# Global existence results for second order neutral functional differential equation with state-dependent delay

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*Abstract.* Our aim in this work is to provide sufficient conditions for the existence of global solutions of second order neutral functional differential equation with state-dependent delay. We use the semigroup theory and Schauder's fixed point theorem.

*Keywords:* neutral functional differential equation of second order; mild solution; infinite delay; state-dependent delay fixed point; semigroup theory; cosine function

*Classification:* 34G20, 34K20, 34K30

## 1. Introduction

In this work we prove the existence of solutions of a class of semilinear functional evolution equations of second order with state-dependent delay. Our investigations will be situated in the Banach space of real continuous and bounded functions on $\mathbb{R}$. More precisely, we will consider the following problem

$$\frac{d}{dt}[y'(t) - g(t, y_{\rho(t,y_t)})] = Ay(t) + f(t, y_{\rho(t,y_t)}), \quad \text{a.e. } t \in J := [0, +\infty) \tag{1}$$

$$y(t) = \phi(t), \ t \in (-\infty, 0], \quad y'(0) = \varphi, \tag{2}$$

where $f, g : J \times \mathcal{B} \to E$ are given functions, $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on $E$, $\phi \in \mathcal{B}$, $\rho : J \times \mathcal{B} \to (-\infty, +\infty)$, and $(E, |\cdot|)$ is a real Banach space. We denote by $y_t$ the element of $\mathcal{B}$ defined by $y_t(\theta) = y(t+\theta)$, $\theta \in (-\infty, 0]$. We assume that the histories $y_t$ belong to some abstract phases $\mathcal{B}$.

We will use Schauder's fixed theorem, combined with the Corduneanu's compactness criteria.

The cosine function theory is related to abstract linear second order differential equations in the same manner that the semigroup theory of bounded linear operators is related to first order partial differential equations. For basic concepts and applications of this theory, we refer the reader to Fattorini [15], Travis and Webb [32].

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Our purpose in this work is to consider a simultaneous generalization of the classical second order abstract Cauchy problem studied by Travis and Webb in [31], [32]. Additionally, we observe that the ideas and techniques in this paper permit the reformulation of the problems studied in [4], [6], [24], [26], [27] to the context of partial second order differential equations, see [31] and the referred papers for details.

Functional differential equations with state-dependent delay appear frequently in applications as a model of equations and for this reason the study of this type of equations has received great attention in the last years. The literature devoted to this subject is concerned fundamentally with the first order functional differential equations for which the state belongs to some finite dimensional space, see among another works [2], [5], [9], [11], [12], [13], [14].

The problem of the existence of solutions for first and second order partial functional differential equations with state-dependent delay have been treated recently in [1], [3], [18], [19], [20], [25], [28], [29]. The literature relative to second order impulsive differential systems with state-dependent delay is very restrict, and we can only cite [30] for ordinary differential systems and [18] for abstract partial differential systems. In [7], [8] the authors consider global existence of first order functional and neutral functional differential equations.

To the best of our knowledge, the study of the existence of solutions for abstract second order neutral functional differential equations with state-dependent delay on unbounded intervals is an untreated topic in the literature and this fact is the main motivation of the present work.

## 2. Preliminaries

In this section we present briefly some notations, definitions, and a theorem that are used throughout this work. In this paper, we will employ an axiomatic definition of the phase space $\mathcal{B}$ introduced by Hale and Kato in [17] and follow the terminology used in [22]. Thus, $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ will be a seminormed linear space of functions mapping $(-\infty, 0]$ into $E$, and satisfying the following axioms.

$(A_1)$ If $y : (-\infty, b) \to E, b > 0$, is continuous on $J$ and $y_0 \in \mathcal{B}$, then for every $t \in J$ the following conditions hold:

(i) $y_t \in \mathcal{B}$;

(ii) there exists a positive constant $H$ such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;

(iii) there exist two functions $L(\cdot), M(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ independent of $y$ such that $L$ is continuous and bounded, and $M$ is locally bounded, and

$$\|y_t\|_{\mathcal{B}} \leq L(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

$(A_2)$ For the function $y$ in $(A_1)$, $y_t$ is a $\mathcal{B}$-valued continuous function on $J$.

