On preimages of ultrafilters in ZF

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This article is dedicated to the memory of Horst Herrlich, a friend and a collaborator of both of us.

Abstract. We show that given infinite sets X, Y and a function $f : X \to Y$ which is onto and n-to-one for some $n \in \mathbb{N}$, the preimage of any ultrafilter $\mathcal{F}$ of Y under f extends to an ultrafilter. We prove that the latter result is, in some sense, the best possible by constructing a permutation model $\mathcal{M}$ with a set of atoms A and a finite-to-one onto function $f : A \to \omega$ such that for each free ultrafilter of ω its preimage under f does not extend to an ultrafilter. In addition, we show that in $\mathcal{M}$ there exists an ultrafilter compact pseudometric space $\mathbf{X}$ such that its metric reflection $\mathbf{X}^*$ is not ultrafilter compact.

Keywords: Boolean Prime Ideal Theorem; weak forms of the axiom of choice; ultrafilters

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1. Notation and terminology

Let $\mathbf{X} = (X, T)$ be a topological space. Then $\mathbf{X}$ is said to be *ultrafilter compact* iff every ultrafilter $\mathcal{F}$ of X *converges* to some point x in $\mathbf{X}$, i.e., for every neighborhood V of x, there exists $F \in \mathcal{F}$ with $V \supseteq F$.

Let $\mathcal{A} = (A_i)_{i \in I}$ be a family of non-empty sets. We say that a function $f : \mathcal{A} \to \mathcal{P}(\bigcup \mathcal{A})$ is a *Kinna-Wagner selection function* for $\mathcal{A}$ iff for every $i \in I$, $\varnothing \neq f(A_i) \subseteq A_i$ and if $|A_i| > 1$ then $f(A_i) \neq A_i$.

Let X be an infinite set. A *filterbase* $\mathcal{F}$ of X is a collection of subsets of X satisfying all but the superset requirement of a filter. i.e., $\varnothing \notin \mathcal{F}$ and $\mathcal{F}$ is closed under finite intersections.

A filter $\mathcal{F}$ of X is called *uniform* iff each of its members has size $|X|$.

If (X, ρ) is a pseudometric space then its metric reflection (X^*, ρ^*) is the set X^* of all equivalence classes in X of the equivalence relation $\sim$ given by:

$$x \sim y \text{ iff } \rho(x, y) = 0$$

and $\rho^* : X^* \times X^* \to \mathbb{R}$ is given by

$$\rho^*([x], [y]) = \rho(x, y),$$

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where $[x]$ denotes the equivalence class of the element x.
In the paper, we use the following principles.

- $\mathbf{BPI}(X)$: Every filterbase of X is included in an ultrafilter of X. ($\mathbf{BPI}(\omega)$ is Form 225 in [4]).
- $\mathbf{UF}(X)$: There is a free ultrafilter on X. ($\mathbf{UF}(\omega)$ is Form 70 in [4]).
- $\mathbf{PUU}(X)$: For every partition P of X, if $\mathcal{F}$ is an ultrafilter of P then the filterbase $\{\bigcup F : F \in \mathcal{F}\}$ of X extends to an ultrafilter. Equivalently, for every set Y, for every onto function $f : X \to Y$, for every ultrafilter $\mathcal{F}$ of Y, $f^{-1}(\mathcal{F}) = \{f^{-1}(F) : F \in \mathcal{F}\}$ extends to an ultrafilter of X.
- $\mathbf{PUU}_\omega(X)$: For every countable partition P of X, if $\mathcal{F}$ is an ultrafilter of P then the filterbase $\{\bigcup F : F \in \mathcal{F}\}$ of X extends to an ultrafilter. Equivalently, for every onto function $f : X \to \omega$ the preimage of every ultrafilter of ω extends to an ultrafilter of X.
- $\mathbf{SPUU}_\omega(X)$: For every onto function $f : X \to \omega$, for every ultrafilter $\mathcal{F}$ of ω, every filter extending the preimage of $\mathcal{F}$ under f extends to an ultrafilter $\mathcal{H}$ of X.

By universal quantifying over X each of the above notions gives rise to a choice principle. For example, the Boolean Prime Ideal theorem **BPI** (Form 14 of [4]) is the statement $\forall X$, $\mathbf{BPI}(X)$. Similarly one defines **UF**, **PUU**, $\mathbf{PUU}_\omega$ and $\mathbf{SPUU}_\omega$.

Besides the above-mentioned principles, there are four more weak forms of choice that we will use in this paper:

- $\mathbf{C}(\aleph_0, < \aleph_0)$ (Form 10 of [4]): Every family $\mathcal{A} = (A_i)_{i\in\omega}$ of non-empty finite sets has a choice function.
- $\mathbf{C}(\aleph_0, \infty)$ (Form 8 of [4]): Every family $\mathcal{A} = (A_i)_{i\in\omega}$ of non-empty sets has a choice function.
- $\mathbf{C}(\aleph_1, < \aleph_0)$: Every family $\mathcal{A} = (A_i)_{i\in\omega_1}$ of non-empty finite sets has a choice function.
- $\mathbf{UUF}(\omega_1)$: There is a uniform ultrafilter on ω_1.

2. Introduction and some preliminary results

The principle **PUU** (: For every infinite set X, Y, for every onto function $f : X \to Y$, for every ultrafilter $\mathcal{F}$ of Y, $f^{-1}(\mathcal{F}) = \{f^{-1}(F) : F \in \mathcal{F}\}$ extends to an ultrafilter of X) was introduced in [2] in order to prove:

(A) For every pseudometric space $\mathbf{X}$, if $\mathbf{X}$ is ultrafilter compact then so is its metric reflection $\mathbf{X}^*$.

