Selections and approaching points in products

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Abstract. The present paper aims to furnish simple proofs of some recent results about selections on product spaces obtained by García-Ferreira, Miyazaki and Nogura. The topic is discussed in the framework of a result of Katětov about complete normality of products. Also, some applications for products with a countably compact factor are demonstrated as well.

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1. Introduction

All spaces in this paper are Hausdorff topological spaces. For a set Z, let

$$\mathscr{F}_2(Z) = \{S \subset Z : 1 \leq |S| \leq 2\} \quad \text{and} \quad [Z]^2 = \{S \subset Z : |S| = 2\}.$$

A map $\sigma : \mathscr{F}_2(Z) \to Z$ is a *weak selection* for Z if $\sigma(S) \in S$ for every $S \in \mathscr{F}_2(Z)$. Every weak selection σ generates an order-like relation $\preceq_\sigma$ on Z defined by $y \preceq_\sigma z$ if $\sigma(\{y,z\}) = y$ [14, Definition 7.1]. The relation $\preceq_\sigma$ is emulating a linear order being both total and antisymmetric, but is not necessarily transitive. Motivated by this, we often write $y \prec_\sigma z$ if $y \preceq_\sigma z$ and $y \neq z$. If Z is a topological space, then σ is *continuous* if it is continuous with respect to the Vietoris topology on $\mathscr{F}_2(Z)$. This can be expressed only in terms of $\preceq_\sigma$ by the property that for every $y, z \in Z$ with $y \prec_\sigma z$, there are open sets $U, V \subset Z$ such that $y \in U$, $z \in V$ and $s \prec_\sigma t$ for every $s \in U$ and $t \in V$ (i.e. $U \prec_\sigma V$), see [10, Theorem 3.1]. Thus, σ is continuous if and only if so is the restriction $\sigma \restriction [Z]^2$, which is behind the reason that often selections for $[Z]^2$ are also called weak selections for Z.

For a non-isolated point p of a space X, $\mathrm{a}(p,X)$ denotes the least cardinal λ such that there exists $S \subset X \setminus \{p\}$ with $|S| \leq \lambda$ and $p \in \overline{S}$, see [4], [11]. Whenever p is isolated in X, set $\mathrm{a}(p,X) = 0$. The cardinal number $\mathrm{a}(p,X)$ stands for the *approaching number* of X in p, and can be compared with the *tightness* $\mathrm{t}(p,X)$ of X at p, see [4], [11]. Originally, $\mathrm{a}(p,X)$ was defined as the *selection approaching number* of X at p (abbreviated "sa", see [4]), but is not depending on weak selections. Finally, we will use $\psi(p,X)$ to denote the *pseudocharacter* of p in X.

The cardinal invariants $\mathrm{a}(p,X)$ and $\psi(p,X)$ are not global and depend only on the topology of X at the point p. In this regard, we will broadly use X_p to

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denote a space X with only one non-isolated point $p \in X$. For instance, for a non-isolated point $p \in X$, we have such a space X_p obtained from X by promoting the points of $X \setminus \{p\}$ to be isolated and preserving the same local base at p. Thus, we have both $\mathrm{a}(p, X_p) = \mathrm{a}(p, X)$ and $\psi(p, X_p) = \psi(p, X)$. Furthermore, if X has a continuous weak selection, then so does the space X_p, see [10, Corollary 3.2]. Accordingly, investigating local properties induced by weak selections, it makes sense to consider at first spaces with only one non-isolated point. The following theorems were proved in [5].

Theorem 1.1. *Let X_p and Y_q be such that $X_p \times Y_q$ has a continuous weak selection. Then $\psi(q, Y_q) \leq \mathrm{a}(p, X_p)$.*

Theorem 1.2. *If S is a stationary set in a regular uncountable cardinal and $\mathrm{a}(p, X_p) < |S|$, then $X_p \times S$ has no continuous weak selection.*

In Theorem 1.2, a subset $S \subset \lambda$ of a regular uncountable cardinal λ is called *stationary* if it intersects any closed unbounded subset of λ. Here, and in the rest of the paper, an ordinal λ will be always equipped with the open-interval topology, and called simply an *ordinal space*.