$(A_3)$ The space $\mathcal{B}$ is complete.

Denote

$$l = \sup\{L(t) : t \in J\},$$

and

$$m = \sup\{M(t) : t \in J\}.$$

**Remark 2.1.** 1. (ii) is equivalent to $|\phi(0)| \leq H\|\phi\|_{\mathcal{B}}$ for every $\phi \in \mathcal{B}$.

2. Since $\|\cdot\|_{\mathcal{B}}$ is a seminorm, two elements $\phi, \psi \in \mathcal{B}$ can verify $\|\phi - \psi\|_{\mathcal{B}} = 0$ without necessarily $\phi(\theta) = \psi(\theta)$ for all $\theta \leq 0$.
3. From the equivalence in the first remark, we can see that for all $\phi, \psi \in \mathcal{B}$ such that $\|\phi - \psi\|_{\mathcal{B}} = 0$ we necessarily have that $\phi(0) = \psi(0)$.

**Example 2.2** (The phase space $(\mathbf{C_r} \times \mathbf{L^p(h, E)})$)**.** Let $h : (-\infty, -r) \to \mathbb{R}$ be a positive Lebesgue integrable function and assume that there exists a non-negative and locally bounded function $\gamma$ on $(-\infty, 0]$ such that $h(\xi + \theta) \leq \gamma(\xi)h(\theta)$, for all $\xi \leq 0$ and $\theta \in (-\infty, -r) \setminus N_\xi$, where $N_\xi \subseteq (-\infty, -r)$ is a set with Lebesgue measure zero. The space $C_r \times L^p(h, E)$ consists of all classes of functions $\varphi(-\infty, 0] \to \mathbb{R}$ such that $\phi$ is continuous on $[-r, 0]$, Lebesgue-measurable and $h\|\phi\|^p$ is Lebesgue integrable on $(-\infty, -r)$. The seminorm in $C_r \times L^p(h, E)$ is defined by

$$\|\phi\|_{\mathcal{B}} := \sup\{\|\phi(\theta)\| : -r \leq \theta \leq 0\} + \left(\int_{-\infty}^{-r} h(\theta)\|\phi(\theta)\|^p d\theta\right)^{\frac{1}{p}}.$$

Assume that $h(\cdot)$ verifies the condition $(g-5)$, $(g-6)$ and $(g-7)$ in the nomenclature [22]. In this case, $\mathcal{B} = C_r \times L^p(h, E)$ verifies assumptions $(A_1)$, $(A_2)$, $(A_3)$ see [22, Theorem 1.3.8] for details. Moreover, when $r = 0$ and $p = 2$ we have that $H = 1$, $M(t) = \gamma(-t)^{\frac{1}{2}}$ and $L(t) = 1 + (\int_{-t}^{0} h(\theta)d\theta)^{\frac{1}{2}}$ for $t \geq 0$.

By $BUC$ we denote the space of bounded uniformly continuous functions defined from $(-\infty, 0]$ to $E$.

Next we mention a few results and notations in the cosine function theory which are needed to establish our results. Along this section, $A$ is the infinitesimal generator of a strongly continuous cosine function of bounded linear operators $(C(t))_{t\in\mathbb{R}}$ on Banach space $(E, \|\cdot\|)$. We denote by $(S(t))_{t\in\mathbb{R}}$ the sine function associated with $(C(t))_{t\in\mathbb{R}}$ which is defined by

$$S(t)y = \int_0^t C(s)y\, ds,\ y \in E,\ t \in \mathbb{R}.$$

The notation $[D(A)]$ stands for the domain of the operator $A$ endowed with the graph norm

$$\|y\|_A = \|y\| + \|Ay\|,\ y \in D(A).$$

Moreover, in this work, $X$ is the space formed by the vector $y \in E$ for which $C(\cdot)y$ is of class $C^1$ on $\mathbb{R}$. It was proved by Kisinsky [23] that $X$ endowed with the norm