The question whether **PUU** is necessary for the proof of (A) was left unanswered in [2]. The latter question leads to the following additional two:

(i) Is **PUU** a theorem of **ZF**?

(ii) Is (A) a Theorem of **ZF**?

In the forthcoming Theorem 5 we show that if in **PUU** we require that the function f satisfies in addition that, for every $y \in Y$, $|f^{-1}(y)| \leq n$, then the conclusion of **PUU** holds true. In view of this development it is plausible to ask the following question:

(iii) Let X, Y be two infinite sets and $f : X \to Y$ be a finite-to-one function. Does the preimage of an ultrafilter $\mathcal{F}$ of Y under f extend to an ultrafilter of X?

The main target of this project is to show that the answer to (i), (ii) and (iii) is in the negative. (i) is answered in Theorem 4 and (ii), (iii) in Theorem 6.

Regarding implications, non-implications and equivalent forms of the principles $\mathbf{BPI}(\omega)$ and $\mathbf{UF}(\omega)$ we refer the interested reader to [6], [3] and [1]. All principles involving ultrafilters in their definition are easily seen to be consequences of **BPI**. In Theorem 3 we show that **BPI** is equivalent to $\mathbf{SPUU}_\omega$. Since $\mathbf{PUU} \to \mathbf{PUU}_\omega$ is clear and $\mathbf{PUU}_\omega$ differs slightly from $\mathbf{SPUU}_\omega$ (in $\mathbf{SPUU}_\omega$ we require that every filter extending the preimage of an ultrafilter of ω extends to an ultrafilter and not just the preimage as we do in $\mathbf{PUU}_\omega$) one might ask whether the implication $\mathbf{PUU}_\omega \to \mathbf{PUU}$ holds true in **ZF**. The rest of our results are subsidiary to our second target which is to show that $\mathbf{PUU}_\omega \nrightarrow \mathbf{PUU}$ in **ZF**.

Before we proceed any further, let us scrutinize a little bit on preimages of filterbases and ultrafilters. Let X, Y be any two infinite sets and $f : X \to Y$ be a function. It is easy to see that the image of a filterbase $\mathcal{F}$ under f is a filterbase. In contrast with the image of a filterbase, the preimage of a filterbase need not be a filterbase. Indeed, if f is not onto then $f^{-1}(F)$ might be empty for some non-empty set F. If f is onto, then it is clear that

$$f^{-1}(\mathcal{F}) = \{f^{-1}(F) : F \in \mathcal{F}\}$$

is a filterbase of X. However, even in case where f is onto, if $\mathcal{F}$ is a filter of Y, $f^{-1}(\mathcal{F})$ need not be a filter of X. Indeed, if f is not one-to-one then for some $y \in Y, f^{-1}(y)$ has at least two elements, say a, b. If $F \in \mathcal{F}$ is such that $y \notin F$ and $A = F \backslash \{y\} \in \mathcal{F}$ then $B = (f^{-1}(A) \cup \{a\}) \supseteq f^{-1}(A)$ but $B \notin f^{-1}(\mathcal{F})$. So, $f^{-1}(\mathcal{F})$ is not a filter.

Given an onto function $f : X \to Y$ and a free ultrafilter $\mathcal{F}$ of Y, even though $f^{-1}(\mathcal{F})$ need not be a filter of X, $f^{-1}(\mathcal{F})$ always extends to a filter of X. So, one may ask whether $f^{-1}(\mathcal{F})$ extends to an ultrafilter of X. Of course, $\mathbf{BPI}(X)$ implies that the job can be done. So, $\mathbf{BPI} \to \mathbf{PUU}$ and one may ask if $\mathbf{PUU} \to \mathbf{BPI}$. The answer to the last question is no. Indeed, in any **ZF** model without free ultrafilters, such as the Feferman/Blass Model $\mathcal{M}15$ in [4], **PUU** holds. To see this, fix infinite sets X, Y, an onto function $f : X \to Y$ and an ultrafilter $\mathcal{F}$ of Y. Since in $\mathcal{M}15$ no infinite set has a free ultrafilters, it follows that $\mathcal{F} = \{F \subseteq Y : y \in F\}$ for some $y \in Y$. Then, it is easy to see that for every $x \in f^{-1}(y)$, $\mathcal{F}^* = \{A \subseteq X : x \in A\}$ is an ultrafilter of X extending the filterbase $\mathcal{W} = \{f^{-1}(F) : F \in \mathcal{F}\}$. However, $\mathbf{UF}(\omega)$ and **BPI** fail in $\mathcal{M}15$.

In the next proposition and the diagram that follows we summarize the easy, as well as known implications and non-implications between some of the principles defined in the first section.

Proposition 1. (i) $\mathbf{PUU}$ *and consequently* $\mathbf{PUU}_\omega$ *does not imply any one of the principles* $\mathbf{UF}(\omega)$, $\mathbf{UF}$ *and* $\mathbf{BPI}$ *in* $\mathbf{ZF}$.

(ii) *The statement "for every infinite set* X*, every filterbase* $\mathcal{F}$ *of* X *of size* $|\mathcal{F}| \leq |\mathbb{R}|$ *extends to an ultrafilter" implies* $\mathbf{PUU}_\omega$*. The reverse implication fails in* $\mathbf{ZF}$.