The main purpose of this paper is to give simple self-contained proofs of these theorems, and discuss also some natural relations with other results. Both proofs are based on the following interpretation of continuity of weak selections. For subsets $S, T \subset Z$ and a weak selection σ for a set (space) Z, we will write that $S \parallel_\sigma T$ if $S \prec_\sigma T$ or $T \prec_\sigma S$. If $S = \{y\}$ and $T = \{z\}$ are different singletons, we always have $\{y\} \parallel_\sigma \{z\}$, written simply $y \parallel_\sigma z$. Hence, in these terms, σ is continuous if and only if for every $\{y, z\} \in [Z]^2$ there are open sets $U, V \subset Z$ such that $y \in U$, $z \in V$ and $U \parallel_\sigma V$.

The proof of Theorem 1.1 is given in the next section. In Section 3, this theorem is related to a classical result of Katětov [13] about complete normality of products. This interpretation leads to another alternative proof of Theorem 1.1, see Propositions 3.3 and 3.4. Theorem 1.2 is proved in Section 4. Whenever λ is an ordinal of uncountable cofinality, the ordinal space λ is countably compact. In the last Section 5, we consider the problem in the realm of countably compact spaces and show that a regular countably compact space X is compact, first countable and zero-dimensional provided its product with a nontrivial convergent sequence has a continuous weak selection, see Theorem 5.2. This is then applied to show that a regular countably compact space X is zero-dimensional and metrizable if and only if X^2 has a continuous weak selection, see Corollary 5.3.

2. Proof of Theorem 1.1

Suppose that $X_p \times Y_q$ has a continuous weak selection σ, but $\psi(q, Y_q) > \mathrm{a}(p, X_p)$. Take a subset $A \subset X_p \setminus \{p\}$ with $|A| = \mathrm{a}(p, X_p)$ and $p \in \overline{A}$. Whenever $s, t \in A$ are different points, we have that $\langle s, q\rangle \parallel_\sigma \langle t, q\rangle$. Hence, by the continuity of σ, for every $a = \{s, t\} \in [A]^2$ there is an open set $U_a \subset Y_q$ with $q \in U_a$ and $\{s\} \times U_a \parallel_\sigma \{t\} \times U_a$. Take distinct points $y, z \in \left(\bigcap_{a \in [A]^2} U_a\right) \setminus \{q\}$ which is possible because $|[A]^2| = |A| = \mathrm{a}(p, X_p) < \psi(q, Y_q)$. Since $\langle p, y\rangle \parallel_\sigma \langle p, z\rangle$, just like before, there is an open set $V \subset X_p$ with $p \in V$ and $V \times \{y\} \parallel_\sigma V \times \{z\}$. Finally, use that $p \in \overline{A}$ to take distinct points $s, t \in V \cap A$. We now have that $\{s, t\} \times \{y\} \parallel_\sigma \{s, t\} \times \{z\}$, which implies that $\langle s, y\rangle \prec_\sigma \langle t, z\rangle$ if and only if $\langle t, y\rangle \prec_\sigma \langle s, z\rangle$. However, $y, z \in U_a$ for $a = \{s, t\}$, and we must also have that $\{s\}\times\{y, z\} \parallel_\sigma \{t\}\times\{y, z\}$, accordingly $\langle s, y\rangle \prec_\sigma \langle t, z\rangle$ if and only if $\langle s, z\rangle \prec_\sigma \langle t, y\rangle$. A contradiction!

Remark 2.1. In contrast to the proof of Theorem 1.1 in [5], the above arguments do not use the corner point $r = \langle p, q\rangle$ of the product $X_p \times Y_q$. Hence, they provide a slight generalisation showing that even the subspace $X_p \times Y_q \setminus \{\langle p, q\rangle\}$ has no continuous weak selection provided $\psi(q, Y_q) > \mathrm{a}(p, X_p)$.

3. Separating sets in products

Subsets $A, B \subset Z$ of a space Z are *separated* if $\overline{A} \cap B = \emptyset = A \cap \overline{B}$; and Z is called *completely normal* (or, *hereditarily normal*) if every pair of separated sets can be separated by open sets. The following interesting result was proved by Katětov [13].

