not contained in a 3-cycle and  $a, b, c, d$  are vertices, then the vertices  $a$  and  $d$  are distinct and are adjacent to each other. Clearly  $a \neq d$ . Assume that  $a, d$  are not adjacent. Since  $a + b \in I$ ,  $(a + d) + b \in I$ ; hence  $a + d \in S_I(R)$ . Thus  $a + d - b - c - a + d$  is a 3-cycle, a contradiction.

Now let  $a$  be an end vertex in  $\Gamma_I(R)$  and  $b$  be a vertex in  $\Gamma_I(R)$  such that  $a$  and  $b$  are adjacent. Since  $\text{gr}(\Gamma_I(R)) < \infty$ ,  $\Gamma_I(R)$  is not a star graph by (1). By Theorem 2.8(1),  $\Gamma_I(R)$  is connected, hence there is a path  $a - b - c - d$  in  $\Gamma_I(R)$  with  $c, d \notin \{a, b\}$ , since  $\Gamma_I(R)$  has at least 4 elements. As  $\text{gr}(\Gamma_I(R)) = 4$ , the edge  $b - c$  is not contained in a 3-cycle. By the above considerations,  $a \neq d$  and  $a, d$  are adjacent to each other which is contradiction.  $\square$

**Example 2.14.** (1) Let  $R = (\{0, 1, 2, 4, 5, 10, 20, 25, 50, 100\}, \text{gcd}, \text{lcm})$  (take  $\text{gcd}(0, 0) = 0$  and  $\text{lcm}(0, 0) = 0$ ) and  $I = \{1, 2\}$  a strong co-ideal of  $R$ . Then  $S_I(R) = \{4, 5, 10, 25, 50\}$ . It can be easily seen that  $\Gamma_I(R)$  is a star graph and  $\text{gr}(\Gamma_I(R)) = \infty$  (see Example 2.10).

(2) Let  $X = \{a, b, c\}$  and  $R = (P(X), \cup, \cap)$ . Then  $I = \{X, \{a, b\}\}$  is a strong co-ideal of  $R$ ,  $\Gamma_I(R)$  is a complete bipartite graph and  $\text{gr}(\Gamma_I(R)) = 4$ .

(3) Let  $R = (\{0, 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}, \text{gcd}, \text{lcm})$ . Then  $I = \{1, 2\}$  is a strong co-ideal of  $R$  and  $\Gamma_I(R)$  is a graph with odd cycle. It can be easily seen that  $\text{gr}(\Gamma_I(R)) = 3$ .

### 3. Complete $r$ -partite graph

In this section we state some theorems, which characterize the complete bipartite identity-summand graph  $\Gamma_I(R)$  with respect to strong co-ideal  $I$  of a semiring  $R$ .

**Theorem 3.1.** *Let  $I$  be a strong co-ideal of a semiring  $R$ . If there exist two prime strong co-ideals  $P_1$  and  $P_2$  of  $R$  such that  $I = P_1 \cap P_2$ , then  $\Gamma_I(R)$  is a complete bipartite graph, and the converse is true when  $I$  is a subtractive co-ideal of  $R$ .*

PROOF: We show that  $\Gamma_I(R)$  is a complete bipartite graph with two parts  $V_1 = P_1 \setminus I$  and  $V_2 = P_2 \setminus I$ . Let  $a, b \in R \setminus I$  with  $a + b \in I$ ; so  $a + b \in P_1 \cap P_2$ . Since  $P_1, P_2$  are prime and  $a, b \notin I$ , either  $a \in P_1 \setminus I, b \in P_2 \setminus I$  or  $a \in P_2 \setminus I, b \in P_1 \setminus I$ .

Let  $a, b \in S_I(R)$  be such that  $a \in P_2 \setminus I, b \in P_1 \setminus I$ . Then  $a + b \in P_1 \cap P_2 = I$ ; hence  $a, b$  are adjacent. Now we show that each two elements of  $V_i$  are not adjacent. Let  $c, d \in V_1$  (so  $c, d \notin I$ ). If  $c + d \in I$ , then  $c + d \in P_2$  gives  $c \in P_2$  or  $d \in P_2$ . As  $c, d \in V_1 \subset P_1, c \in I$  or  $d \in I$ , a contradiction. Similarly, each two elements of  $V_2$  are not adjacent. So  $\Gamma_I(R)$  is complete bipartite with two parts  $V_1$  and  $V_2$ .

