Some results on (n,d)-injective modules, (n,d)-flat modules and n-coherent rings

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Abstract. Let n, d be two non-negative integers. A left R-module M is called (n,d)-injective, if $\mathrm{Ext}^{d+1}(N,M) = 0$ for every n-presented left R-module N. A right R-module V is called (n,d)-flat, if $\mathrm{Tor}_{d+1}(V,N) = 0$ for every n-presented left R-module N. A left R-module M is called weakly n-FP-injective, if $\mathrm{Ext}^n(N,M) = 0$ for every $(n+1)$-presented left R-module N. A right R-module V is called weakly n-flat, if $\mathrm{Tor}_n(V,N) = 0$ for every $(n+1)$-presented left R-module N. In this paper, we give some characterizations and properties of (n,d)-injective modules and (n,d)-flat modules in the cases of $n \geq d+1$ or $n > d+1$. Using the concepts of weakly n-FP-injectivity and weakly n-flatness of modules, we give some new characterizations of left n-coherent rings.

Keywords: (n,d)-injective modules; (n,d)-flat modules; n-coherent rings

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1. Introduction

Throughout this paper, R denotes an associative ring with identity, all modules considered are unitary and n, d are non-negative integers unless otherwise specified. For any R-module M, $M^+ = \mathrm{Hom}(M, \mathbb{Q}/\mathbb{Z})$ will be the character module of M.

Recall that a left R-module A is said to be *finitely presented* if there is an exact sequence $F_1 \to F_0 \to A \to 0$ in which F_1, F_0 are finitely generated free left R-modules, or equivalently, if there is an exact sequence $P_1 \to P_0 \to A \to 0$, where P_1, P_0 are finitely generated projective left R-modules. Let n be a positive integer. Then a left R-module M is called *n-presented* [2] if there is an exact sequence of left R-modules $F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$ in which every F_i is a finitely generated free (or equivalently projective) left R-module. A left R-module M is said to be *FP-injective* [7] if $\mathrm{Ext}^1(A,M) = 0$ for every finitely presented left R-module A. FP-injective modules are also called *absolutely pure modules* [5]. FP-injective modules and their generations have been studied by many authors. For example, following [1], a left R-module M is called *n-FP-injective* if $\mathrm{Ext}^n(N,M) = 0$ for every n-presented left R-module N; a right R-module M is called *n-flat* if $\mathrm{Tor}_n(M,N) = 0$ for every n-presented left R-module N. Following [8], a left R-module M is called *(n,d)-injective*, if

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$\mathrm{Ext}^{d+1}(N, M) = 0$ for every n-presented left R-module N; a right R-module V is called *(n,d)-flat*, if $\mathrm{Tor}_{d+1}(V, N) = 0$ for every n-presented left R-module N. We recall also that a ring R is called *left n-coherent* [2] if every n-presented left R-module is $(n+1)$-presented. In [1], left n-coherent rings are characterized by n-FP-injective modules and n-flat modules. In this paper, we shall give some new characterizations and properties of (n,d)-injective modules and (n,d)-flat modules in the cases of $n \geq d+1$ or $n > d+1$. Moreover, we shall extend the concepts of n-FP-injective modules and n-flat modules to *weakly n-FP-injective modules* and *weakly n-flat modules*, respectively. Using the concepts of weakly n-FP-injectivity and weakly n-flatness of modules, we shall give some new characterizations of left n-coherent rings.

2. Weakly n-FP-injective modules and weakly n-flat modules

We first extend the concepts of n-FP-injective modules and n-flat modules as follows.

Definition 2.1. Let n be a positive integer. Then a left R-module M is called weakly n-FP-injective, if $\mathrm{Ext}^n(N, M) = 0$ for every $(n+1)$-presented left R-module N. A right R-module V is called weakly n-flat, if $\mathrm{Tor}_n(V, N) = 0$ for every $(n+1)$-presented left R-module N.