Theorem 3.1 (Katětov [13])**.** *Let λ be an infinite cardinal number and X and Y be spaces such that $X \times Y$ is completely normal. Then either each subset of X of cardinality $\le \lambda$ is closed, or each closed subset of Y is G_λ.*

A subset of Y is G_λ if it is an intersection of λ many open sets. It is evident that $\psi(q, Y) \le \lambda$ if and only if $\{q\}$ is a G_λ-set. If X has the property that S is closed for every $S \subset X$ with $|S| \le \lambda$, then $\mathrm{a}(p, X) > \lambda$ for every non-isolated point $p \in X$. Accordingly, we have the following consequence.

Corollary 3.2. *Let X and Y be such that $X \times Y$ is completely normal. If $p \in X$ is a non-isolated point and $q \in Y$, then $\psi(q, Y) \le \mathrm{a}(p, X)$.*

Since $\mathrm{a}(p, X_p) = \mathrm{a}(p, X)$ and $\psi(q, Y_q) = \psi(q, Y)$, Corollary 3.2 is reduced to the associated spaces X_p and Y_q. For such spaces, complete normality of $X_p \times Y_q$ makes sense only to ensure that the separated sets $(X_p \setminus \{p\}) \times \{q\}$ and $\{p\} \times (Y_q \setminus \{q\})$ can be separated by open sets. Indeed, we now have the following interpretation of Corollary 3.2 without any explicit mentioning of complete normality.

Proposition 3.3. *Let X_p and Y_q be such that $\psi(q, Y_q) > \mathrm{a}(p, X_p)$. Then the sets $(X_p \setminus \{p\}) \times \{q\}$ and $\{p\} \times (Y_q \setminus \{q\})$ cannot be separated by open sets.*

Proof: Suppose $U \subset X_p \times Y_q$ is open such that $(X_p \setminus \{p\}) \times \{q\} \subset U$. Since p is a non-isolated point of X_p, there exists $S \subset X_p \setminus \{p\}$ such that $|S| = \mathrm{a}(p, X_p)$ and $p \in \overline{S}$. For every $x \in S$ there exists an open $V_x \subset Y_q$ containing q such that $\{x\} \times V_x \subset U$. Since $\psi(q, Y_q) > \mathrm{a}(p, X_p) = |S|$, it follows that $\bigcap_{x\in S} V_x$ contains a point $y \neq q$. Since $S \times \{y\} \subset U$, we get that $\langle p, y\rangle \in \overline{S} \times \{y\} \subset \overline{U}$ and, therefore, $\overline{U} \cap (\{p\} \times (Y_q \setminus \{q\})) \neq \emptyset$. $\square$

Complementary to Proposition 3.3 is the following observation showing that, in the same setting, "existence of continuous weak selections" is quite similar to "complete normality".

Proposition 3.4. *Let X_p and Y_q be such that $\psi(q, Y_q) > \mathrm{a}(p, X_p)$. If $X_p \times Y_q$ has a continuous weak selection, then there are sets $p \in A \subset X_p$ and $q \in B \subset Y_q$ such that $\psi(q, B) > \mathrm{a}(p, A) > 0$ and $(A \setminus \{p\}) \times \{q\}$ and $\{p\} \times (B \setminus \{q\})$ can be separated by open sets.*

PROOF: Let $r = \langle p, q\rangle$, and σ be a continuous weak selection for $Z = X_p \times Y_q$. Then the $\preceq_\sigma$-open intervals

$$(\leftarrow, r)_{\preceq_\sigma} = \big\{z \in Z : z \prec_\sigma r\big\} \quad \text{and} \quad (r, \rightarrow)_{\preceq_\sigma} = \big\{z \in Z : r \prec_\sigma z\big\}$$

are disjoint open sets forming a partition of $Z\setminus\{r\}$. We are going to show that they must separate some subsets of the "corner" sides $(X_p\setminus\{p\})\times\{q\}$ and $\{p\}\times(Y_q\setminus\{q\})$ of the product. Indeed, $(X_p\setminus\{p\})\times\{q\} \subset Z\setminus\{r\}$ and there exists $S \subset X_p\setminus\{p\}$ such that $|S| = \mathrm{a}(p, X_p)$, $p \in \overline{S}$ and either $S \times \{q\} \subset (\leftarrow, r)_{\preceq_\sigma}$ or $S \times \{q\} \subset (r, \rightarrow)_{\preceq_\sigma}$, say $S \times \{q\} \subset (\leftarrow, r)_{\preceq_\sigma}$. Take $A = S \cup \{p\}$ and $B = \{y \in Y_q : r \preceq_\sigma \langle p, y\rangle\}$. Since $A \times \{q\} \subset (\leftarrow, r]_{\preceq_\sigma} = (\leftarrow, r)_{\preceq_\sigma} \cup \{r\}$, it follows from [4, Theorem 4.1] that $\psi(r, (\leftarrow, r]_{\preceq_\sigma}) \le |A| = \mathrm{a}(p, X_p) < \psi(q, Y_q)$. Hence, $\psi(q, B) = \psi(q, Y_q)$ because $\{p\} \times B \subset [r, \rightarrow)_{\preceq_\sigma} = Z \setminus (\leftarrow, r)_{\preceq_\sigma}$. These A and B are as required because $(A \setminus \{p\}) \times \{q\} \subset (\leftarrow, r)_{\preceq_\sigma}$ and $\{p\} \times (B \setminus \{q\}) \subset (r, \rightarrow)_{\preceq_\sigma}$. □

