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On the class of positive almost weak* Dunford-Pettis operators

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Abstract. In this paper, we introduce and study the class of almost weak* Dunford-Pettis operators. As consequences, we derive the following interesting results: the domination property of this class of operators and characterizations of the wDP* property. Next, we characterize pairs of Banach lattices for which each positive almost weak* Dunford-Pettis operator is almost Dunford-Pettis.

Keywords: almost weak* Dunford-Pettis operator; almost Dunford-Pettis operator; weak Dunford-Pettis* property; positive Schur property; order continuous norm

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1. Introduction and notation

Let us recall from [2] that a norm bounded subset A of a Banach lattice E is said to be almost limited if every disjoint weak* null sequence (f_n) of E' converges uniformly on A, that is, $\lim_{n\to\infty} \sup_{x\in A} |f_n(x)| = 0$.

An operator T from a Banach lattice E into a Banach space Y is said to be almost Dunford-Pettis if $\|T(x_n)\| \to 0$ in Y for every weakly null sequence (x_n) consisting of pairwise disjoint elements in E [6].

A Banach space X has the Dunford-Pettis* property (DP* property for short), if $x_n \xrightarrow{w} 0$ in X and $f_n \xrightarrow{w^\star} 0$ in X' imply $f_n(x_n) \to 0$.

A Banach lattice E has

- the positive Schur property, if $\|f_n\| \to 0$ for every weakly null sequence $(f_n) \subset E^+$, equivalently, $\|f_n\| \to 0$ for every weakly null sequence $(f_n) \subset E^+$ consisting of pairwise disjoint terms (see page 16 of [9]);

- the weak Dunford-Pettis* property (wDP* property for short), if every relatively weakly compact set in E is almost limited, equivalently, whenever $f_n(x_n) \to 0$ for every weakly null sequence (x_n) in E and for every disjoint weak* null sequence (f_n) in E' [2].

Recall from [4] that an operator T from a Banach space X into another Banach space Y is called weak* Dunford-Pettis if $f_n(T(x_n)) \to 0$ for every weakly null sequence $(x_n) \subset X$, and every weak* null sequence $(f_n) \subset Y'$. In this paper,

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we introduce and study the disjoint version of this class of operators, that we call almost weak* Dunford-Pettis operators (Definition 2.1). It is a class which contains that of weak* Dunford-Pettis (resp. almost Dunford-Pettis).

The main results are some characterizations of almost weak* Dunford-Pettis operators (Theorem 2.3). Next, we derive the following interesting consequences: the domination property of this class of operators (Corollary 2.4), a characterization of wDP* property (Corollary 2.5). After that, we prove that each positive almost weak* Dunford-Pettis operator from a Banach lattice E into a σ- Dedekind complete Banach lattice F is almost Dunford-Pettis if and only if E has the positive Schur property or the norm of F is order continuous (Theorem 2.7). As consequence, we will give some interesting results (Corollaries 2.8 and 2.9).

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A norm $\|\cdot\|$ of a Banach lattice E is order continuous if for each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 in the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$. A Riesz space is said to be σ-Dedekind complete if every countable subset that is bounded above has a supremum, equivalently, whenever $0 \leq x_n \uparrow \leq x$ implies the existence of $\sup(x_n)$.

We will use the term operator $T : E \longrightarrow F$ between two Banach lattices to mean a bounded linear mapping. It is positive if $T(x) \geq 0$ in F whenever $x \geq 0$ in E. If T is an operator from a Banach lattice E into another Banach lattice F then its dual operator T' is defined from F' into E' by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. We refer the reader to [1] for unexplained terminology of Banach lattice theory and positive operators.

2. Main results

Next we give the definition of almost weak* Dunford-Pettis operator between Banach lattices, which is a different version of the weak* Dunford-Pettis operator.