$$\|y\|_X = \|y\| + \sup_{0\leq t\leq 1} \|AS(t)y\|,\ y \in X,$$

is a Banach space. The operator valued function

$$G(t) = \begin{pmatrix} C(t) & S(t) \\ AS(t) & C(t) \end{pmatrix}$$

is a strongly continuous group of bounded linear operators on the space $X \times E$ generated by the operator

$$\mathcal{A}\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}$$

defined on $D(A) \times X$. It follows that $AS(t) : X \to E$ is a bounded linear operator and that $AS(T)y \to 0$, $t \longrightarrow 0$, for each $y \in X$. Furthermore, if $y : [0, +\infty) \to E$ is a locally integrable function, then $z(t) = \int_0^t S(t-s)y(s)\,ds$ defines an $X$-valued continuous function. This is a consequence of the fact that

$$\int_0^t G(t-s)\begin{pmatrix} 0 \\ y(s) \end{pmatrix} ds = \begin{pmatrix} \displaystyle\int_0^t S(t-s)y(s)\,ds \\ \displaystyle\int_0^t C(t-s)y(s)\,ds \end{pmatrix}$$

defines an $X \times E$-valued continuous function. The existence of solutions for the second order abstract Cauchy problem

$$\begin{cases} y''(t) = Ay(t) + h(t), & t \in J := [0, +\infty), \\ y(0) = y_0, & y'(0) = y_1, \end{cases} \tag{3}$$

where $h : J \to E$ is an integrable function has been discussed in [31]. Similarly, the existence of solutions of the semilinear second order abstract Cauchy problem has been treated in [32].

**Definition 2.3.** The function $y(\cdot)$ given by

$$y(t) = C(t)y_0 + S(t)y_1 + \int_0^t S(t-s)h(s)\,ds, \; t \in J,$$

is called a mild solution of (3).

**Remark 2.4.** When $y_0 \in X$, $y(\cdot)$ is continuously differentiable and we have

$$y'(t) = AS(t)y_0 + C(t)y_1 + \int_0^t C(t-s)h(s)\,ds.$$

For additional details about cosine function theory, we refer the reader to [31], [32].

**Definition 2.5.** A map $f : J \times \mathcal{B} \to E$ is said to be Carathéodory if

(i) $t \to f(t, y)$ is measurable for all $y \in \mathcal{B}$,
(ii) $y \to f(t, y)$ is continuous for almost each $t \in J$.

**Theorem 2.6** (Schauder's fixed point [16])**.** *Let $B$ be a closed, convex and nonempty subset of a Banach space $E$. Let $N : B \to B$ be a continuous mapping such that $N(B)$ is a relatively compact subset of $E$. Then $N$ has at least one fixed point in $B$ that is, there exists $y \in B$ such that $Ny = y$.*

**Lemma 2.7** (Corduneanu [10])**.** *Let $D \subset BC([0, +\infty), E)$. Then $D$ is relatively compact if the following conditions hold.*

(a) *$D$ is bounded in $BC$.*
(b) *The function belonging to $D$ is almost equicontinuous on $[0, +\infty)$, i.e., equicontinuous on every compact of $[0, +\infty)$.*
(c) *The set $D(t) := \{y(t) : y \in D\}$ is relatively compact on every compact of $[0, +\infty)$.*
(d) *The function from $D$ is equiconvergent, that is, given $\epsilon > 0$, responds $T(\epsilon) > 0$ such that $|u(t) - \lim_{t\to+\infty} u(t)| < \epsilon$, for any $t \geq T(\epsilon)$ and $u \in D$.*

In this section by $BC := BC(-\infty, +\infty)$ we denote the Banach space of all bounded and continuous functions from $(-\infty, +\infty)$ into $E$ equipped with the standard norm

$$\|y\|_{BC} = \sup_{t\in(-\infty,+\infty)} |y(t)|.$$

Finally, by $BC' := BC'([0, +\infty))$ we denote the Banach space of all bounded and continuous functions from $[0, +\infty)$ into $E$ equipped with the standard norm

$$\|y\|_{BC'} = \sup_{t\in[0,+\infty)} |y(t)|.$$

## 3. Existence of mild solutions

Now we give our main existence result for problem (1)–(2). Before starting and proving this result, we give the definition of the mild solution.