Conversely, suppose that  $I$  is a subtractive co-ideal and let  $V_1, V_2$  be two parts of  $\Gamma_I(R)$ . Set  $P_1 = V_1 \cup I$  and  $P_2 = V_2 \cup I$ . One can easily see that  $I = P_1 \cap P_2$ . First we show that  $P_1, P_2$  are strong co-ideals of  $R$ . Let  $a, b \in P_1$ . If  $a, b \in I$ , then  $ab \in I \subseteq P_1$ . So we may assume that  $a \notin I$  or  $b \notin I$ . If  $a, b \in V_1$ , we have  $a + c \in I$  and  $b + c \in I$  for each  $c \in V_2$ , since  $\Gamma_I(R)$  is complete bipartite. By Proposition 2.1,  $a, b \in (I : c)$  gives  $ab \in (I : c)$ . If  $ab \in I$ , then  $a \in I$  and  $b \in I$  by Proposition 2.1 which is a contradiction. Thus  $ab \in S_I(R)$ . Since  $ab + c \in I$  for each  $c \in V_2, ab \in V_1$ ; so  $ab \in P_1$ . If  $a \in V_1$  and  $b \in I$ , then  $a + c, b + c \in I$  for each  $c \in V_2$  and  $ab \notin I$ . As  $I$  is subtractive,  $ab + c \in I$  by Proposition 2.1, which gives  $ab \in V_1$ . Now suppose that  $a \in P_1$  and  $r \in R$ ; we show that  $a + r \in P_1$ . If  $a \in I$ , then  $a + r \in I \subseteq P_1$ . If  $a \in V_1$ , then  $a + c \in I$  for each  $c \in V_2$ . Since  $I$  is a co-ideal of  $R, (a + r) + c \in I$  for each  $r \in R$ . If  $a + r \notin I$ , then  $a + r \in V_1 \subseteq P_1$  (because  $c \in V_2$  and  $\Gamma_I(R)$  is bipartite). If  $a + r \in I$ , then  $a + r \in P_1$ . Therefore  $P_1$  is a co-ideal of  $R$ . As  $I$  is a strong co-ideal and  $1 \in I \subseteq P_1, P_1$  is a strong co-ideal of  $R$ . Similarly,  $P_2$  is a strong co-ideal.

Now we claim that  $P_1$  is prime. Let  $a + b \in P_1$  such that  $a, b \notin P_1$ ; so  $a, b \notin I$ . If  $a + b \in I$ , then either  $a \in V_1$  and  $b \in V_2$  or  $a \in V_2$  and  $b \in V_1$  which is a contradiction, since  $a, b \notin P_1$ . Thus  $a + b \notin I$ . If  $a + b \in V_1$ , then  $a + b + c \in I$  for each  $c \in V_2$ . We claim that  $b + c \notin I$ . If  $b + c \in I$ , then  $c \in V_2$  gives  $b \in V_1$ , a contradiction. Hence  $b + c \notin I$ . By the similar way,  $a + c \notin I$ . Since  $a + (b + c) \in I$  and  $a \notin V_1$ , we have  $a \in V_2$  and  $b + c \in V_1$ . Likewise,  $b \in V_2$  and  $a + c \in V_1$ . Because  $a + b + c \in I, (a + c) + (b + c) \in I$ . It shows that two vertices  $a + c$  and  $b + c$  of  $V_1$  are adjacent, a contradiction. Thus  $P_1$  is a prime strong co-ideal of  $R$ . Similarly,  $P_2$  is a prime strong co-ideal of  $R$ .  $\square$

**Example 3.2.** Let  $X = \{a, b, c\}$ ,  $R = (P(X), \cup, \cap)$  and  $I = \{X, \{b, c\}\}$ . Consider

$$P_1 = (I : \{a, b\}) = \{\{c\}, \{b, c\}, \{a, c\}, X\},$$

$$P_2 = (I : \{a, c\}) = \{\{b\}, \{a, b\}, \{c, b\}, X\}.$$

An inspection will show that  $P_1$  and  $P_2$  are prime strong co-ideals of  $R$  and  $I = P_1 \cap P_2$ . It is easy to see that  $\Gamma_I(R)$  is a complete bipartite graph with  $S_I(R) = \{\{b\}, \{c\}, \{a, b\}, \{a, c\}\}$ .

**Theorem 3.3.** *Let  $I$  be a subtractive co-ideal of a semiring  $R$ . If  $\Gamma_I(R)$  is complete  $r$ -partite, then  $\Gamma_I(R)$  is a complete bipartite graph.*

PROOF: Let  $V_1, V_2, \dots, V_r$  be parts of  $\Gamma_I(R)$ ,  $r \geq 3$  and  $c_i \in V_i$  for each  $i$ . Since  $\Gamma_I(R)$  is complete  $r$ -partite,  $c_1 + c_3 \in I$  and  $c_3 + c_2 \in I$ . Since  $c_1, c_2 \in (I : c_3)$  and  $(I : c_3)$  is a strong co-ideal by Proposition 2.1,  $c_1c_2 \in (I : c_3)$ . As  $c_1, c_2 \notin I$ ,  $c_1c_2 \notin I$  by Proposition 2.1(1). We claim that  $c_1c_2 \in V_1$ . If not, then  $c_1(c_2 + 1) = c_1c_2 + c_1 \in I$  because  $c_1 \in V_1$ . Since  $I$  is subtractive, we get  $c_1 \in I$ , a contradiction. Therefore  $c_1c_2$  and  $c_1$  are not adjacent (because  $c_1, c_1c_2 \in V_1$ ). As  $c_2 \in V_2$ ,  $c_2(c_1 + 1) = c_1c_2 + c_2 \in I$ . Since  $I$  is subtractive,  $c_2 \in I$ , a contradiction. Hence  $r = 2$ . □

The connectivity of a graph  $G$ , denoted by  $k(G)$ , is defined to be the minimum number of vertices that are necessary to remove from  $G$  in order to produce a disconnected graph.