Definition 2.1. An operator T from a Banach lattice E to a Banach lattice F is almost weak* Dunford-Pettis if $f_n(T(x_n)) \to 0$ for every weakly null sequence (x_n) in E consisting of pairwise disjoint terms, and for every weak* null sequence (f_n) in F' consisting of pairwise disjoint terms.

For proof of the next theorem, we need the following lemma which is just Lemma 2.2 of Chen in [2].

Lemma 2.2. *Let E be a σ-Dedekind complete Banach lattice, and let (f_n) be a weak* convergent sequence of E'. If (g_n) is a disjoint sequence of E' satisfying $|g_n| \leq |f_n|$ for each n, then the sequences $(g_n), (|g_n|), (g_n)^+, (g_n)^-$ are all weak* convergent to zero. In particular, if (f_n) is a disjoint weak* convergent sequence in its own right, then the sequences $(f_n), (|f_n|), (f_n)^+, (f_n)^-$ are all weak* null.*

Now, for positive operators between two Banach lattices, we give a characterization of almost weak⋆ Dunford-Pettis operators.

Theorem 2.3. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. For every positive operator T from E into F, the following assertions are equivalent.*

(1) *T is almost weak⋆ Dunford-Pettis operator.*
(2) *For every disjoint weakly null sequence $(x_n) \subset E^+$, and every disjoint weak⋆ null sequence $(f_n) \subset (F')^+$ it follows that $f_n(T(x_n)) \to 0$.*
(3) *For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak⋆ null sequence $(f_n) \subset F'$ it follows that $f_n(T(x_n)) \to 0$.*
(4) *For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak⋆ null sequence $(f_n) \subset (F')^+$ it follows that $f_n(T(x_n)) \to 0$.*
(5) *For every weakly null sequence $(x_n) \subset E^+$, and every weak⋆ null sequence $(f_n) \subset (F')^+$ it follows that $f_n(T(x_n)) \to 0$.*

PROOF: (1) $\Rightarrow$ (2) Obvious.

(2) $\Rightarrow$ (3) Assume by way of contradiction that there exists a disjoint weakly null sequence $(x_n) \subset E^+$, and a weak⋆ null sequence $(f_n) \subset F'$ such that $f_n(T(x_n))$ does not converge to 0. The inequality $|f_n(T(x_n))| \leq |f_n|(T(x_n))$ implies $|f_n|(T(x_n))$ does not converge to 0. Then there exist some $\epsilon > 0$ and a subsequence of $|f_n|(T(x_n))$ (which we shall denote by $|f_n|(T(x_n))$ again) satisfying $|f_n|(T(x_n)) > \epsilon$ for all n.

On the other hand, since $x_n \to 0$ weakly in E, then $T(x_n) \to 0$ weakly in F. Now an easy inductive argument shows that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that

$$|g_n|(T(z_n)) > \epsilon$$

and

$$\left(4^n \textstyle\sum_{i=1}^{n} |g_i|\right)(T(z_{n+1})) < \tfrac{1}{n}$$

for all $n \geq 1$. Put $h = \sum_{i=1}^{\infty} 2^{-n} |g_n|$ and $h_n = (|g_{n+1}| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n}h)^+$. By Lemma 4.35 of [1] the sequence (h_n) is disjoint. Since $0 \leq h_n \leq |g_{n+1}|$ for all $n \geq 1$ and (g_n) is weak⋆ null in F', then from Lemma 2.2 (h_n) is weak⋆ null in F'. From the inequality

$$\begin{aligned} h_n(T(z_{n+1})) &\geq \left(|g_{n+1}| - 4^n \sum_{i=1}^{n} |g_i| - 2^{-n}h\right)(T(z_{n+1})) \\ &\geq \epsilon - \frac{1}{n} - 2^{-n}h(T(z_{n+1})) \end{aligned}$$

we see that $h_n(T(z_{n+1})) \geq \frac{\epsilon}{2}$ must hold for all n sufficiently large (because $2^{-n}h(T(z_{n+1})) \to 0$), which contradicts with our hypothesis (2).