**Definition 3.1.** We say that a continuous function $y : (-\infty, +\infty) \to E$ is a mild solution of problem (1)–(2) if $y(t) = \phi(t)$, $t \in (-\infty, 0]$, $y'(0) = \varphi$ and

$$\begin{aligned} y(t) &= C(t)\phi(0) + S(t)[\varphi - g(0, \phi)] \\ &\quad + \int_0^t C(t-s)g(s, y_{\rho(t,y_t)})\, ds + \int_0^t S(t-s)f(s, y_{\rho(t,y_t)})\, ds,\ t \in J. \end{aligned} \tag{4}$$

Set

$$\mathcal{R}(\rho^-) = \{\rho(s, \phi) : (s, \phi) \in J \times \mathcal{B},\ \rho(s, \phi) \leq 0\}.$$

We always assume that $\rho : J \times \mathcal{B} \to \mathbb{R}$ is continuous. Additionally, we introduce following hypothesis:

($H_\phi$) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$ and there exists a continuous and bounded function $\mathcal{L}^\phi : \mathcal{R}(\rho^-) \to (0, \infty)$ such that

$$\|\phi_t\| \leq \mathcal{L}^\phi(t)\|\phi\| \ \text{ for every } \ t \in \mathcal{R}(\rho^-).$$

**Remark 3.2.** The condition $(H_\phi)$ is frequently verified by functions continuous and bounded. For more details, see for instance [22].

**Lemma 3.3** ([21, Lemma 2.4])**.** *If $y : (-\infty, +\infty) \to E$ is a function such that $y_0 = \phi$, then*

$$\|y_s\|_{\mathcal{B}} \leq (m + \mathcal{L}^\phi)\|\phi\|_{\mathcal{B}} + l \sup\{|y(\theta)|; \theta \in [0, max\{0, s\}]\}, \ s \in \mathcal{R}(\rho^-) \cup J,$$

*where* $\mathcal{L}^\phi = \sup_{t \in \mathcal{R}(\rho^-)} \mathcal{L}^\phi(t)$.

Let us introduce the following hypotheses.

$(H_1)$ $C(t), S(t)$ are compact for $t > 0$ in the Banach space $E$. Let $M = \sup\{\|C\|_{B(E)} : t \geq 0\}$, and $M' = \sup\{\|S\|_{B(E)} : t \geq 0\}$.

$(H_2)$ The function $f : J \times \mathcal{B} \to E$ is Carathéodory.

$(H_3)$ There exists a continuous function $k : J \to [0, +\infty)$ such that

$$|f(t, u) - f(t, v)| \leq k(t)\|u - v\|, \ t \in J, \ u, v \in \mathcal{B}$$

and

$$k^* := \sup_{t \in J} \int_0^t k(s)\, ds < \infty.$$

$(H_4)$ The function $t \to f(t, 0) = f_0 \in L^1(J, [0, +\infty))$ with $F^* = \|f_0\|_{L^1}$.

$(H_5)$ The function $g : J \times \mathcal{B} \to E$ is Carathéodory and there exists a function $k_g : J \to [0, +\infty)$ such that

$$|g(t, u) - g(t, v)| \leq k_g(t)\|u - v\|_{\mathcal{B}}, \ \text{ for each }, \ u, v \in \mathcal{B}$$

and

$$k_g^* := \sup_{t \in J} \int_0^t k_g(s)\, ds < \infty.$$

$(H_6)$ The function $t \to g(t, 0) = g_0 \in L^1(J, [0, +\infty))$ with $g^* = \|g_0\|_{L^1}$.

$(H_7)$ For each bounded $B \subset BC'$ and $t \in J$ the set

$$\Big\{S(t)[\varphi - g(0, \phi)] + \int_0^t C(t - s) g(s, y_{\rho(t, y_t)})\, ds + \int_0^t S(t - s) f(s, y_{\rho(t, y_t)})\, ds : y \in B\Big\}$$

is relatively compact in $E$.