**Theorem 3.4.** *Let  $I$  be a  $Q$ -strong co-ideal of a semiring  $R$ . If  $\Gamma(R/I)$  is the graph on only two vertices  $q_1I, q_2I$ , then*

- (1)  $\Gamma_I(R)$  is a complete bipartite graph and  $k(\Gamma_I(R)) = \min\{|q_1I|, |q_2I|\}$ ;
- (2)  $I = P_1 \cap P_2$ , where  $P_1 = q_1I \cup I$  and  $P_2 = q_2I \cup I$  are prime strong co-ideals of  $R$ .

PROOF: (1) Since  $q_1I$  and  $q_2I$  are the only vertices of  $\Gamma(R/I)$  and  $\Gamma(R/I)$  has no loop,  $q_1I \oplus q_2I = I$ ; so  $q_1a + q_2b \in I$  for each  $a, b \in I$ . Since by Proposition 2.5, all elements of  $q_1I$  and  $q_2I$  are adjacent and none of elements of  $q_iI$  are adjacent together, we get  $\Gamma_I(R)$  is a complete bipartite graph. The other statement is clear.

(2) It is clear by the proof of Theorem 3.1. □

**Example 3.5.** Let  $R = (P(X), \cup, \cap)$ , where  $X = \{a, b, c\}$ . By Example 2.12,  $I = \{X, \{a, b\}\}$  is a  $Q$ -strong co-ideal of  $R$  with  $Q = \{q_1 = \{c\}, q_2 = \{a, c\}, q_3 = \{b, c\}, q_e = X\}$  and  $S^*(R/I) = \{q_2I, q_3I\}$ . Since  $\Gamma(R/I)$  has only two vertices,  $I = P_1 \cap P_2$ , where  $P_1 = q_2I \cup I$  and  $P_2 = q_3I \cup I$ . Moreover  $k(\Gamma_I(R)) = 2$ .

For every nonnegative integer  $r$ , the graph  $G$  is called  $r$ -regular if the degree of each vertex of  $G$  is equal to  $r$ .

**Theorem 3.6.** *Let  $I$  be a subtractive co-ideal of a semiring  $R$ , and let  $\Gamma_I(R)$  be a finite regular graph. Then  $\Gamma_I(R)$  is  $K_{n,n}$  for some  $n \in \mathbb{N}$ .*

PROOF: The proof is similar to [8, Theorem 4.8]. □

#### 4. Chromatic number, clique number and planar property

In this section we collect some basic properties concerning chromatic number and clique number of the graph  $\Gamma_I(R)$ .

**Proposition 4.1.** *Let  $I$  be a co-ideal of a semiring  $R$ .*

- (1) *If  $I$  is a  $Q$ -strong co-ideal, then  $w(\Gamma_I(R)) \leq |Q| - 2$ .*
- (2) *If  $I$  is a subtractive co-ideal with  $w(\Gamma_I(R))$  being finite, then  $R$  has a.c.c on co-ideals of the form  $(I : a)$ , where  $a \in R$ . Moreover, if  $(I : a_i)$  and  $(I : a_j)$  are distinct maximal elements of  $\Delta = \{(I : a) : a \in R \setminus I\}$ , then  $a_i$  is adjacent to  $a_j$  in  $\Gamma_I(R)$ .*

PROOF: (1) If  $w(\Gamma_I(R)) = \infty$ , then  $Q$  must be infinite by Proposition 2.5. Assume that  $w(\Gamma_I(R)) = n$  and let  $x_1, x_2, \dots, x_n$  be the vertices of the greatest complete subgraph of  $\Gamma_I(R)$ . Since  $I$  is a  $Q$ -strong co-ideal, there exist unique elements  $q_i \in Q$  such that  $x_i \in q_i I$  ( $1 \leq i \leq n$ ). By Proposition 2.5,  $q_i \neq q_0, q_e$  and  $q_i \neq q_j$  for each  $1 \leq i \neq j \leq n$ . Thus  $w(\Gamma_I(R)) \leq |Q| - 2$ .

(2) The proof of the first statement is similar to [8, Lemma 5.1]. Now, if  $(I : a_i)$  and  $(I : a_j)$  are distinct maximal elements of  $\Delta = \{(I : a) : a \in R \setminus I\}$  (partially ordered by inclusion), then by the usual argument, one can show that  $(I : a_i)$  and  $(I : a_j)$  are prime. We show  $a_i + a_j \in I$ . If not, then  $(I : a_i) \subseteq (I : a_i + a_j)$  and  $(I : a_j) \subseteq (I : a_i + a_j)$ , and hence  $(I : a_i) = (I : a_i + a_j) = (I : a_j)$ , a contradiction.  $\square$

Note that the condition that  $w(\Gamma_I(R))$  is finite is necessary in Proposition 4.1 as the following example shows:

**Example 4.2.** Let  $X = \{x_i : i \in \mathbb{N}\}$  and  $R = (P(X), \cup, \cap)$ . Let  $X_2 = \{x_i : i \geq 2\}$  and  $I = \{X_2, X\}$ . It is clear that  $I$  is a subtractive co-ideal of  $R$ . Set  $Y_j = X - \{x_j\}$ . Then  $A = \{Y_i : i \in \mathbb{N}\}$  is an infinite clique in  $\Gamma_I(R)$ . An inspection shows that the following chain is infinite:

$$(I : \{x_2\}) \subseteq (I : \{x_2, x_3\}) \subseteq (I : \{x_2, x_3, x_4\}) \subseteq \dots$$

The next theorem does establish a relation between the clique numbers of  $\Gamma_I(R)$  and  $\Gamma(R/I)$ .