Theorem 2.2. *Let M be a left R-module and $n \geq d+1$. Then the following statements are equivalent:*

(1) *M is (n,d)-injective;*

(2) *if $F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} N \to 0$ is exact and each F_i is finitely generated and free, then $\mathrm{Ext}^1(\mathrm{Ker}\,(f_{d-1}), M) = 0$;*

(3) *if $F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} N \to 0$ is exact and each F_i is finitely generated and free, then every homomorphism from $\mathrm{Ker}\,(f_d)$ to M extends to F_d.*

PROOF: (1) $\Leftrightarrow$ (2) It follows from the isomorphism

$$\mathrm{Ext}^{d+1}(N, M) \cong \mathrm{Ext}^1(\mathrm{Ker}\,(f_{d-1}), M).$$

(2) $\Leftrightarrow$ (3) It follows from the exact sequence

$$\mathrm{Hom}(F_d, M) \to \mathrm{Hom}(\mathrm{Ker}\,(f_d), M) \to \mathrm{Ext}^1(\mathrm{Ker}\,(f_{d-1}), M) \to 0. \qquad \square$$

Corollary 2.3. *Let $n \geq d+1$. Then FP-injective module is (n,d)-injective. In particular, FP-injective module is n-FP-injective.*

PROOF: Let M be FP-injective and let $F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} N \to 0$ be exact and each F_i be finitely generated and free. Then $K_{d-1} = \mathrm{Ker}\,(f_{d-1})$ is $(n-d)$-presented and so finitely presented since $n \geq d+1$. And thus $\mathrm{Ext}^1(K_{d-1}, M) = 0$. By Theorem 2.2, M is (n,d)-injective. $\square$

Let B be a left R-module and A be a submodule of B, k be a positive integer. Recall that A is said to be a pure submodule of B if for right R-module M, the induced map $M\otimes_R A \to M\otimes_R B$ is monic, or equivalently, every finitely presented left R-module is projective with respect to the exact sequence $0 \to A \to B \to B/A \to 0$. In this case, the exact sequence $0 \to A \to B \to B/A \to 0$ is called pure. It is well known that a left R-module M is FP-injective if and only if it is pure in every module containing it as a submodule. According to [9], A is said to be k-pure in B if every k-presented left R-module N is projective with respect to the exact sequence $0 \to A \to B \to B/A \to 0$. Clearly, a submodule A of a module B is pure in B if and only if A is 1-pure in B, and a k-pure submodule is $(k+1)$-pure. By [9, Theorem 2.2], A is $(k,0)$-injective if and only if A is k-pure in every module containing A if and only if A is k-pure in $E(A)$.

Proposition 2.4. *If $n \geq d+1$, then the class of (n,d)-injective left R-modules is closed under $(n-d)$-pure submodules.*

Proof: Let A be an $(n-d)$-pure submodule of an (n,d)-injective left R-module B. Let $F_n \overset{f_n}{\to} F_{n-1} \overset{f_{n-1}}{\to} \cdots \to F_1 \overset{f_1}{\to} F_0 \overset{\epsilon}{\to} N \to 0$ be exact with each F_i finitely generated and free. Write $K_{d-1} = \mathrm{Ker}\,(f_{d-1})$. Then K_{d-1} is $(n-d)$-presented. Since B is (n,d)-injective, $\mathrm{Ext}^1(K_{d-1},B) = 0$ by Theorem 2.2. So we have an exact sequence

$$\mathrm{Hom}(K_{d-1},B) \to \mathrm{Hom}(K_{d-1},B/A) \to \mathrm{Ext}^1(K_{d-1},A) \to 0.$$

Observing that A is $(n-d)$-pure in B, the sequence

$$\mathrm{Hom}(K_{d-1},B) \to \mathrm{Hom}(K_{d-1},B/A) \to 0$$

is exact. Hence $\mathrm{Ext}^1(K_{d-1},A) = 0$, and so A is (n,d)-injective by Theorem 2.2 again. □

Corollary 2.5 ([8, Proposition 2.4(1)])**.** *If $n \geq d+1$, then every pure submodule of an (n,d)-injective left R-module is (n,d)-injective.*