**Remark 3.4.** By conditions $(H_3)$–$(H_6)$ we deduce that

$$|f(t, u)| \leq k(t)\|u\|_{\mathcal{B}} + F^*, \ t \in J, \ u \in \mathcal{B},$$
$$|g(t, u)| \leq k_g(t)\|u\|_{\mathcal{B}} + g^*, \ t \in J, \ u \in \mathcal{B},$$

and

$$|g(0,\phi)| \le k_g(0)\|\phi\|_{\mathcal{B}} + g^*.$$

**Theorem 3.5.** *Assume that* $(H_1)$–$(H_7)$, $(H_\phi)$ *hold. If* $K^*Ml < 1$, *then the problem* (1)–(2) *has at least one mild solution on* $BC$.

Proof: We transform the problem (1)–(2) into a fixed point problem. Consider the operator $N : BC \to BC$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty,0], \\ C(t)\phi(0) + S(t)[\varphi - g(0,\phi)] & \\ + \int_0^t C(t-s)\ g(s, y_{\rho(t,y_t)})\, ds & \\ + \int_0^t S(t-s)\ f(s, y_{\rho(t,y_t)})\, ds, & \text{if } t \in J. \end{cases}$$

Let $x(\cdot) : (-\infty,+\infty) \to E$ be the function defined by

$$x(t) = \begin{cases} \phi(t), & \text{if } t \in (-\infty,0], \\ C(t)\ \phi(0), & \text{if } t \in J, \end{cases}$$

then $x_0 = \phi$. For each $z \in BC$ with $z(0) = 0$, $y'(0) = \varphi = z'(0) = \varphi_1$, we denote by $\overline{z}$ the function

$$\overline{z}(t) = \begin{cases} 0, & \text{if } t \in (-\infty,0], \\ z(t), & \text{if } t \in J. \end{cases}$$

If $y$ satisfies (4), we can decompose it as $y(t) = z(t) + x(t)$, $t \in J$, which implies $y_t = z_t + x_t$ for every $t \in J$ and the function $z(\cdot)$ satisfies

$$z(t) = S(t)[\varphi_1 - g(0,\phi)] + \int_0^t C(t-s)\ g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\, ds$$
$$+ \int_0^t S(t-s)\ f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\, ds,\ t \in J.$$

Set $BC_0' = \{z \in BC' : z(0) = 0\}$ and let

$$\|z\|_{BC_0'} = \sup\{|z(t)| : t \in J\},\ z \in BC_0'.$$

Then $BC_0'$ is a Banach space with the norm $\|\cdot\|_{BC_0'}$. We define the operator $\mathcal{A} : BC_0' \to BC_0'$ by

$$\mathcal{A}(z)(t) = S(t)[\varphi_1 - g(0,\phi)]$$
$$+ \int_0^t C(t-s)\ g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\, ds$$

$$+\int_0^t S(t-s)\, f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})\, ds, \ t \in J.$$

We shall show that the operator $\mathcal{A}$ satisfies all conditions of Schauder's fixed point theorem. The operator $A$ maps $BC_0'$ into $BC_0'$, indeed the map $\mathcal{A}(z)$ is continuous on $[0,+\infty)$ for any $z \in BC_0'$, and for each $t \in J$ we have

$$\begin{aligned}
|\mathcal{A}(z)(t)| &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}} + g^*] \\
&\quad + M' \int_0^t |f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - f(s,0) + f(s,0)|\, ds \\
&\quad + M \int_0^t |g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}) - g(s,0) + g(s,0)|\, ds \\
&\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}} + g^*] + M' \int_0^t |f(s,0)|\, ds \\
&\quad + M' \int_0^t k(s) \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\, ds \\
&\quad + M \int_0^t |g(s,0)|\, ds + M \int_0^t k_g(s) \|z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}\|_{\mathcal{B}}\, ds \\
&\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}} + g^*] + M'F^* \\
&\quad + M' \int_0^t k(s)(l|z(s)| + (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}})\, ds \\
&\quad + Mg^* + M \int_0^t k_g(s)(l|z(s)| + (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}})\, ds.
\end{aligned}$$