(iii) $\mathbf{PUU} \to \mathbf{PUU}_\omega$, $\mathbf{BPI}(\omega) \to \mathbf{UF}(\omega)$, $\mathbf{BPI}(\omega_1) \to \mathbf{BPI}(\omega)$, $\mathbf{BPI}(\omega_1) \to \mathbf{UUF}(\omega_1)$, $\mathbf{UUF}(\omega_1) \to \mathbf{UF}(\omega_1)$ *and* $\mathbf{UF}(\omega_1) \leftrightarrow \mathbf{UUF}(\omega_1) \vee \mathbf{UF}(\omega)$ *but,* $\mathbf{UF}(\omega) \nrightarrow \mathbf{BPI}(\omega)$, $\mathbf{BPI}(\omega) \nrightarrow \mathbf{BPI}(\omega_1)$, $\mathbf{UF}(\omega) \nrightarrow \mathbf{UUF}(\omega_1)$ *and* $\mathbf{UF}(\omega_1) \nrightarrow \mathbf{UUF}(\omega_1)$ *in* $\mathbf{ZF}$.

(iv) $\mathbf{UF}(\omega)$ *iff* $\mathbf{UF}(\mathbb{R})$ *iff* $\mathbf{CBPI}(\omega)$ (*: Every countable filterbase of* ω *extends to an ultrafilter*).

Proof: (i) This follows from the discussion preceding the statement of this proposition.

(ii) Fix X an infinite set and an onto function $f : X \to \omega$. Let $\mathcal{F}$ be an ultrafilter of ω and $\mathcal{W} = \{f^{-1}(F) : F \in \mathcal{F}\}$. Since $|\mathcal{W}| = |\mathcal{F}| \leq |\mathbb{R}|$, it follows by our hypothesis that $\mathcal{W}$ extends to an ultrafilter of X.

The second assertion follows from the fact that $\mathbf{PUU}_\omega$ holds true but $\mathbf{UF}(\omega)$ fails in $\mathcal{M}15$ and the observation that the statement "for every infinite set X, every filterbase of $\mathcal{F}$, $|\mathcal{F}| \leq |\mathbb{R}|$ of X extends to an ultrafilter" implies $\mathbf{UF}(\omega)$ (the set of all cofinite subsets of ω is countable and by our hypothesis extends to a necessarily free ultrafilter).

(iii) $\mathbf{PUU} \to \mathbf{PUU}_\omega$, $\mathbf{BPI}(\omega) \to \mathbf{UF}(\omega)$, $\mathbf{BPI}(\omega_1) \to \mathbf{BPI}(\omega)$, $\mathbf{BPI}(\omega_1) \to \mathbf{UUF}(\omega_1)$ and $\mathbf{UUF}(\omega_1) \to \mathbf{UF}(\omega_1)$ are left as an easy exercise for the reader.

$\mathbf{UF}(\omega_1) \leftrightarrow \mathbf{UUF}(\omega_1) \vee \mathbf{UF}(\omega)$. It suffices to show ($\to$) as the opposite implication is obvious. Assume that $\mathbf{UUF}(\omega_1)$ fails. We show that $\mathbf{UF}(\omega)$ holds true. Fix, by our hypothesis, a free ultrafilter $\mathcal{F}$ of ω_1. Since $\mathbf{UUF}(\omega_1)$ fails, it follows that there exists $K \in \mathcal{F}$ such that $|K| = \aleph_0$. Since the trace $\{K \cap F : F \in \mathcal{F}\}$ of $\mathcal{F}$ to K is clearly a free ultrafilter of K it follows that $\mathbf{UF}(\omega)$ holds true as required.

$\mathbf{UF}(\omega) \nrightarrow \mathbf{BPI}(\omega)$ has been established in [3].

For the non-implications we refer the reader to [6] where a symmetric model $\mathcal{N}$ has been constructed in which $|\mathbb{R}| = \aleph_1$ but the set C of all co-countable subsets of ω_1 is included in no ultrafilter of ω_1 meaning that $\mathbf{UUF}(\omega_1)$ and $\mathbf{BPI}(\omega_1)$ fail in $\mathcal{N}$. Since $|\mathbb{R}| = \aleph_1$ implies $\mathbf{BPI}(\omega)$ and $\mathbf{UF}(\omega)$ hence, $\mathbf{UF}(\omega_1)$ also, it follows that $\mathcal{N}$ satisfies $\mathbf{BPI}(\omega)$, $\mathbf{UF}(\omega)$, $\mathbf{UF}(\omega_1)$ and the negations of $\mathbf{BPI}(\omega_1)$ and $\mathbf{UUF}(\omega_1)$.

(iv) See Theorems 3.1 and 3.3 in [3]. □

$$
\begin{array}{ccc}
\mathbf{BPI}(\omega_1) & \begin{array}{c}\rightarrow\\ \not\leftarrow\end{array} & \mathbf{BPI}(\omega) \\
\downarrow \quad (\uparrow?) & \not\swarrow\ (\nearrow?) & \downarrow \quad \not\uparrow \\
\mathbf{UUF}(\omega_1) & \begin{array}{c}(\rightarrow?)\\ \not\leftarrow\end{array} & \mathbf{UF}(\omega) \\
 & \searrow \quad (\nwarrow?) & \downarrow \quad (\uparrow?) \\
 & & \mathbf{UF}(\omega_1)
\end{array}
$$

Diagram 1

Question 1. Does the statement **CBPI** (: For every infinite set X, every countable filterbase $\mathcal{F}$ of X extends to an ultrafilter) imply $\mathbf{PUU}_\omega$ (: For every infinite set X, for every onto function $f : X \to \omega$ the preimage of every ultrafilter of ω extends to an ultrafilter of X)?