Corollary 2.6. *Let R be any ring and n be a positive integer. Then*

(1) *pure submodules of n-FP-injective R-modules are n-FP-injective. In particular, pure submodules of FP-injective R-modules are FP-injective;*
(2) *2-pure submodules of weakly n-FP-injective R-modules are weakly n-FP-injective. In particular, pure submodules of weakly n-FP-injective modules are weakly n-FP-injective.*

Corollary 2.7. *If $n \geq d+1$, then every $(n-d,0)$-injective submodule of an (n,d)-injective module is (n,d)-injective.*

Proposition 2.8. *If $n > d+1$, then the class of (n,d)-injective left R-modules is closed under direct limits.*

Proof: See [1, Lemma 2.9(2)]. □

Corollary 2.9. *The class of weakly n-FP-injective left R-modules is closed under direct limits.*

Proposition 2.10. *Let $\{M_i \mid i \in I\}$ be a family of left R-modules. Then the following statements are equivalent:*

(1) *each M_i is (n,d)-injective;*
(2) *$\prod_{i\in I} M_i$ is (n,d)-injective.*

Moreover, if $n \geq d+1$, then the above two conditions are equivalent to

(3) *$\bigoplus_{i\in I} M_i$ is (n,d)-injective.*

Proof: (1) $\Leftrightarrow$ (2) It follows from the isomorphism

$$\mathrm{Ext}^{d+1}(A, \prod_{i\in I} M_i) \cong \prod_{i\in I} \mathrm{Ext}^{d+1}(A, M_i).$$

(1) $\Leftrightarrow$ (3) Let $F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} N \to 0$ be exact and each F_i be finitely generated and free. It is easy to see that $\mathrm{Ker}\,(f_d)$ is $(n-d-1)$-presented. Since $n \geq d+1$, $\mathrm{Ker}\,(f_d)$ is finitely generated, and so the result follows immediately from Theorem 2.2 (3). □

Corollary 2.11 ([8, Lemma 2.9])**.** *If R is a left n-coherent ring, then every direct sum of (n,d)-injective left R-modules is (n,d)-injective.*

Proof: Let $\{M_i \mid i \in I\}$ be a family of (n,d)-injective left R-modules. Then each M_i is $(n+d+1,d)$-injective. By Proposition 2.10, $\bigoplus_{i\in I} M_i$ is $(n+d+1,d)$-injective. Since R is left n-coherent, every n-presented left R-module is $(n+d+1)$-presented. So every $(n+d+1,d)$-injective left R-module is (n,d)-injective, and thus $\bigoplus_{i\in I} M_i$ is (n,d)-injective. □

Corollary 2.12. (1) *If R is a left Noetherian ring, then every direct sum of (n,d)-injective left R-modules is (n,d)-injective for any non-negative integers n and d. In particular, if R is a left Noetherian ring, then for any non-negative integer d, the class of the left R-modules with injective dimensions at most d is closed under direct sums.*

(2) *If R is a left coherent ring, then every direct sum of (n,d)-injective left R-modules is (n,d)-injective for any positive integer n and any non-negative integer d.*

Recall that a right R-module V is called (n,d)-flat [8] if $\mathrm{Tor}_{d+1}(V,N)=0$ for every n-presented left R-module N.

Theorem 2.13. *Let V be a right R-module and $n \geq d+1$. Then the following statements are equivalent:*

(1) *V is (n,d)-flat;*
(2) *if $F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} N \to 0$ is exact and each F_i is finitely generated and free, then $\mathrm{Tor}_1(V, \mathrm{Ker}\,(f_{d-1})) = 0$;*

(3) *if* $F_n \xrightarrow{f_n} F_{n-1} \xrightarrow{f_{n-1}} \cdots \to F_1 \xrightarrow{f_1} F_0 \xrightarrow{\epsilon} N \to 0$ *is exact and each* F_i *is finitely generated and free, then the canonical map* $V \otimes \mathrm{Ker}\,(f_d) \to V \otimes F_d$ *is monic.*