It is evident that Propositions 3.3 and 3.4 offer another alternative proof of Theorem 1.1, now relating this result to Katětov's Theorem 3.1.

Remark 3.5. The proof of Proposition 3.4 relies on [4, Theorem 4.1] that for a continuous weak selection σ for a space Z and $r \in Z$, we have

$$\psi\big(r, (\leftarrow, r]_{\preceq_\sigma}\big) \le \mathrm{a}\big(r, (\leftarrow, r]_{\preceq_\sigma}\big) \quad \text{and} \quad \psi\big(r, [r \rightarrow)_{\preceq_\sigma}\big) \le \mathrm{a}\big(r, [r, \rightarrow)_{\preceq_\sigma}\big).$$

This fact also has a very simple proof. Namely, suppose that $r \in \overline{A}$ for some $A \subset (\leftarrow, r)_{\preceq_\sigma}$, and take a point $s \in \bigcap_{z\in A}(z, r]_{\preceq_\sigma}$. Then $A \subset (\leftarrow, s]_{\preceq_\sigma}$ and, therefore, $r \in \overline{A} \subset (\leftarrow, s]_{\preceq_\sigma}$. So, $s = r$ because $r \preceq_\sigma s \preceq_\sigma r$. Consequently, $\psi(r, (\leftarrow, r]_{\preceq_\sigma}) \le |A|$.

4. Proof of Theorem 1.2

The proof of Theorem 1.2 is based on the same idea as that of Theorem 1.1; in fact, it is almost identical but uses the following observation.

Proposition 4.1. *Let S be a stationary subset of regular uncountable cardinal λ, and η be a continuous weak selection for $\{0, 1\} \times S$. Then S contains a closed unbounded subset T with $\{0\} \times T \parallel_\eta \{1\} \times T$.*

PROOF: Since η is continuous, for every $\alpha \in S \setminus \{0\}$, there exists $f(\alpha) < \alpha$ such that $\{0\}\times(S\cap(f(\alpha), \alpha]) \parallel_\eta \{1\}\times(S\cap(f(\alpha), \alpha])$. This defines a *regressive* function $f : S \to \lambda$, i.e. a function f with the property that $f(\alpha) < \alpha$ for every $\alpha \in S\setminus\{0\}$.

By the pressing down lemma, S contains a stationary subset $H \subset \lambda$ such that f is constant on H. By the properties of f, we have that $\{0\} \times H \parallel_\eta \{1\} \times H$. Since η is continuous, the same is true for the closure $T = \overline{H}$ of H in S. The proof is completed. $\square$