Remark 1. (i) We stress the fact that in case $\mathbb{R}$ has a subset of size $\aleph_1$ then, by employing Proposition 1 part (iv), we can eliminate the question-mark in the implication $\mathbf{UF}(\omega_1) \to \mathbf{UF}(\omega)$ of Diagram 1. However, the statement "$\mathbb{R}$ has an uncountable well-ordered subset", Form 170 in [4], is not a theorem of **ZF**.

Another point we would like to stress is that in contrast to the fact that uniform and free ultrafilters of ω coincide, the set of all uniform ultrafilters of ω_1 is strictly included in the set of the free ones. This explains the question-mark in the implication $\mathbf{UF}(\omega_1) \to \mathbf{UUF}(\omega_1)$ of Diagram 1.

(ii) Regarding Question 1, we can adopt the proof of Proposition 1 (ii) to show that $\mathbf{PUU}_\omega \not\to \mathbf{CBPI}$ in **ZF**. However, in the forthcoming Theorem 4 we show that $\mathbf{PUU}_\omega \wedge \mathbf{UF}(\omega) \to \mathbf{CBPI}$.

In the following Proposition 2 and the forthcoming Theorem 5 we give some instances where the preimage of an ultrafilter always extends to an ultrafilter.

Proposition 2. *Let X, Y be two infinite sets, $f : X \to Y$ be an onto function and $\mathcal{F}$ be an ultrafilter of Y. If*

(a) *for some $H \in \mathcal{F}$, $\{f^{-1}(t) : t \in H\}$ has choice set C, or*
(b) *for every $H \in \mathcal{F}$, $\{f^{-1}(t) : t \in H\}$ has no Kinna-Wagner selection function, then $\mathcal{F}^* = \{f^{-1}(F) : F \in \mathcal{F}\}$ extends to an ultrafilter $\mathcal{W}$ of X.*

Proof: If $\mathcal{F}$ is a fixed ultrafilter of Y then the conclusion is straightforward. So, we assume that $\mathcal{F}$ is free.

(i) Assume (a) holds. Clearly, in this case the restriction $f|C : C \to H$ of f to C is one-to-one and onto. Since, the restriction $\mathcal{F}_H$ of $\mathcal{F}$ to H is an ultrafilter of H, it follows that

$$\mathcal{U} = \{K \subseteq C : (f|C)^{-1}(F) \subseteq K \text{ for some } F \in \mathcal{F}_H\}$$

is an ultrafilter of C. It is easy to see that $\mathcal{U}$ extends to an ultrafilter $\mathcal{W}$ of X including $\mathcal{F}^*$.

(ii) Assume (b) holds. We show that

$$\mathcal{W} = \{K \subseteq X : f^{-1}(F) \subseteq K \text{ for some } F \in \mathcal{F}\}$$

is the required ultrafilter of X.

Since $\mathcal{W}$ is clearly a filter, it suffices to show that if $K \subseteq X$ satisfies that, for all $W \in \mathcal{W}$, $K \cap W \neq \varnothing$, then $K \in \mathcal{W}$. Fix such a set $K \in \mathcal{P}(X)$ and let

$$H^{'} = \{t \in Y : K \cap f^{-1}(t) \neq \varnothing\}.$$

We claim that $H^{'} \in \mathcal{F}$. Indeed, if $H^{'} \notin \mathcal{F}$ then $(Y \backslash H^{'}) \in \mathcal{F}$ and $f^{-1}(Y \backslash H^{'}) \in \mathcal{W}$. However, $K \cap f^{-1}(Y \backslash H^{'}) = \bigcup\{K \cap f^{-1}(t) : t \in Y \backslash H^{'}\} = \varnothing$ contradicts our hypothesis. Hence, $H^{'} \in \mathcal{F}$.

Let $H^* = \{t \in H^{'} : f^{-1}(t) \backslash K \neq \varnothing\}$. By (b), $H^* \notin \mathcal{F}$. Hence, $H^{'} \backslash H^* \in \mathcal{F}$. Since, $f^{-1}(H^{'} \backslash H^*) \subseteq K$ it follows that $K \in \mathcal{W}$ as required. □

3. Main results

Theorem 3. BPI *if and only if* $\mathbf{SPUU}_\omega$ (*: For every infinite set X, for every onto function $f : X \to \omega$, for every ultrafilter $\mathcal{F}$ of ω, every filter extending the preimage of $\mathcal{F}$ under f extends to an ultrafilter*).

Proof: ($\rightarrow$) This is straightforward.

($\leftarrow$) Fix X an infinite set and let $\mathcal{H}$ be a filterbase of X. We show that $\mathcal{H}$ extends to an ultrafilter of X. Let $Y = X \cup \mathbb{N}$. Without loss of generality we may assume that $X \cap \mathbb{N} = \varnothing$. Let $f : Y \to \omega$ be the function given by

$$f(x) = \begin{cases} 0 & \text{if } x \in X, \\ x & \text{otherwise.} \end{cases}$$

Let $\mathcal{F}$ be the fixed ultrafilter of ω of all supersets of $\{0\}$. Clearly, $f^{-1}(F) \cap Y = X$ for every $F \in \mathcal{F}$. Hence, $\mathcal{W} = \mathcal{H} \cup \{f^{-1}(F) : F \in \mathcal{F}\}$ has the finite intersection property and the filter $\mathcal{Q}$ generated by $\mathcal{W}$ extends $\{f^{-1}(F) : F \in \mathcal{F}\}$. By our hypothesis, $\mathcal{Q}$ extends to an ultrafilter $\mathcal{U}$ of Y. Since $X \in \mathcal{U}$, it follows that $\mathcal{V} = \{U \cap X : U \in \mathcal{U}\} \supseteq \mathcal{H}$ is an ultrafilter of X extending $\mathcal{H}$ as required. □

Next we show that the negation of **PUU** is consistent with **ZF** and $\mathbf{PUU}_\omega \nrightarrow$ **PUU** in **ZF**.