**Theorem 4.3.** *Let  $I$  be a  $Q$ -strong co-ideal of a semiring  $R$ . Then  $w(\Gamma_I(R)) = w(\Gamma(R/I))$ .*

PROOF: Assume that  $\{x_i\}_{i \in J}$  is a clique in  $\Gamma_I(R)$  and let  $q_i$  be the unique element of  $Q$  such that  $x_i \in q_i I$  ( $i \in J$ ). Then  $\{q_i I\}_{i \in J}$  is a clique in  $\Gamma(R/I)$  by Proposition 2.5. Hence  $w(\Gamma(R/I)) \geq w(\Gamma_I(R))$ . Now, let  $\{q_i I\}_{i \in K}$  be a clique in  $\Gamma(R/I)$ , then  $\{q_i\}_{i \in K}$  is a clique in  $\Gamma_I(R)$ , by Proposition 2.5. Thus  $w(\Gamma_I(R)) \geq w(\Gamma(R/I))$ . Therefore,  $w(\Gamma_I(R)) = w(\Gamma(R/I))$ .  $\square$

The next several results investigate the relationship between the chromatic number and clique number of the graph  $\Gamma_I(R)$ .

**Theorem 4.4.** *Let  $R$  be a semiring and  $I$  be a subtractive co-ideal of  $R$ . Then the following are equivalent:*

- (1)  $\chi(\Gamma_I(R))$  is finite;
- (2)  $w(\Gamma_I(R))$  is finite;
- (3) the subtractive co-ideal  $I$  is a finite intersection of prime co-ideals.

PROOF: The proof is similar to [8, Theorem 5.2]. □

*Remark 4.5.* Let  $P, I$  be strong co-ideals of a semiring  $R$  with  $P$  prime and  $I \subseteq P$ . Then the non-empty set  $\Delta = \{P' \in \text{co-Spec}(R) : I \subseteq P' \subseteq P\}$  has a minimal element  $P_1$  with respect to inclusion (by partially ordering  $\Delta$  by reverse inclusion and using Zorn's Lemma), so  $P_1$  is an element of  $\min(I)$ , the set of minimal prime strong co-ideals of  $R$  containing  $I$ . Thus if  $P$  is a prime strong co-ideal of the commutative semiring  $R$  and  $P$  contains the strong co-ideal  $I$  of  $R$ , then there exists a minimal prime strong co-ideal  $P'$  of  $R$  with  $I \subseteq P' \subseteq P$ .

**Theorem 4.6.** *Let  $I$  be a subtractive co-ideal of a semiring  $R$ .*

- (1) *If  $\{P_\alpha\}_{\alpha \in \Lambda}$  is the set of all prime strong co-ideals of  $R$  containing  $I$ , then  $I = \bigcap_{\alpha \in \Lambda} P_\alpha$ .*
- (2) *If  $P_1, \dots, P_n$  are the only distinct minimal prime strong co-ideals of  $R$  containing  $I$ , then  $\bigcap_{i=1}^n P_i = I$  and  $I \neq \bigcap_{1 \leq i \leq n, i \neq j} P_i$ , for each  $1 \leq j \leq n$ .*

PROOF: (1) We need to show that  $\bigcap_{\alpha \in \Lambda} P_\alpha \subseteq I$ . Let  $x \in \bigcap_{\alpha \in \Lambda} P_\alpha$  with  $x \notin I$ . Set  $\Sigma = \{J : J \text{ is a subtractive co-ideal of } R \text{ containing } I, x \notin J\}$ . Since  $I \in \Sigma$ ,  $\Sigma \neq \emptyset$ . An inspection will show that the partially ordered set  $(\Sigma, \subseteq)$  has a maximal element by Zorn's Lemma, say  $K$ . Since  $x \notin K$ ,  $K \neq R$ . We show that  $K$  is prime. Let  $a + b \in K$  such that  $a \notin K$ . Hence  $a \in (K : b)$  and  $a \notin K$ . As  $K \subsetneq (K : b)$  and  $(K : b)$  is subtractive by Proposition 2.1,  $x \in (K : b)$ . Hence  $b \in (K : x)$ . It is clear that  $K \subseteq (K : ax)$ . If  $(K : ax) \neq K$ , then  $x \in (K : ax)$ . Hence  $x(1 + a) = x + ax \in K$ . Since  $K$  is subtractive,  $x \in K$ , a contradiction. Therefore  $K = (K : ax)$ . We claim that  $(K : ax) = (K : a) \cap (K : x)$ . Let  $r \in (K : ax)$ . Then  $ax \in (K : r)$ . By Proposition 2.1(3) and 2.1(1),  $a, x \in (K : r)$ . Thus  $r \in (K : x) \cap (K : a)$  and  $(K : ax) \subseteq (K : a) \cap (K : x)$ . For the reverse of inclusion let  $r \in (K : a) \cap (K : x)$ . Then  $a, x \in (K : r)$ . By Proposition 2.1(3),  $ax \in (K : r)$  and so  $r \in (K : ax)$ . Hence the equality holds. As  $b \in (K : a) \cap (K : x)$ ,  $b \in K$ . Thus  $K$  is prime, which implies  $x \in K$ , a contradiction, as needed.