Let $C = (m + \mathcal{L}^\phi + lMH)\|\phi\|_{\mathcal{B}}$. Then, we have

$$\begin{aligned}
|\mathcal{A}(z)(t)| &\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}} + g^*] + M'F^* \\
&\quad + M'C \int_0^t k(s)\, ds + M'l \int_0^t k(s)|z(s)|\, ds \\
&\quad + Mg^* + MC \int_0^t k_g(s)\, ds + Ml \int_0^t k_g(s)|z(s)|\, ds \\
&\leq M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}} + g^*] + M'F^* + M'Ck^* + M'l\|z\|_{BC_0'}k^* \\
&\quad + Mg^* + MCk_g^* + Ml\|z\|_{BC_0'}k_g^*.
\end{aligned}$$

Set

$$C_1 = M'[\|\varphi_1\| + k_g(0)\|\phi\|_{\mathcal{B}} + g^*] + M'F^* + M'Ck^* + Mg^* + MCk_g^*.$$

Hence, $\mathcal{A}(z) \in BC_0'$.

Moreover, let $r > 0$ be such that $r \geq \frac{C_1}{1-Mlk^*}$, and $B_r$ be the closed ball in $BC'_0$ centered at the origin and of radius $r$. Let $y \in B_r$ and $t \in [0, +\infty)$. Then,

$$|\mathcal{A}(z)(t)| \leq C_1 + M'lk^*rMlk_g^*r.$$

Thus, we have

$$\|\mathcal{A}(z)\|_{BC'_0} \leq r,$$

which means that the operator $N$ transforms the ball $B_r$ into itself.

Now we prove that $\mathcal{A} : B_r \to B_r$ satisfies the assumptions of Schauder's fixed theorem. The proof will be given in several steps.

**Step 1**: $\mathcal{A}$ is continuous in $B_r$.
Let $\{z_n\}$ be a sequence such that $z_n \to z$ in $B_r$. At the first, we study the convergence of the sequences $(z^n_{\rho(s,z^n_s)})_{n\in\mathbb{N}}$, $s \in J$.

If $s \in J$ is such that $\rho(s, z_s) > 0$, then we have,

$$\begin{aligned}\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} &\leq \|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z^n_s)}\|_{\mathcal{B}} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} \\ &\leq l\|z_n - z\|_{B_r} + \|z_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}},\end{aligned}$$

which proves that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) > 0$. Similarly, $\rho(s, z_s) < 0$, and we get

$$\|z^n_{\rho(s,z^n_s)} - z_{\rho(s,z_s)}\|_{\mathcal{B}} = \|\phi^n_{\rho(s,z^n_s)} - \phi_{\rho(s,z_s)}\|_{\mathcal{B}} = 0,$$

which also shows that $z^n_{\rho(s,z^n_s)} \to z_{\rho(s,z_s)}$ in $\mathcal{B}$ as $n \to \infty$ for every $s \in J$ such that $\rho(s, z_s) < 0$. Combining the previous arguments, we can prove that $z^n_{\rho(s,z_s)} \to \phi$ for every $s \in J$ such that $\rho(s, z_s) = 0$. Finally,

$$\begin{aligned}&|\mathcal{A}(z_n)(t) - \mathcal{A}(z)(t)| \\ &\leq M' \int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\, ds \\ &\quad + M \int_0^t |g(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) - g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)})|\, ds.\end{aligned}$$

Then by $(H_2)$, $(H_5)$ we have

$$\begin{aligned}&f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) \to f(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \quad \text{as} \quad n \to \infty, \\ &g(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)}) \to g(s, z_{\rho(s,z_s+x_s)} + x_{\rho(s,z_s+x_s)}), \quad \text{as} \quad n \to \infty,\end{aligned}$$

and by the Lebesgue dominated convergence theorem we get,

$$\|\mathcal{A}(z_n) - \mathcal{A}(z)\|_{BC'_0} \to 0, \quad \text{as} \quad n \to \infty.$$

Thus $\mathcal{A}$ is continuous.