Theorem 4. (i) $\mathbf{C}(\aleph_0, \infty)$ (*: Every family $\mathcal{A} = (A_i)_{i \in \omega}$ of non-empty sets has a choice function*) *implies* $\mathbf{PUU}_\omega$ (*: For every countable partition P*

of X, if $\mathcal{F}$ is an ultrafilter of P then the filterbase $\{\bigcup F : F \in \mathcal{F}\}$ of X extends to an ultrafilter).

(ii) $\mathbf{PUU}_\omega \wedge \mathbf{UF}(\omega)$ *(: ω has a free ultrafilter) implies* **CBPI** *(: For every infinite set X, every countable filterbase $\mathcal{F}$ of X extends to an ultrafilter).*

(iii) **CBPI** *implies "for every family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty sets there exists a family $\mathcal{U} = \{\mathcal{U}_i : i \in \omega\}$ such that for every $i \in \omega$, $\mathcal{U}_i$ is an ultrafilter of A_i" which in turn implies* $\mathbf{C}(\aleph_0, < \aleph_0)$ *(: Every family $\mathcal{A} = (A_i)_{i\in\omega}$ of non-empty finite sets has a choice function).*

(iv) $\mathbf{C}(\aleph_0, < \aleph_0) \wedge \mathbf{PUU} \wedge \mathbf{UUF}(\omega_1)$ *(: ω_1 has a uniform ultrafilter) implies* $\mathbf{C}(\aleph_1, < \aleph_0)$.

(v) $\mathbf{PUU} \wedge \mathbf{BPI}(\omega_1)$ *(: Every filterbase of ω_1 extends to an ultrafilter) implies* $\mathbf{C}(\aleph_1, < \aleph_0)$.

(vi) $\mathbf{PUU}_\omega$ *does not imply* **PUU** *in* **ZF**.

(vii) *There is a model $\mathcal{N}$ of* **ZF** *satisfying* $\mathbf{UF}(\omega)$ *and the negation of* $\mathbf{PUU}_\omega$*, hence the negation of* **PUU** *also. In particular,* $\mathbf{UF}(\omega)$ *and* $\mathbf{PUU}_\omega$ *are independent of each other in* **ZF**.

Proof: (i) This follows at once from Proposition 2.

(ii) Fix X an infinite set and let $\mathcal{W} = \{W_n : n \in \omega\}$ be a filterbase of X. If $\bigcap \mathcal{W} \neq \varnothing$ then the conclusion is straightforward. For every $x \in \bigcap \mathcal{W}$ the fixed ultrafilter $\mathcal{F}_x$ generated by $\{x\}$ extends $\mathcal{W}$. So, assume that $\bigcap \mathcal{W} = \varnothing$ and $\mathcal{W}$ is strictly descending. For every $n \in \omega$, let $U_n = W_n \backslash W_{n+1}$. Define a function $f : X \to \omega$ by requiring:

$$f(x) = \begin{cases} n+1 & \text{if } x \in U_n\,, \\ 0 & \text{if } x \in X \backslash W_0. \end{cases}$$

Fix, by $\mathbf{UF}(\omega)$, a free ultrafilter $\mathcal{F}$ of ω. Since $\mathcal{F}$ contains all cofinite subsets of ω, it follows that $\{W_n : n \in \omega\} \subseteq \{f^{-1}(F) : F \in \mathcal{F}\}$. By $\mathbf{PUU}_\omega$, $\{f^{-1}(F) : F \in \mathcal{F}\}$ and consequently $\mathcal{W}$ extends to an ultrafilter $\mathcal{F}$ of X.

(iii) Fix a family $\mathcal{A} = \{A_i : i \in \omega\}$ of non-empty sets. We show that there exists a family $\mathcal{U} = \{\mathcal{U}_i : i \in \omega\}$ such that for every $i \in \omega$, $\mathcal{U}_i$ is an ultrafilter of A_i. For every $i \in \omega$ let $X_i = A_i \cup \{i\}$. Clearly, $X = \prod_{i\in\omega} X_i \neq \varnothing$ and $\mathcal{W} = \{W_n : n \in \omega\}$ where for every $n \in \omega$, $W_n = \bigcap\{\pi_i^{-1}(A_i) : i \leq n\}$ is a countable filterbase of X. Let, by **CBPI**, $\mathcal{F}$ be an ultrafilter of X extending $\mathcal{W}$. Since for every $i \in \omega$, $\mathcal{F}_i = \pi_i(\mathcal{F})$ is an ultrafilter of X_i and $A_i \in \mathcal{F}_i$, it follows that the trace $\mathcal{U}_i$ of $\mathcal{F}_i$ to A_i is an ultrafilter of A_i. Hence, $\mathcal{U} = \{\mathcal{U}_i : i \in \omega\}$ is as required.

The second assertion is a straightforward consequence of the fact that ultrafilters of finite sets are fixed.