(2) By (1) and Remark 4.5,  $\bigcap_{i=1}^n P_i = I$ . To see the other statement, suppose  $I = \bigcap_{1 \leq i \leq n, i \neq j} P_i$  for some  $1 \leq j \leq n$ . Since for each  $i \neq j$ ,  $P_i \not\subseteq P_j$ , there is  $x_i \in P_i$  such that  $x_i \notin P_j$ . As  $\sum_{i \neq j} x_i \in \bigcap_{1 \leq i \leq n, i \neq j} P_i \subseteq P_j$ , it is clear that  $x_i \in P_j$  for some  $i \neq j$ , that is a contradiction. Thus  $I \neq \bigcap_{1 \leq i \leq n, i \neq j} P_i$  for each  $1 \leq j \leq n$ . □

**Theorem 4.7.** *Let  $I$  be a co-ideal of a semiring  $R$ .*

- (1) *If  $I$  is a subtractive co-ideal of  $R$  which is not prime, then  $w(\Gamma_I(R)) = |\min(I)|$ .*

- (2) If  $I$  is a  $Q$ -strong co-ideal with  $|\min(I)|$  finite, then each  $P \in \min(I)$  is of the form  $P = (I : q)$  for some  $q \in Q$ .
- (3) If  $I$  is a subtractive co-ideal of a semiring  $R$ , then  $\chi(\Gamma_I(R)) = w(\Gamma_I(R))$ .

PROOF: (1) First, we prove that  $|\min(I)| = \infty$  if and only if  $w(\Gamma_I(R)) = \infty$ . It suffices to show that  $|\min(I)|$  is finite if and only if  $w(\Gamma_I(R))$  is finite. Let  $|\min(I)|$  be finite. Let  $|\min(I)|$  be finite. Then  $I$  is a finite intersection of prime co-ideals by Theorem 4.6; so by Theorem 4.4,  $w(\Gamma_I(R))$  is finite. Now assume that  $w(\Gamma_I(R))$  is finite. Hence by Theorem 4.4,  $I = \bigcap_{i=1}^n P_i$  for some prime strong co-ideals  $P_i$  of  $R$ . Let  $\{Q_\alpha\}_{\alpha \in \Lambda} = \min(I)$ . For each  $\alpha \in \Lambda$ ,  $I \subseteq Q_\alpha$ , so  $\bigcap_{i=1}^n P_i \subseteq Q_\alpha$  for each  $\alpha \in \Lambda$ . This implies that  $P_i \subseteq Q_\alpha$  for some  $1 \leq i \leq n$ . Since  $Q_\alpha$  is minimal,  $P_i = Q_\alpha$ . This gives  $\Lambda$  is finite, and so  $|\min(I)|$  is finite.

Let  $|\min(I)| = n$ . By Theorem 4.6(2), there exists  $x_j \in (\bigcap_{1 \leq i \leq n, i \neq j} P_i) \setminus P_j$  for each  $1 \leq j \leq n$ . Since each  $P_i$  is a strong co-ideal of  $R$ ,  $x_i + x_j \in I$ ; hence  $X = \{x_1, x_2, \dots, x_n\}$  is a clique in  $\Gamma_I(R)$ . Hence  $w(\Gamma_I(R)) \geq n$ . Now we show that  $w(\Gamma_I(R)) \leq n$ . We do this by induction on  $n$ . If  $n = 2$ , then  $\Gamma_I(R)$  is a complete bipartite graph by Theorem 3.1; hence  $w(\Gamma_I(R)) = 2$ . Suppose  $n > 2$  and the result is true for any integer less than  $n$ . Let  $\{x_1, x_2, \dots, x_m\}$  be a clique in  $\Gamma_I(R)$ . Thus  $x_1 + x_j \in I = \bigcap_{1 \leq i \leq n} P_i$ . Without loss of generality, suppose that  $x_1 \notin P_1$  and  $x_2, x_3, \dots, x_m \in P_1$  and  $x_2, \dots, x_m \notin \bigcap_{2 \leq i \leq n} P_i$ . Let  $J = \bigcap_{2 \leq i \leq n} P_i$ . Hence  $\{x_2, x_3, \dots, x_m\}$  is a clique in  $\Gamma_J(R)$ . By induction hypothesis  $m - 1 \leq n - 1$  and so  $m \leq n$ .