**Step 2**: $\mathcal{A}(B_r) \subset B_r$, this is clear.

**Step 3**: $\mathcal{A}(B_r)$ is equicontinuous on every compact interval $[0, b]$ of $[0, +\infty)$ for $b > 0$. Let $\tau_1, \tau_2 \in [0, b]$ with $\tau_2 > \tau_1$, we have

$$
\begin{aligned}
&|\mathcal{A}(z)(\tau_2) - \mathcal{A}(z)(\tau_1)| \\
&\leq \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}[\|\varphi_1\| - g(0, \phi)] \\
&\quad + \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} |f(s, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)})| \, ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} |f(s, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)})| \, ds \\
&\quad + \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} |g(s, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)})| \, ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} |g(s, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)})| \, ds \\
&\leq \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}[\|\varphi_1\| - g(0, \phi)] \\
&\quad + \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} |f(s, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)}) - f(s, 0)| \, ds \\
&\quad + \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} |f(s, 0)| \, ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} |f(s, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)}) - f(s, 0)| \, ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} |f(s, 0)| \, ds \\
&\quad + \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} |g(s, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)}) - g(s, 0)| \, ds \\
&\quad + \int_0^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} |g(s, 0)| \, ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} |g(s, z^n_{\rho(s, z^n_s + x_s)} + x_{\rho(s, z^n_s + x_s)}) - g(s, 0)| \, ds \\
&\quad + \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} |g(s, 0)| \, ds \\
&\leq \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)}[\|\varphi_1\| - g(0, \phi)] \\
&\quad + C \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} k(s) \, ds \\
&\quad + lr \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} k(s) \, ds \\
&\quad + \int_0^{\tau_1} \|S(\tau_2 - s) - S(\tau_1 - s)\|_{B(E)} |f(s, 0)| \, ds
\end{aligned}
$$

$$
\begin{aligned}
&+ C\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} k(s)\, ds \\
&+ lr\int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} k(s)\, ds \\
&+ \int_{\tau_1}^{\tau_2} \|S(\tau_2 - s)\|_{B(E)} |f(s,0)|\, ds \\
&+ C\int_{0}^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} k_g(s)\, ds \\
&+ lr\int_{0}^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} k_g(s)\, ds \\
&+ \int_{0}^{\tau_1} \|C(\tau_2 - s) - C(\tau_1 - s)\|_{B(E)} |g(s,0)|\, ds \\
&+ C\int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} k_g(s)\, ds \\
&+ lr\int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} k_g(s)\, ds \\
&+ \int_{\tau_1}^{\tau_2} \|C(\tau_2 - s)\|_{B(E)} |g(s,0)|\, ds.
\end{aligned}
$$

When $\tau_2 \to \tau_1$, the right-hand side of the above inequality tends to zero, since $C(t), S(t)$ are a strongly continuous operator and the compactness of $C(t)$, $S(t)$ for $t > 0$ implies the continuity in the uniform operator topology (see [31], [32]). This proves the equicontinuity.

**Step 4**: $N(B_r)$ is relatively compact on every compact interval of $[0, \infty)$. This is a consequence of $(H_7)$.

**Step 5**: $N(B_r)$ is equiconvergent.
Let $y \in B_r$, we have

$$
\begin{aligned}
|\mathcal{A}(z)(t)| &\le M'\|\varphi_1\| + M'\int_0^t |f(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|\, ds \\
&\quad + M\int_0^t |g(s, z^n_{\rho(s,z^n_s+x_s)} + x_{\rho(s,z^n_s+x_s)})|\, ds \\
&\le C_1 + M'rl\int_0^t k(s)\, ds + Mrl\int_0^t k_g(s)\, ds.
\end{aligned}
$$

Then

$$|\mathcal{A}(z)(t)| \to C_2, \quad \text{as} \quad t \to +\infty,$$

where

$$C_2 \le M'\|\varphi_1\| + MF^* + Mk^*(C + lr).$$

Hence,

$$|\mathcal{A}(z)(t) - \mathcal{A}(z)(+\infty)| \to 0, \quad \text{as} \quad t \to +\infty.$$

As a consequence of Steps 1–5 and, by Lemma 2.7, we can conclude that $\mathcal{A} : B_r \to B_r$ is continuous and compact. We deduce that $\mathcal{A}$ has a fixed point $z^*$. Then $y^* = z^* + x$ is a fixed point of the operator $N$, which is a mild solution of the problem (1)–(2).