(iv) Fix $\mathcal{A} = \{A_i : i \in \aleph_1\}$ a family of non-empty sets. Assume for contradiction that there is no choice function for $\mathcal{A}$. Let ∞ be a new point and put

$$X = \prod_{\alpha\in\aleph_1} (A_\alpha \cup \{\infty\}).$$

For every $\alpha \in \aleph_1$ define

$$P_\alpha = \{x \in X : x(\alpha) = \infty \text{ and for every } \beta \in \alpha\ x(\beta) \neq \infty\}.$$

By $\mathbf{C}(\aleph_0, < \aleph_0)$ each P_α is non-empty and since there is no choice function they partition X. Since there is a uniform ultrafilter on ω_1 by **PUU**, there is an ultrafilter $\mathcal{U}$ on X such that for every $\alpha \in \omega_1$ the set $\bigcup\{P_\beta : \alpha \in \beta\}$ is in $\mathcal{U}$. Since A_α is finite there is a unique $a_\alpha \in A_\alpha \cup \{\infty\}$ such that

$$\{x \in X : x(\alpha) = a_\alpha\} \in \mathcal{U}.$$

But since $\bigcup\{P_\beta : \alpha \in \beta\} \in \mathcal{U}$, it must be that $a_\alpha \neq \infty$ and so $f : \aleph_1 \to \bigcup \mathcal{A}$, $f(\alpha) = a_\alpha$ is a choice function for $\mathcal{A}$. Contradiction!

(v) This follows from (ii), (iii) and (iv) of the present theorem and Proposition 1.

(vi) We recall that Jech's Model $\mathcal{N}2(\aleph_1)$ in [4] is specified by a set A of atoms of size $\aleph_1$, the group G of all permutations of A leaving the set

$$B = \{\{a_i, b_i\} : i \in \aleph_1\}$$

pointwise fixed where B is a disjointed set having union A and the set S of supports is all countable subsets of A. It is known, see, e.g., [4], that in $\mathcal{N}2(\aleph_1)$, $\mathbf{C}(\aleph_0, \infty)$, hence by part (i) $\mathbf{PUU}_\omega$ also, holds true but B has no choice set meaning that $\mathbf{C}(\aleph_1, < \aleph_0)$ fails. Since in permutation models the power set of a well-ordered cardinal number is well-orderable, we can use transfinite induction on $\aleph = |\mathcal{P}(\omega_1)|$ to extend every filterbase of ω_1 to an ultrafilter. Hence, $\mathbf{BPI}(\omega_1)$ holds true in $\mathcal{N}2(\aleph_1)$. Thus, by part (v), it follows that **PUU** fails in $\mathcal{N}2(\aleph_1)$. Finally, an application of the Jech-Sochor Embedding Theorem (Theorem 6.1 in [5]) yields a **ZF** model satisfying $\mathbf{PUU}_\omega$ and the negation of **PUU** meaning that $\mathbf{PUU}_\omega \nrightarrow \mathbf{PUU}$ in **ZF**.

(vii) It is known that in the model $\mathcal{N}[\Gamma]$ in [3], $\mathbf{UF}(\omega)$ holds but $\mathbf{C}(\aleph_0, < \aleph_0)$ fails. Hence, by parts (ii) and (iii) of the present theorem, $\mathbf{PUU}_\omega$ and **PUU** fail in $\mathcal{N}[\Gamma]$.

The second assertion follows from the first part and Proposition 1. □

Theorem 5. *Let X, Y be two infinite sets, $n \in \mathbb{N}$ and $f : X \to Y$ be an onto function such that for every $y \in Y$, $|f^{-1}(y)| \leq n$. Then, for every ultrafilter $\mathcal{F}$ of Y, the preimage $\mathcal{F}^* = \{f^{-1}(F) : F \in \mathcal{F}\}$ of $\mathcal{F}$ extends to an ultrafilter $\mathcal{W}$ of X.*

Proof: We get a proof by induction that

"$\forall n \in \omega \backslash \{0\}$, if X and Y are infinite sets, $f : X \to Y$ is an onto function such that for every $y \in Y$, $|f^{-1}(y)| \leq n$ and $\mathcal{F}$ is an ultrafilter of Y then the preimage of $\mathcal{F}$ under f extends to an ultrafilter of X".

Assume that the statement is true for every $k < n$ and let X, Y be two infinite sets, $\mathcal{F}$ be an ultrafilter of Y and $f : X \to Y$ be an onto function with $|f^{-1}(y)| \leq n$

for every $y \in Y$. We show that $\{f^{-1}(H) : H \in \mathcal{F}\}$ extends to an ultrafilter on X. This, in case $\mathcal{F}$ is fixed, follows from the discussion preceding Proposition 1. So, we assume that $\mathcal{F}$ is free. By Proposition 2 part (b), if for every $H \in \mathcal{F}$, $\{f^{-1}(t) : t \in H\}$ has no Kinna-Wagner selection function then we are done. So, assume that for some $H_0 \in \mathcal{F}$, $\{f^{-1}(t) : t \in H_0\}$ has a Kinna-Wagner selection function C_1. Then, for all $t \in H_0$, $0 < |C_1(f^{-1}(t))| \leq n-1$.