(2) Let  $I$  be a  $Q$ -strong co-ideal of  $R$  with  $|\min(I)| = n$ . By Theorem 4.6(2),  $I = \bigcap_{i=1}^n P_i$  where  $\min(I) = \{P_1, \dots, P_n\}$ . Then by (1),  $w(\Gamma_I(R)) = n$ . Let  $\{x_1, x_2, \dots, x_n\}$  be a clique in  $\Gamma_I(R)$  where  $x_j \in (\bigcap_{1 \leq i \leq n, i \neq j} P_i) \setminus P_j$ . Since  $I$  is a  $Q$ -strong co-ideal of  $R$ , there exists unique element  $q_j \in Q$  such that  $x_j \in q_j I$  for each  $1 \leq j \leq n$ . As  $\{x_1, x_2, \dots, x_n\}$  is a clique in  $\Gamma_I(R)$ ,  $\{q_1, q_2, \dots, q_n\}$  is a clique in  $\Gamma_I(R)$  by Proposition 2.5. Let  $x_j = q_j a_j$  for some  $a_j \in I$ . We show that  $q_j \in \bigcap_{1 \leq i \leq n, i \neq j} P_i \setminus P_j$ . It suffices to show that  $q_j \notin P_j$  and there is no  $i \neq j$  such that  $q_j \notin P_i$ . If  $q_j \in P_j$ , then  $x_j = q_j a_j \in P_j$ , a contradiction (because  $a_j \in I \subseteq P_j$ ). So  $q_j \notin P_j$ . Also if  $q_j \notin P_i$  for some  $i \neq j$ , then  $q_i + q_j \notin P_i$  and hence  $q_i + q_j \notin I$ , a contradiction (similarly, as  $x_i \notin P_i$ ,  $q_i \notin P_i$ ). Therefore  $q_j \in \bigcap_{1 \leq i \leq n, i \neq j} P_i \setminus P_j$ . We claim that  $(I : q_j) = P_j$ . Let  $x \in (I : q_j)$ . Then  $x + q_j \in I$ , and so  $x + q_j \in P_j$ . Since  $q_j \notin P_j$ ,  $x \in P_j$ . Hence  $(I : q_j) \subseteq P_j$ . For the reverse of inclusion, let  $x \in P_j$ . Then  $q_j \in \bigcap_{1 \leq i \leq n, i \neq j} P_i \setminus P_j$  gives  $x + q_j \in I$ . Therefore  $P_j \subseteq (I : q_j)$  and we have equality.

(3) By Theorem 4.4,  $w(\Gamma_I(R)) = \infty$  if and only if  $\chi(\Gamma_I(R)) = \infty$ . Hence, we assume that  $\chi(\Gamma_I(R))$  is finite. It is known that  $w(\Gamma_I(R)) \leq \chi(\Gamma_I(R))$ . Let  $w(\Gamma_I(R)) = n$ . By Theorem 4.6,  $I = P_1 \cap \dots \cap P_n$ , where for each  $i$ ,  $P_i$  is a minimal prime strong co-ideal. By an argument like that in Theorem 4.4 ((3)  $\Rightarrow$  (1)),  $\chi(\Gamma_I(R)) \leq n$ . Therefore  $\chi(\Gamma_I(R)) = w(\Gamma_I(R))$ .  $\square$

A graph is said to be planar if it can be drawn in the plane so that its edges intersect only at their ends. A subdivision of a graph is a graph obtained by replacing edges of this graph with pairwise internally-disjoint paths. A remarkably

simple characterization of planar graphs was given by Kuratowski in 1930, that says that a graph is planar if and only if it contains no subdivision of  $K_5$  or  $K_{3,3}$  [4]. It is natural to ask for which strong co-ideal  $I$  of  $R$  the  $\Gamma_I(R)$  is planar.

**Proposition 4.8.** *Let  $I$  be a  $Q$ -strong co-ideal of  $R$ .*

- (1) *If  $\Gamma_I(R)$  is planar, then for each edge  $q_1I$ - $q_2I$  of  $\Gamma(R/I)$ ,  $|q_iI| \leq 2$  for some  $1 \leq i \leq 2$ .*
- (2) *If  $\Gamma_I(R)$  is planar, then  $\Gamma(R/I)$  is planar.*

PROOF: (1) Assume that  $\Gamma_I(R)$  is planar and  $q_1I$  and  $q_2I$  are two vertices of  $\Gamma(R/I)$  such that  $|q_iI| \geq 3$  for each  $i = 1, 2$ . Let  $V_1 = \{x_1, x_2, x_3\} \subseteq q_1I$  and  $V_2 = \{y_1, y_2, y_3\} \subseteq q_2I$ . As  $q_1I$  and  $q_2I$  are adjacent in  $\Gamma(R/I)$ ,  $x_i$  and  $y_j$  are adjacent in  $\Gamma_I(R)$  by Proposition 2.5. Then  $V_1$  and  $V_2$  are two parts of a complete bipartite graph as a subgraph of  $\Gamma_I(R)$ . Hence  $\Gamma_I(R)$  is not planar.

(2) Let  $\Gamma_I(R)$  be planar. By Proposition 2.5, two vertices  $q_1I$  and  $q_2I$  are adjacent in  $\Gamma(R/I)$  if and only if  $q_1$  and  $q_2$  are adjacent in  $\Gamma_I(R)$ . Hence we can take  $\Gamma(R/I)$  as a subgraph of  $\Gamma_I(R)$ . If  $\Gamma(R/I)$  is not planar, then  $\Gamma_I(R)$  is not planar, a contradiction. So  $\Gamma(R/I)$  is planar.  $\square$

The following example shows that the converse of Proposition 4.8 is not true.