**3.1 An example.** Take $E = L^2[0, \pi]; \mathcal{B} = C_0 \times L^2(h, E)$ and define $A : E \to E$ by $A\omega = \omega''$ with domain

$$D(A) = \{\omega \in E; \omega, \omega' \text{ are absolutely continuous, } \omega'' \in E, \ \omega(0) = \omega(\pi) = 0\}.$$

It is well known that $A$ is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in\mathbb{R}}$ on $E$. Moreover, $A$ has discrete spectrum, the eigenvalues are $-n^2$, $n \in \mathbb{N}$ with corresponding normalized eigenvectors

$$z_n(\tau) := \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sin n\tau,$$

and the following properties hold.

(a) $\{z_n : n \in \mathbb{N}\}$ is an orthonormal basis of $E$.
(b) If $y \in E$, then $Ay = -\sum_{n=1}^{\infty} n^2 \langle y, z_n\rangle z_n$.
(c) For $y \in E, C(t)y = \sum_{n=1}^{\infty} \cos(nt)\langle y, z_n\rangle z_n$, and the associated sine family is

$$S(t)y = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n} \langle y, z_n\rangle z_n,$$

which implies that the operator $S(t)$ is compact for all $t \in J$ and that

$$\|C(t)\| = \|S(t)\| \le 1, \quad \text{for all} \quad t \in \mathbb{R}.$$

(d) If $\Phi$ denotes the group of translations on $E$ defined by

$$\Phi(t)y(\xi) = \tilde{y}(\xi + t),$$

where $\tilde{y}$ is the extension of $y$ with period $2\pi$. Then

$$C(t) = \frac{1}{2}\left(\Phi(t) + \Phi(-t)\right), \ A = B^2,$$

where $B$ is the infinitesimal generator of the group $\Phi$ on

$$X = \{y \in H^1(0, \pi) : y(0) = y(\pi) = 0\}.$$

For more details see [15].

Consider the functional partial differential equation of the second order

$$\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}z(t,x)+\int_{-\infty}^{0} b(s-t)z(s-\rho_1(t)\rho_2(|z(t)|),x)\,ds\right]=\frac{\partial^2}{\partial x^2}z(t,x)$$
$$+\int_{-\infty}^{0} a(s-t)z(s-\rho_1(t)\rho_2(|z(t)|),x)\,ds,$$
$$x\in[0,\pi],\ t\in J:=[0,+\infty), \tag{5}$$
$$z(t,0)=z(t,\pi)=0,\ t\in[0,+\infty), \tag{6}$$
$$z(t,x)=\phi(t,x),\quad \frac{\partial z(0,x)}{\partial t}=\varphi(x),\ t\in[-r,0],\ x\in[0,\pi], \tag{7}$$

where $\phi\in\mathcal{B}$, $\rho_i:[0,\infty)\to[0,\infty)$, $a,b:\mathbb{R}\to\mathbb{R}$ are continuous, and

$$L_f=\int_{-\infty}^{0}\frac{a^2(s)}{2h(s)}\,ds<\infty,\quad L_g=\int_{-\infty}^{0}\frac{b^2(s)}{2h(s)}\,ds<\infty.$$

Under these conditions, we define the functions $f:J\times\mathcal{B}\to E$, $\rho:J\times\mathcal{B}\to\mathbb{R}$ by

$$f(t,\psi)(x)=\int_{-\infty}^{0} a(s)\psi(s,x)\,ds,$$
$$g(t,\psi)(x)=\int_{-\infty}^{0} b(s)\psi(s,x)\,ds,$$
$$\rho(s,\psi)=s-\rho_1(s)\rho_2(|\psi(0)|),$$

we have

$$\|f(t,\cdot)\|_{\mathfrak{B}(\mathcal{B},E)}\le L_f,\quad\text{and}\quad\|g(t,\cdot)\|_{\mathfrak{B}(\mathcal{B},E)}\le L_g.$$

Then the problem (1)–(2) is an abstract formulation of the problem (5)–(7). If conditions $(H_2)$–$(H_7)$, $(H_\phi)$ are satisfied, Theorem 3.5 implies that the problem (5)–(7) has at least one mild solution on $BC$.

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