Letting $\mathcal{F}_1 = \{H \cap H_0 : H \in \mathcal{F}\}$ and $X_1 = \bigcup\{C_1(f^{-1}(t)) : t \in H_0\}$ we have that X_1 and H_0 are infinite (since $\mathcal{F}$ is free every element of $\mathcal{F}$ must be infinite), that $f_1 = f|X_1 : X_1 \to H_0$ is onto with the property that for all $y \in H_0, |f_1^{-1}(y)| < n$ and that $\mathcal{F}_1$ is an ultrafilter on H_0. Hence, by the induction hypothesis, $\{f_1^{-1}(F) : F \in \mathcal{F}_1\}$ extends to an ultrafilter $\mathcal{U}$ on X_1. Clearly,

$$\{K \subseteq X : \exists Z \in \mathcal{U} \text{ such that } Z \subseteq K\}$$

is the required ultrafilter on X extending $\{f^{-1}(H) : H \in \mathcal{F}\}$. □

Theorem 6. (i) *It is consistent with* **ZF** *the existence of an infinite set X and a finite-to-one function $f : X \to \omega$ such that the preimage of an ultrafilter $\mathcal{F}$ of ω under f does not extend to an ultrafilter of X.*

(ii) *The negation of the statement "If the pseudometric space* $\mathbf{X}$ *is ultrafilter compact then so is its metric reflection* $\mathbf{X}^*$*" is consistent with* **ZF***.*

Proof: (i) We will construct a model $\mathcal{M}$ of $\mathbf{ZF}^0$ with a set A of atoms such that there is a finite-to-one function $f : A \to \omega$ with the property that if $\mathcal{F}$ is an ultrafilter of $\omega, f^{-1}(\mathcal{F})$ does not extend to an ultrafilter of A.

Assume that the ground model has a countable set of atoms A. Write A as a countable union of disjoint sets $A = \bigcup\{A_i : i \in \omega\}$ such that for each $i \in \omega$, $|A_i| = 2^i$. This can be conveniently done if we index atoms by finite sequences of zeros and ones as follows. Let $2^{<\omega} = \bigcup_{n\in\omega} 2^n$ be the set of all finite sequences of elements of $2 = \{0, 1\}$. Let $\sigma \mapsto a_\sigma$ be a one to one function from $2^{<\omega}$ onto A. For each $i \in \omega$ let $A_i = \{a_\sigma : \sigma \in 2^i\}$. We call A_i the ith level zero blocks and define a "sub-block" structure on each A_i as follows. For $i > 0$, partition A_i into two level one sub-blocks each of cardinality 2^{i-1}. Assuming $2^{i-1} > 1$ partition each level one sub-block into two level two sub-blocks each of cardinality 2^{i-2}. Assuming $2^{i-2} > 1$ partition the level two sub-blocks into level three sub-blocks, etc. This can be done more precisely using the indexing of the atoms by elements of $2^{<\omega}$: If $i, n \in \omega$, $n \leq i$ and $\sigma \in 2^n$ then the set $A_{i,\sigma} = \{a_\gamma : \gamma \in 2^i \text{ and } \gamma \restriction n = \sigma\}$ is called the *level n sub-block of A_i determined by σ*. (Note that, if $n = 0$, then $2^n = \{\emptyset\}$ and so there is only one level zero sub-block of A_i, namely $A_{i,\emptyset} = \{a_\gamma : \gamma \in 2^i \text{ and } \emptyset \subseteq \gamma\} = A_i$.) To say this in a slightly different way, if $\gamma \in 2^i$ and $n \leq i$, then a_γ is in the level n sub-block $A_{i,\sigma}$ of A_i where $\sigma = \gamma \restriction n$. It follows that for $i \geq n$, A_i is the disjoint union

$$A_i = \bigcup\{A_{i,\sigma} : \sigma \in 2^n\}.$$

For example,

$$A_3 = \{a_{(0,0,0)}, a_{(0,0,1)}, a_{(0,1,0)}, a_{(0,1,1)}, a_{(1,0,0)}, a_{(1,0,1)}, a_{(1,1,0)}, a_{(1,1,1)}\}$$

is a level zero block. Its level one sub-blocks are

$$A_{3,(0)} = \{a_{(0,0,0)}, a_{(0,0,1)}, a_{(0,1,0)}, a_{(0,1,1)}\}$$

and $A_{3,(1)} = \{a_{(1,0,0)}, a_{(1,0,1)}, a_{(1,1,0)}, a_{(1,1,1)}\}$. Its level two sub-blocks are $A_{3,(0,0)} = \{a_{(0,0,0)}, a_{(0,0,1)}\}$, $A_{3,(0,1)} = \{a_{(0,1,0)}, a_{(0,1,1)}\}$, $A_{3,(1,0)} = \{a_{(1,0,0)}, a_{(1,0,1)}\}$ and $A_{3,(1,1)} = \{a_{(1,1,0)}, a_{(1,1,1)}\}$. Its level three sublocks are its singleton subsets. The element $a_{(1,0,1)}$ is in the level one sub-block $A_{3,(1)}$.

For $n \in \omega$ we let $\mathcal{B}_n$ be the set of all level n sub-blocks, that is

$$\mathcal{B}_n = \{A_{i,\sigma} : n \leq i \text{ and } \sigma \in 2^n\}.$$

We now describe the group G and the filter Γ of subgroups of G that will determine the model $\mathcal{M}$.

For every $F \in [\omega]^{<\omega}$ let ϕ_F denote the permutation of A given by $\phi_F(a_\sigma) = a_\rho$ where $\sigma, \rho \in 2^{<\omega}$ have the same length and

$$\rho(i) = \begin{cases} \sigma(i) & \text{if } i \notin F \\ 1 + \sigma(i) \bmod 2 & \text{if } i \in F \end{cases}.$$

Since for every $F, H \in [\omega]^{<\omega}, \phi_F \circ \phi_H = \phi_{F \triangle H} = \phi_H \circ \phi_F$ we see that the group $(G, \circ)$, $G = \{\phi_F : F \in [\omega]^{<\omega}\}$ is commutative. Hence, the subgroups

$$G_n = \{\phi_F : F \in [\omega]^{<\omega}, F \cap n = \varnothing\},\ n \in \omega$$

are normal.