(3) $\Rightarrow$ (4) Obvious.

(4) $\Rightarrow$ (5) Assume by way of contradiction that there exists a weakly null sequence $(x_n) \subset E^+$ and a weak* null sequence $(f_n) \subset (F')^+$ such that $f_n(T(x_n))$ does not converge to 0. Then there exists some $\epsilon > 0$ and a subsequence of $f_n(T(x_n))$ (which we shall denote by $f_n(T(x_n))$ again) satisfying $f_n(T(x_n)) \geq \epsilon$ for all n.

On the other hand, since (f_n) is a weak* null sequence in (F'), then $T'(f_n) \to 0$ weak* in E'. Now an easy inductive argument shows that there exist a subsequence (z_n) of (x_n) and a subsequence (g_n) of (f_n) such that

$$T'(g_n)(z_n) > \epsilon$$

and

$$T'(g_{n+1})\left(4^n \textstyle\sum_{i=1}^n z_i\right) < \tfrac{1}{n}$$

for all $n \geq 1$. Put $z = \sum_{n=1}^\infty 2^{-n} z_n$ and $y_n = (z_{n+1} - 4^n \sum_{i=1}^n z_i - 2^{-n} z)^+$. By Lemma 4.35 of [1] the sequence (y_n) is disjoint. Since $0 \leq y_n \leq z_{n+1}$ for all $n \geq 1$ and (z_n) is weakly null in E, then from Theorem 4.34 of [1] $(y_n) \to 0$ weakly in E. From the inequality

$$\begin{aligned} T'(g_{n+1})(y_n) &\geq T'(g_{n+1})\left(z_{n+1} - 4^n \sum_{i=1}^n z_i - 2^{-n} z\right) \\ &\geq \epsilon - \frac{1}{n} - 2^{-n} T'(g_{n+1})(z) \end{aligned}$$

we see that $g_{n+1}(T(y_n)) = T'(g_{n+1})(y_n) \geq \frac{\epsilon}{2}$ must hold for all n sufficiently large (because $2^{-n} T'(g_{n+1})(z) \to 0$), which contradicts with our hypothesis (4).

(5) $\Rightarrow$ (1) Let (x_n) be a weak null sequence in E consisting of pairwise disjoint terms, and let (f_n) be a weak* null sequence in F' consisting of pairwise disjoint terms, it follows from Remark(1) of [6] that $(|x_n|)$ is weakly null in E, and from lemma 2.2 that $(|f_n|)$ is weak* null in F'. So by our hypothesis (5), $|f_n|\,(T\,|x_n|) \to 0$. Now, from the inequality $|f_n(T(x_n))| \leq |f_n|\,(T(|x_n|))$ for each n, we deduce that $f_n(T(x_n)) \to 0$, and this completes the proof. □

The domination property for almost weak* Dunford-Pettis operators can be derived from Theorem 2.3.

Corollary 2.4. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. If S and T are two positive operators from E into F such that $0 \leq S \leq T$ and T is an almost weak* Dunford-Pettis, then S is also almost weak* Dunford-Pettis.*

Proof: Let (x_n) be a weakly null sequence in E^+ and (f_n) be a weak* null sequence in $(F')^+$. According to (5) of Theorem 2.3, it suffices to show that $f_n(S(x_n)) \to 0$. Since T is almost weak* Dunford-Pettis, then Theorem 2.3 implies that $f_n(T(x_n)) \to 0$. Now, by the inequality $0 \leq f_n(S(x_n)) \leq f_n(T(x_n))$ for each n, we conclude that $f_n(S(x_n)) \to 0$. □

As consequence of Theorem 2.3 and Theorem 3.2 of Chen [2], other characterizations of Banach lattices with the wDP* property are given in the following Corollary.