Proof: (1) $\Leftrightarrow$ (2) It follows from the isomorphism

$$\mathrm{Tor}_{d+1}(V,N) \cong \mathrm{Tor}_1(V, \mathrm{Ker}\,(f_{d-1})).$$

(2) $\Leftrightarrow$ (3) It follows from the exact sequence

$$0 \to \mathrm{Tor}_1(V, \mathrm{Ker}\,(f_{d-1})) \to V \otimes \mathrm{Ker}\,(f_d) \to V \otimes F_d. \qquad \square$$

Proposition 2.14. *Let* $\{V_i \mid i \in I\}$ *be a family of right* R*-modules. Then the following statements are equivalent:*

(1) *each* V_i *is* (n,d)*-flat;*
(2) $\bigoplus_{i\in I} V_i$ *is* (n,d)*-flat.*

Moreover, if $n > d+1$*, then the above two conditions are equivalent to*

(3) $\prod_{i\in I} V_i$ *is* (n,d)*-flat.*

Proof: (1) $\Leftrightarrow$ (2) It follows from the isomorphism $\mathrm{Tor}_{d+1}(\bigoplus_{i\in I} V_i, A) \cong \bigoplus_{i\in I} \mathrm{Tor}_{d+1}(V_i, A)$.

(1) $\Leftrightarrow$ (3) Since $n > d+1$, by [1, Lemma 2.10(2)], for any n-presented left R-module A, we have $\mathrm{Tor}_{d+1}(\prod_{i\in I} V_i, A) \cong \prod_{i\in I} \mathrm{Tor}_{d+1}(V_i, A)$, so the conditions (1) and (3) are equivalent. $\square$

Corollary 2.15. *If* R *is a left* n*-coherent ring, then every direct product of* (n,d)*-flat right* R*-modules is* (n,d)*-flat.*

Proof: Let $\{V_i \mid i \in I\}$ be a family of (n,d)-flat right R-modules. Then each V_i is $(n+d+2,d)$-flat. By Proposition 2.14, $\prod_{i\in I} V_i$ is $(n+d+2,d)$-flat. Since R is left n-coherent, every n-presented left R-module is $(n+d+2)$-presented. So every $(n+d+2,d)$-flat right R-module is (n,d)-flat, and thus $\prod_{i\in I} V_i$ is (n,d)-flat. $\square$

Corollary 2.16. *If* R *is a left coherent ring, then the class of right* R*-modules with flat dimension at most* d *is closed under direct product. In particular, if* R *is a left coherent ring, then direct product of flat right* R*-modules is flat.*

Lemma 2.17 ([8, Proposition 2.3])**.** *We have that* V *is an* (n,d)*-flat right* R*-module if and only if* V^+ *is an* (n,d)*-injective left* R*-module.*

Proposition 2.18. *If* $n > d+1$*, then the following are true for any ring* R*:*

(1) *a left* R*-module* M *is* (n,d)*-injective if and only if* M^+ *is* (n,d)*-flat;*
(2) *the class of* (n,d)*-injective left* R*-modules is closed under pure submodules, pure quotients, direct sums, direct summands, direct products and direct limits;*

(3) *the class of (n,d)-flat right R-modules is closed under pure submodules, pure quotients, direct sums, direct summands, direct products and direct limits.*

Proof: (1) Let A be an n-presented left R-module. Since $n > d+1$, by [1, Lemma 2.7(2)], we have

$$\mathrm{Tor}_{d+1}(M^+, A) \cong \mathrm{Ext}^{d+1}(A, M)^+,$$

and so (1) follows.

(2) By Corollary 2.5 and Proposition 2.10, we need only to prove that the class of (n,d)-injective left R-modules is closed under pure quotients and direct limits. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of left R-modules with B being (n,d)-injective. Then we get the split exact sequence $0 \to C^+ \to B^+ \to A^+ \to 0$ by [3, Proposition 5.3.8]. Since B^+ is (n,d)-flat by (1), C^+ is also (n,d)-flat, and so C is (n,d)-injective by (1) again. Moreover, since $n > d+1$, by [1, Lemma 2.9(2)], we have that

$$\mathrm{Ext}^{d+1}(N, \varinjlim M_k) \cong \varinjlim \mathrm{Ext}^{d+1}(N, M_k)$$

for every n-presented left R-module N, and so the class of (n,d)-injective left R-modules is closed under direct limits.