Let Γ be the filter of subgroups of G generated by $\{G_n : n \in \omega\}$. That is,

$$\Gamma = \{H : H \text{ is a subgroup of } G \text{ and } \exists n \in \omega : G_n \subseteq H\}.$$

In order for Γ to yield a model of $\mathbf{ZF}^0$, Γ must be closed under conjugation by elements of G, a fact which follows trivially by the commutativity of G. Let $\mathcal{M}$ be the model determined by G and Γ. (An element x of the ground model is in $\mathcal{M}$ if and only if every element y of $\{x\} \cup \mathrm{TC}(x)$ has the property that for some $n \in \omega$, $G_n \subseteq \mathrm{Sym}_G(y)$. Here we have used $\mathrm{TC}(x)$ for the transitive closure of x and $\mathrm{Sym}_G(y)$ for $\{\phi \in G : \phi(y) = y\}$.)

Lemma 7. *In the model $\mathcal{M}$ there is no free ultrafilter on A.*

Proof: Toward a proof by contradiction assume that $\mathcal{F}$ is a free ultrafilter on A which is in $\mathcal{M}$. Then there is an $n \in \omega$ such that for all $F \in [\omega]^{<\omega}$ if $F \cap n = \varnothing$ then $\phi_F(\mathcal{F}) = \mathcal{F}$. Since $\mathcal{F}$ is free and $\bigcup_{i \leq n} A_i$ is finite we may conclude that $\bigcup_{i > n} A_i \in \mathcal{F}$. Our plan is to partition $\bigcup_{i > n} A_i$ into two sets B_0 and B_1, both in $\mathcal{M}$, such that $\phi_{\{n\}}(B_0) = B_1$ and $\phi_{\{n\}}(B_1) = B_0$. Since exactly one of B_0

or B_1 is in $\mathcal{F}$ and $\phi_{\{n\}} \in G_n$ this will contradict the assumption that for all $\phi \in G_n, \phi(\mathcal{F}) = \mathcal{F}$.

Clearly, for every $n \in \omega$, $\phi_{\{n\}}$ is the permutation of A which fixes $\bigcup_{i \leq n} A_i$ pointwise and for $i > n$ interchanges A_{i,σ_0} and A_{i,σ_1} for every level n block $A_{i,\sigma} \subseteq A_i$ where for every $v \leq n$,

$$\sigma_0(v) = \begin{cases} \sigma(v) & \text{if } v \in n, \\ 0 & \text{if } v = n, \end{cases} \quad \text{and} \quad \sigma_1(v) = \begin{cases} \sigma(v) & \text{if } v \in n, \\ 1 & \text{if } v = n. \end{cases}$$

Using σ_0 and σ_1 we partition the set $\mathcal{B}_{n+1}$ of level $n+1$ blocks into $\mathcal{B}^0_{n+1} = \{A_{i,\sigma_0} : i > n \text{ and } \sigma \in 2^n\}$ and $\mathcal{B}^1_{n+1} = \{A_{i,\sigma_1} : i > n \text{ and } \sigma \in 2^n\}$. Since $G_{n+1} \subseteq \mathrm{Sym}_G(B)$ for every level $n+1$ block B, it follows that every subset of $\mathcal{B}_{n+1}$ is in $\mathcal{M}$ and therefore both $\mathcal{B}^0_{n+1}$ and $\mathcal{B}^1_{n+1}$ are in $\mathcal{M}$. We let $B_0 = \bigcup \mathcal{B}^0_{n+1}$ and $B_1 = \bigcup \mathcal{B}^1_{n+1}$. Both of these sets are in $\mathcal{M}$.

Since for every $\sigma \in 2^n$ and every $i > n$, $\phi_{\{n\}}$ interchanges A_{i,σ_0} and A_{i,σ_1} we have that $\phi_{\{n\}}$ interchanges $\mathcal{B}^0_{n+1}$ and $\mathcal{B}^1_{n+1}$. Therefore $\phi_{\{n\}}$ interchanges B_0 and B_1 and the proof of the lemma is complete. □

To complete the proof of (i) we define the function $f : A \to \omega$ by $f(a) = i$ where $a \in A_i$. Note that f is finite-to-one. Let $\mathcal{F}$ be any free ultrafilter on ω. Any ultrafilter in A extending $f^{-1}(\mathcal{F})$ must be free and by Lemma 7 no such ultrafilters on A exist.

(ii) Let $d : A \times A \to \mathbb{R}$ be the pseudometric given by

$$d(a,b) = \begin{cases} 1 & \text{if } a \in A_i, b \in A_j, \ i,j \in \omega \text{ and } i \neq j, \\ 0 & \text{otherwise.} \end{cases}$$

Since by Lemma 7 A has only principal ultrafilters, it follows that $\mathbf{A} = (A, d)$ is ultrafilter compact. The fact that its metric reflection $\mathbf{A}^*$ is not ultrafilter compact follows from the observation that $\mathbf{A}^*$ is homeomorphic with ω taken with the discrete topology and no free ultrafilter of ω converges.

Finally, an application of the Jech-Sochor Embedding Theorem (Theorem 6.1 in [5]) shows that (i) and (ii) are transferable to **ZF**. (The forcing used in the Jech-Sochor Embedding Theorem is always at least countably closed so these models will always have a free ultrafilter on ω). □

Corollary 8. *The Model $\mathcal{M}$ of Theorem 6 satisfies the negation of $\mathbf{PUU}_\omega$, hence the negation of **PUU** as well.*

PROOF: See the last two lines of the proof of Theorem 6 part (i). □

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