**Example 4.9.** Let  $X = \{a, b, c, d\}$  and  $R = (P(X), \cup, \cap)$ . An inspection will show that

$$P_1 = \{Y \subseteq X | b \in Y\},$$

$$P_2 = \{Y \subseteq X | a \in Y\}$$

are prime strong co-ideals of  $R$ . Let  $I = P_1 \cap P_2$ . By Theorem 3.1,  $\Gamma_I(R)$  is a complete bipartite graph with parts  $V_1$  and  $V_2$  and  $|V_1| = |P_1 \setminus I| = 4$  and  $|V_2| = |P_2 \setminus I| = 4$ . Hence  $K_{3,3}$  is a subgraph of  $\Gamma_I(R)$ , and so  $\Gamma_I(R)$  is not planar.

Set  $Q = \{q_0 = \{d, c\}, q_e = X, q_1 = \{b, c, d\}, q_2 = \{d, c, a\}\}$ , then  $q_0I = \{\{d, c\}, \{c\}, \{d\}, \emptyset\}$ ,  $q_eI = I$ ,  $q_1I = \{\{b, c, d\}, \{b, c\}, \{b, d\}, \{b\}\}$  and  $q_2I = \{\{d, c, a\}, \{d, a\}, \{a, c\}, \{a\}\}$ . By usual argument,  $I$  is a  $Q$ -strong co-ideal of  $R$  and  $R/I = \{q_0I, q_eI, q_1I, q_2I\}$ . Since  $q_1 + q_2 \in I$ ,  $q_1I \oplus q_2I = I$  by Proposition 2.5. Hence  $S^*(R/I) = \{q_1I, q_2I\}$ . Therefore  $\Gamma(R/I)$  is planar.

**Theorem 4.10.** *Let  $I$  be a subtractive co-ideal of semiring  $R$ . If  $|\min(I)| \geq 4$ , then  $\Gamma_I(R)$  is not planar.*

PROOF: If  $|\min(I)| \geq 5$ , then by Theorem 4.7(1),  $w(\Gamma_I(R)) \geq 5$ . Hence  $\Gamma_I(R)$  is not planar.

If  $|\min(I)| = 4$ , then Theorem 4.7(1) implies that  $w(\Gamma_I(R)) = 4$ . Hence there exists  $\{x_1, x_2, x_3, x_4\} \subseteq S_I(R)$  such that  $\{x_1, \dots, x_4\}$  forms a clique in  $\Gamma_I(R)$ . Let  $x_{ij} = x_i x_j$ , where  $1 \leq i, j \leq 4$ ,  $i \neq j$ . Suppose that  $k \in \{1, 2, 3, 4\} \setminus \{i, j\}$ . Since  $x_i, x_j \in (I : x_k)$ ,  $x_{ij} \in (I : x_k)$ . If  $x_{ij} \in I$ , then  $x_i(x_j + 1) = x_{ij} + x_i \in I$ , hence  $x_i \in I$ , which is a contradiction. This implies that  $x_{ij} \in S_I(R)$ . We claim that

$x_{ij} \notin \{x_1, x_2, x_3, x_4\}$ . Assume that  $x_{ij} = x_s$  for some  $1 \leq s \leq 4$ . If  $s = i$ , then  $x_{ij} + x_j \in I$ . This implies that  $x_i \in I$  which is a contradiction. Similarly, for  $s = j$ . If  $s \neq j$  and  $s \neq i$ , then  $x_{ij} + x_s \in I$ ; hence  $x_s + x_s \in I$ . It follows that  $x_s \in I$  by Proposition 2.1(1), a contradiction. Therefore  $x_{ij} \notin \{x_1, x_2, x_3, x_4\}$ . Let  $s \neq k$  and  $s, k \in \{1, 2, 3, 4\} - \{i, j\}$ . Since  $x_{ij} + x_s \in I$  and  $x_{ij} + x_k \in I$ , we have  $x_s, x_k \in (I : x_{ij})$ ; thus  $x_{sk} \in (I : x_{ij})$ . Set  $V_1 = \{x_1, x_{13}, x_3\}$  and  $V_2 = \{x_2, x_{24}, x_4\}$ . Then  $V_1$  and  $V_2$  are two parts of a complete 2-partite subgraph of  $\Gamma_I(R)$ . Therefore  $\Gamma(R)$  is not planar.  $\square$

In the following example, it is shown that if  $|\min(I)| = 3$ , then  $\Gamma_I(R)$  may be planar.

**Example 4.11.** (1) Let  $R = \{p_1^i p_2^j p_3^k p_4^t : i \in \{0, 1, 2, 3\}, j \in \{0, 1, 2, 3\}, k \in \{0, 1\}, t \in \{0, 1\}\} \cup \{0\}$  where  $p_i$ 's are prime integer. Then  $(R, \gcd, \text{lcm})$  is a semiring and  $I = \{1, p_4\}$  is a subtractive strong co-ideal of  $R$ . Since for each  $1 \leq m, n \leq 3$  where  $m \neq n$ ,  $\gcd(p_m, p_n) = 1 \in I$ ,  $\{p_1, p_2, p_3\}$  is a clique in  $\Gamma_I(R)$  and  $w(\Gamma_I(R)) = 3$ . Hence  $|\min(I)| = 3$  by Theorem 4.7. Set  $V_1 = \{p_1, p_1^2, p_1^3\}$  and  $V_2 = \{p_2, p_2^2, p_2^3\}$ . Then  $K_{3,3}$  is a subgraph of  $\Gamma_I(R)$  with two parts  $V_1$  and  $V_2$ . Hence  $\Gamma_I(R)$  is not planar.