(3) Since $n > d+1$, by Proposition 2.14, the class of (n,d)-flat right R-modules is closed under direct sums, direct summands and direct products. Let $0 \to A \to B \to C \to 0$ be a pure exact sequence of right R-modules with B being (n,d)-flat. Since B^+ is (n,d)-injective by Lemma 2.17, A^+ and C^+ are also (n,d)-injective, and so A and C are (n,d)-flat by Lemma 2.17 again. So the class of (n,d)-flat right R-modules is closed under pure submodules and pure quotients. Moreover, by the isomorphism formula

$$\mathrm{Tor}_{d+1}(N, \varinjlim M_k) \cong \varinjlim \mathrm{Tor}_{d+1}(N, M_k)$$

we see that the class of (n,d)-flat right R-modules is closed under direct limits. □

Theorem 2.19. *Let n be a positive integer. Then the following statements are equivalent for a ring R:*

(1) *R is left n-coherent;*
(2) *for each $m \geq n$ and each $d \geq 0$, every (m,d)-injective left R-module is (n,d)-injective;*
(3) *for each $m \geq n$ and each $d \geq 0$, every (m,d)-flat right R-module is (n,d)-flat;*
(4) *every weakly n-FP-injective left R-module is n-FP-injective;*
(5) *every weakly n-flat right R-module is n-flat.*

Proof: (1) ⇒ (2) ⇒ (4) and (1) ⇒ (3) ⇒ (5) are obvious.

(4) $\Rightarrow$ (5) Let M be a weakly n-flat right R-module. Then by Lemma 2.17, M^+ is weakly n-FP-injective, so M^+ is n-FP-injective by (2). And thus M is n-flat by Lemma 2.17 again.

(5) $\Rightarrow$ (1) Assume (5). Then since the direct products of weakly n-flat right R-modules are weakly n-flat by Proposition 2.14, the direct products of n-flat right R-modules are n-flat, and so R is left n-coherent by [1, Theorem 3.1]. □

Let $\mathcal{F}$ be a class of left (right) R-modules and M a left (right) R-module. Following [3], we say that a homomorphism $\varphi : M \to F$ where $F \in \mathcal{F}$ is an $\mathcal{F}$-preenvelope of M if for any morphism $f : M \to F'$ with $F' \in \mathcal{F}$, there is a $g : F \to F'$ such that $g\varphi = f$. An $\mathcal{F}$-preenvelope $\varphi : M \to F$ is said to be an $\mathcal{F}$-envelope if every endomorphism $g : F \to F$ such that $g\varphi = \varphi$ is an isomorphism. Dually, we have the definitions of $\mathcal{F}$-precovers and $\mathcal{F}$-covers. $\mathcal{F}$-envelopes ($\mathcal{F}$-covers) may not exist in general, but if they exist, they are unique up to isomorphism.

Theorem 2.20. *If $n > d+1$, then the following hold for any ring R:*

1. *every left R-module has an (n,d)-injective cover and an (n,d)-injective preenvelope;*
2. *every right R-module has an (n,d)-flat cover and an (n,d)-flat preenvelope;*
3. *if $A \to B$ is an (n,d)-injective (resp. (n,d)-flat) preenvelope of a left (resp. right) R-module A, then $B^+ \to A^+$ is an (n,d)-flat (resp. (n,d)-injective) precover of A^+.*

Proof: (1) Since $n > d+1$, the class of (n,d)-injective left R-modules is closed under direct sums and pure quotients by Proposition 2.18(2), and so every left R-module has an (n,d)-injective cover by [4, Theorem 2.5]. Since the class of (n,d)-injective left R-modules is closed under direct summands, direct products and pure submodules by Proposition 2.18(2), every left R-module has an (n,d)-injective preenvelope by [6, Corollary 3.5(c)].