Corollary 2.5. *Let E be a σ-Dedekind complete Banach lattice. Then, the following assertions are equivalent.*

(1) *E has the wDP* property.*
(2) *The solid hull of every relatively weakly compact set in E is almost limited.*
(3) *The identity operator $Id_E : E \to E$ is almost weak* Dunford-Pettis.*
(4) *For every disjoint weakly null sequence $(x_n) \subset E$, and every disjoint weak* null sequence $(f_n) \subset E'$ it follows that $f_n(x_n) \to 0$.*
(5) *For every disjoint weakly null sequence $(x_n) \subset E^+$, and every disjoint weak* null sequence $(f_n) \subset (E')^+$ it follows that $f_n(x_n) \to 0$.*
(6) *For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset E'$ it follows that $f_n(x_n) \to 0$.*
(7) *For every disjoint weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (E')^+$ it follows that $f_n(x_n) \to 0$.*
(8) *For every weakly null sequence $(x_n) \subset E^+$, and every weak* null sequence $(f_n) \subset (E')^+$ it follows that $f_n(x_n) \to 0$.*

Proof: (3) $\Leftrightarrow$ (4) Obvious.

(3) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) follows from Theorem 2.3.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (4) follows from Theorem 3.2 of [2]. □

The proof of the next theorem is based on the following proposition.

Proposition 2.6. *Let E, F and G be three Banach lattices such that G has the DP* property. Then, each operator $T : E \to F$ that admits a factorization through the Banach lattice G is almost weak* Dunford-Pettis.*

Proof: Let $P : E \to G$ and $Q : G \to F$ be two operators such that $T = Q \circ P$. Let (x_n) be a disjoint weakly null sequence in E and let (f_n) be a disjoint weak* null sequence in F'. It is clear that $P(x_n) \xrightarrow{w} 0$ in G and $Q'(f_n) \xrightarrow{w^\star} 0$ in G'. As G has the DP* property, then

$$f_n(Tx_n) = f_n(Q \circ P(x_n)) = (Q'f_n)(P(x_n)) \to 0.$$

This proves that T is almost weak* Dunford-Pettis. □

Note that every almost Dunford-Pettis operator is almost weak* Dunford-Pettis, but the converse is not true in general. In fact, $Id_{\ell^\infty} : \ell^\infty \to \ell^\infty$ is almost weak* Dunford-Pettis operator because ℓ^∞ has the wDP* property, but it fails to be almost Dunford-Pettis because ℓ^∞ does not have the positive Schur property.

Now, we characterize Banach lattices such that each positive almost weak* Dunford-Pettis operator is almost Dunford-Pettis.

Theorem 2.7. *Let E and F be two Banach lattices such that F is σ-Dedekind complete. Then the following assertions are equivalent.*

(1) *Each positive almost weak* Dunford-Pettis operator $T : E \to F$ is almost Dunford-Pettis.*

(2) *One of the following assertions is valid:*

(a) *E has the positive Schur property,*

(b) *the norm of F is order continuous.*

Proof: (1) $\Rightarrow$ (2) Assume by way of contradiction that E does not have the positive Schur property and the norm of F is not order continuous. We have to construct a positive almost weak* Dunford-Pettis operator which is not almost Dunford-Pettis. As E does not have the positive Schur property, then there exists a disjoint weakly null sequence (x_n) in E^+ which is not norm null. By choosing a subsequence we may suppose that there is $\epsilon > 0$ with $\|x_n\| > \epsilon > 0$ for all n. From the equality $\|x_n\| = \sup\{f(x_n) : f \in (E')^+, \|f\| = 1\}$, there exists a sequence $(f_n) \subset (E')^+$ such that $\|f_n\| = 1$ and $f_n(x_n) \geq \epsilon$ holds for all n. Now, consider the operator $R : E \to \ell^\infty$ defined by

$$R(x) = (f_n(x))_{n=1}^{\infty}$$

On the other hand, since the norm of F is not order continuous, it follows from Theorem 4.51 of [1] that ℓ^∞ is lattice embeddable in F, i.e., there exists a lattice homomorphism $S : \ell^\infty \to F$ and there exist tow positive constants M and m satisfying