(2) Let  $R = (\{0, 1, 2, 3, 5, 6, 10, 15, 30\}, \gcd, \text{lcm})$ . Then  $I = \{1\}$  is a subtractive strong co-ideal of  $R$  and  $S_I(R) = \{2, 3, 5, 6, 10, 15\}$ . By drawing  $\Gamma_I(R)$ , one can see that  $w(\Gamma_I(R)) = 3$ . Hence  $|\min(I)| = 3$  by Theorem 4.7. Also  $\Gamma_I(R)$  is planar.

*Remark 4.12.* Let  $I$  be a subtractive strong co-ideal of a semiring  $R$ .

(1) If  $|\min(I)| = 1$ , then by Theorem 4.6(2),  $I$  is a prime strong co-ideal of  $R$ . Hence  $\Gamma_I(R) = \emptyset$  by Proposition 2.3.

(2) If  $|\min(I)| = 2$ , then  $I = P_1 \cap P_2$  for some prime strong co-ideals  $P_1$  and  $P_2$  by Theorem 4.6. Hence by Theorem 3.1,  $\Gamma_I(R)$  is  $K_{n,m}$  for some integer  $n$  and  $m$ , where  $|P_1 \setminus I| = n$  and  $|P_2 \setminus I| = m$ . If  $n, m \geq 3$ , then  $K_{3,3}$  is a subgraph of  $\Gamma_I(R)$  and so  $\Gamma_I(R)$  is not planar.

(3) If  $|\min(I)| \geq 4$ , then by Theorem 4.10,  $\Gamma_I(R)$  is not planar.

(4) If  $R$  and  $I$  are the semiring and co-ideal as in Example 4.11(2), then  $|\min(I)| = 3$  and  $\Gamma_I(R)$  is planar. However there exist a semiring  $R$  and a strong co-ideal  $I$  of  $R$  that have only three minimal prime co-ideals and  $\Gamma_I(R)$  is not planar as Example 4.11(1) shows. It is not entirely clear for us for which strong co-ideals  $I$  with  $|\min(I)| = 3$ , the  $\Gamma_I(R)$  is planar.

**Acknowledgment.** The authors express their deep gratitude to the referee for her/his helpful suggestions for the improvement of this work.

### REFERENCES

[1] Anderson D.F., Livingston P.S., *The zero-divisor graph of a commutative ring*, J. Algebra **217** (1999), 434–447.  
 [2] Akbari S., Maimani H.R., Yassemi S., *When a zero-divisor graph is planar or a complete r-partite graph*, J. Algebra **270** (2003), 169–180.  
 [3] Beck I., *Coloring of commutative rings*, J. Algebra **116** (1988), 208–226.

- [4] Bondy J. A., Murty U.S.R., *Graph Theory*, Graduate Texts in Mathematics, 244, Springer, New York, 2008.
- [5] Ebrahimi Atani S., *The zero-divisor graph with respect to ideals of a commutative semiring*, Glas. Mat. **43(63)** (2008), 309–320.
- [6] Ebrahimi Atani S., *An ideal-based zero-divisor graph of a commutative semiring*, Glas. Mat. **44(64)** (2009), 141–153.
- [7] Ebrahimi Atani S., Dolati Pish Hesari S., Khoramdel M., *Strong co-ideal theory in quotients of semirings*, J. Adv. Res. Pure Math. **5(3)** (2013), 19–32.
- [8] Ebrahimi Atani S., Dolati Pish Hesari S., Khoramdel M., *The identity-summand graph of commutative semirings*, J. Korean Math. Soc. **51** (2014), 189–202.
- [9] Dolžan D., Oblak P., *The zero-divisor graphs of rings and semirings*, Internat. J. Algebra Comput. **22** (2012), no. 4, 1250033, 20 pp.
- [10] Golan J.S., *Semirings and Their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.
- [11] Maimani H.R., Pournaki M.R., Yassemi S., *Zero-divisor graph with respect to an ideal*, Comm. Algebra **34** (2006), 923–929.
- [12] Redmond P., *An ideal-based zero-divisor graph of a commutative ring*, Comm. Algebra **31** (2003), 4425–4443.
- [13] Spiroff S., Wickham C., *A zero divisor graph determined by equivalence classes of zero divisors*, Comm. Algebra **39** (2011), 2338–2348.
- [14] Wang H., *On rational series and rational language*, Theoret. Comput. Sci. **205** (1998), 329–336.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GUILAN, P.O.BOX 1914, RASHT, IRAN

*E-mail:* ebrahimi@guilan.ac.ir  
saboura\_dolati@yahoo.com  
mehdikhoramdel@gmail.com

(Received March 23, 2014, revised January 31, 2015)