Having the above property, the proof of Theorem 1.2 goes precisely in the same way as that of Theorem 1.1. Namely, let $\mathrm{a}(p, X_p) < |S| = \lambda$, and contrary to the claim, suppose that $X_p \times S$ has a continuous weak selection σ. Just like before, take a subset $A \subset X_p \setminus \{p\}$ such that $|A| = \mathrm{a}(p, X_p)$ and $p \in \overline{A}$. Since σ is continuous, by Proposition 4.1, for every $a = \{s, t\} \in [A]^2$, there exists a closed unbounded subset $T_a \subset S$ such that $\{s\} \times T_a \parallel_\sigma \{t\} \times T_a$. Let C_a be the closure of T_a in λ. Then $\{C_a : a \in [A]^2\}$ is a collection of closed unbounded subsets of λ. Since $|[A]^2| = |A| = \mathrm{a}(p, X_p) < \lambda$, the intersection $C = \bigcap_{a \in [A]^2} C_a$ is also a closed unbounded subset of λ. Since S is stationary and each T_a is closed in S, there are distinct $\alpha, \beta \in S \cap C \subset \bigcap_{a \in [A]^2} T_a$. Having $\langle p, \alpha\rangle \parallel_\sigma \langle p, \beta\rangle$ and using the continuity of σ, there is an open set $V \subset X_p$ with $p \in V$ and $V \times \{\alpha\} \parallel_\sigma V \times \{\beta\}$. Since $p \in \overline{A}$, there are distinct points $s, t \in V \cap A$ such that $\{s, t\} \times \{\alpha\} \parallel_\sigma \{s, t\} \times \{\beta\}$. However, $\alpha, \beta \in S \cap C \subset T_a$ for this particular $a = \{s, t\}$, and we must also have that $\{s\} \times \{\alpha, \beta\} \parallel_\sigma \{t\} \times \{\alpha, \beta\}$, which is impossible. A contradiction!

5. Countable compactness and products

The following is an immediate consequence of Theorem 1.2. In particular, it furnishes a very simple proof of [3, Example 3.1].

Corollary 5.1. *The space $(\omega + 1) \times \omega_1$ has no continuous weak selection.*

Here, ω is the first infinite ordinal, and ω_1 — the first uncountable one. The ordinal space ω_1 is certainly regular and countably compact. The following theorem now provides a natural generalisation of Corollary 5.1.

Theorem 5.2. *Let X be a regular countably compact space such that $(\omega+1)\times X$ has a continuous weak selection. Then X is a compact zero-dimensional first countable space.*

Proof: Consider the nontrivial case when X is infinite. According to Theorem 1.1, $\psi(p, X) \le \omega$ for every $p \in X$, i.e., each point of X is a G_δ-point. Since X is regular, each point is the intersection of the closure of countably many neighbourhoods, hence the space is first countable being countably compact. Thus, $\mathrm{a}(p, X) \le \omega$ for every $p \in X$ and, by [2, Corollary 5.4], X will be both Tychonoff and suborderable (in particular, pseudocompact). By [5, Theorem 3.4], X will be totally disconnected. It remains to show that X is also compact. We will actually show that $X = \beta X$, where βX is the Čech-Stone compactification of X. To this end, let us observe that $Y = (\omega + 1) \times X$ is pseudocompact because so is X. Since Y has a continuous weak selection, by [7, Theorem 2.3], Y^2 is also pseudocompact.

Accordingly, the Čech-Stone compactification βY of Y has a continuous weak selection [1], [16], see also [9, Corollary 3.6]. However, by Glicksberg's theorem [8], $\beta Y = \beta((\omega + 1) \times X) = (\omega + 1) \times \beta X$. Thus, by the same reasoning as before, each point of βX must be a G_δ-point. Since X is pseudocompact, by a result of Hewitt [12, Theorem 28], the remainder $\beta X \setminus X$ does not contain any nonempty closed G_δ-subset of βX. Therefore, $X = \beta X$. □

We now have the following interesting consequence.

Corollary 5.3. *A regular countably compact space X is zero-dimensional and metrizable if and only if X^2 has a continuous weak selection.*

Proof: If X is zero-dimensional and metrizable, then so is X^2. Moreover, X^2 is a subset of the Cantor set, hence it has a continuous weak selection because so does the Cantor set. Conversely, suppose X is an infinite countably compact regular space and X^2 has a continuous weak selection. Then X has a continuous weak selection (because so does X^2), and it follows from [18, Theorem 2] that X is sequentially compact. Hence, X contains a nontrivial convergent sequence being infinite. So, it also contains a copy of $(\omega + 1)$; accordingly, $(\omega + 1) \times X$ has a continuous weak selection as well. Thus, by Theorem 5.2, X is compact and zero-dimensional. Then X^2 will be orderable being compact and having itself a continuous weak selection [15, Theorem 1.1]. Finally, by a result of Treybig [17], X will be also metrizable. □

Since every Tychonoff pseudocompact space with a continuous weak selection is countably compact (see, e.g., [9, Corollary 3.9]), Corollary 5.3 is a natural generalisation of [6, Theorem 2.18]. It also answers [6, Question 2.22] in the affirmative.

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