$$m \left\|(\lambda_k)_k\right\|_\infty \leq \left\|S((\lambda_k)_k)\right\|_F \leq M \left\|(\lambda_k)_k\right\|_\infty$$

for all $(\lambda_k)_k \in \ell^\infty$. Put $T = S \circ R$, and note by Proposition 2.6 that T is a positive almost weak* Dunford-Pettis operator because ℓ^∞ has DP* property. However, for the disjoint weakly null sequence $(x_n) \subset E^+$, we have

$$\|T(x_n)\| = \|S((f_k(x_n))_k)\| \geq m \left\|(f_k(x_n))_k\right\|_\infty \geq m f_n(x_n) \geq m\epsilon$$

for every n. This shows that T is not almost Dunford-Pettis, and we are done.

(a) $\Rightarrow$ (1) In this case, each operator $T : E \to F$ is almost Dunford-Pettis.

(b) $\Rightarrow$ (1) Let $(x_n) \subset E$ be a positive disjoint weakly null sequence. We shall show that $\|T(x_n)\| \to 0$. By Corollary 2.6 of [3], it suffices to prove that $|T(x_n)| \xrightarrow{w} 0$ and $f_n(T(x_n)) \to 0$ for every disjoint and norm bounded sequence $(f_n) \subset (F')^+$. Let $f \in (F')^+$ and by Theorem 1.23 of [1] there exists some $g \in [-f, f]$ with $f\,|Tx_n| = g(Tx_n)$. Since $x_n \xrightarrow{w} 0$ then $f\,|Tx_n| = g(Tx_n) = (T'g)(x_n) \to 0$, thus $|T(x_n)| \xrightarrow{w} 0$. On the other hand, let $(f_n) \subset (F')^+$ be a disjoint and norm bounded sequence. As the norm of F is order continuous, then by Corollary 2.4.3 of [5] $f_n \xrightarrow{w^\star} 0$. Now, since T is positive almost weak* Dunford-Pettis then, $f_n(T(x_n)) \to 0$. This completes the proof. $\square$

Remark 1. The assumption that F is σ-Dedekind complete is essential in Theorem 2.7. In fact, if we consider $E = \ell^\infty$ and $F = c$, the Banach lattice of all convergent sequences, it is clear that $F = c$ is not σ-Dedekind complete, and it follows from the proof of Proposition 1 of [7] and Theorem 5.99 of [1] that each operator from ℓ^∞ into c is Dunford-Pettis (and hence is almost Dunford-Pettis). But ℓ^∞ does not have the positive Schur property and the norm of c is not order continuous.

As consequences of Theorem 2.7, we have the following characterization.

Corollary 2.8. *Let E be a σ-Dedekind complete Banach lattice. Then the following assertions are equivalent.*

(1) *Each positive almost weak⋆ Dunford-Pettis operator $T : E \to E$ is almost Dunford-Pettis.*
(2) *The norm of E is order continuous.*

Proof: The result follows from Theorem 2.7 by noting that if E has the positive Schur property then the norm of E is order continuous. □

Now, from Corollary 2.8 and Theorem 4.9 (Nakano) of [1], we obtain the following result, which is just Proposition 3.3 of [2].

Corollary 2.9. *Let E be a Banach lattice. Then E has the positive Schur property if and only if E has the wDP⋆ property and its norm is order continuous.*

Proof: The "only if" part is trivial.

For the "if" part, since E has wDP⋆ property, then $Id_E : E \to E$ is almost weak⋆ Dunford-Pettis operator. As the norm of E is order continuous, it follows from Theorem 4.9 (Nakano) of [1] that E is σ-Dedekind complete, and by Corollary 2.8 we have that $Id_E : E \to E$ is almost Dunford-Pettis. This proves that E has the positive Schur property. □

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