(2) is similar to (1).

(3) Let $A \to B$ be an (n,d)-injective preenvelope of a left R-module A. Then B^+ is (n,d)-flat by Proposition 2.18(1). For any (n,d)-flat right R-module V, V^+ is an (n,d)-injective left R-module by Lemma 2.17, and so $\mathrm{Hom}(B,V^+) \to \mathrm{Hom}(A,V^+)$ is epic. Consider the following commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}(B,V^+) \arrow[r] \arrow[d, "\tau_1"'] & \mathrm{Hom}(A,V^+) \arrow[d, "\tau_2"] \\
\mathrm{Hom}(V,B^+) \arrow[r] & \mathrm{Hom}(V,A^+)
\end{tikzcd}$$

Since τ_1 and τ_2 are isomorphisms, $\mathrm{Hom}(V,B^+) \to \mathrm{Hom}(V,A^+)$ is an epimorphism. So $B^+ \to A^+$ is an (n,d)-flat precover of A^+. The other is similar. □

Proposition 2.21. *Let $n > d+1$. Then the following statements are equivalent for a ring R:*

(1) *${}_RR$ is (n,d)-injective;*
(2) *every left R-module has an epic (n,d)-injective cover;*
(3) *every right R-module has a monic (n,d)-flat preenvelope;*
(4) *every injective right R-module is (n,d)-flat;*
(5) *every FP-injective right R-module is (n,d)-flat.*

Proof: (1)⇒(2) Let M be a left R-module. Then M has an (n,d)-injective cover $\varphi : C \to M$ by Theorem 2.20(1). On the other hand, there is an exact sequence $A \xrightarrow{\alpha} M \to 0$ with A free. Note that A is (n,d)-injective by (1), there exists a homomorphism $\beta : A \to C$ such that $\alpha = \varphi\beta$. It shows that φ is epic.

(2)⇒(1) Let $f : N \to {}_RR$ be an epic (n,d)-injective cover. Then the projectivity of ${}_RR$ implies that ${}_RR$ is isomorphic to a direct summand of N, and so ${}_RR$ is (n,d)-injective.

(1)⇒(3) Let M be any right R-module. Then M has an (n,d)-flat preenvelope $f : M \to F$ by Theorem 2.20(2). Since $({}_RR)^+$ is a cogenerator, there exists an exact sequence $0 \to M \xrightarrow{g} \prod({}_RR)^+$. Since ${}_RR$ is (n,d)-injective, by Proposition 2.18(1) and Proposition 2.18(3), $\prod({}_RR)^+$ is (n,d)-flat. So there exists a right R-homomorphism $h : F \to \prod({}_RR)^+$ such that $g = hf$, which shows that f is monic.

(3)⇒(4) Assume (3). Then for every injective right R-module E, E has a monic (n,d)-flat preenvelope F, so E is isomorphic to a direct summand of F, and thus E is (n,d)-flat.

(4)⇒(1) Since $({}_RR)^+$ is injective, by (4), it is (n,d)-flat. Thus ${}_RR$ is (n,d)-injective by Proposition 2.18(1).

(4)⇒(5) Let M be an FP-injective right R-module. Then M is a pure submodule of its injective envelope $E(M)$. By (4), $E(M)$ is (n,d)-flat. So M is (n,d)-flat by Corollary 2.5.

(5) ⇒ (4) is clear. □

Remark 2.22. It is easy to see that if R is a left n-coherent ring, then a left R-module M is (n,d)-injective if and only if M is (m,d)-injective for every $m > n$ if and only if M is (m,d)-injective for some $m > n$. A right R-module V is (n,d)-flat if and only if V is (m,d)-flat for every $m > n$ if and only if V is (m,d)-flat for some $m > n$. So, if R is a left n-coherent ring, then the results from Theorem 2.2 to Proposition 2.21 hold without the conditions "$n \geq d+1$" or "$n > d+1$".

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