Comaximal graph of $C(X)$

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Abstract. In this article we study the comaximal graph $\Gamma'_2C(X)$ of the ring $C(X)$. We have tried to associate the graph properties of $\Gamma'_2C(X)$, the ring properties of $C(X)$ and the topological properties of X. Radius, girth, dominating number and clique number of the $\Gamma'_2C(X)$ are investigated. We have shown that $2 \leq \operatorname{Rad}\Gamma'_2C(X) \leq 3$ and if $|X| > 2$ then girth $\Gamma'_2C(X) = 3$. We give some topological properties of X equivalent to graph properties of $\Gamma'_2C(X)$. Finally we have proved that X is an almost P-space which does not have isolated points if and only if $C(X)$ is an almost regular ring which does not have any principal maximal ideals if and only if $\operatorname{Rad}\Gamma'_2C(X) = 3$.

Keywords: rings of continuous functions; comaximal graph; radius; girth; dominating number; clique number; zero cellularity; P-space; almost P-space; connected space; regular ring

Classification: 54C40

1. Introduction

Throughout this paper, G stands for an undirected graph. Distance between two vertices u and v is defined as the length of shortest path between u and v, and is denoted by $d(u, v)$, then the *diameter* of G is denoted by $\operatorname{diam}(G)$, and is defined to be the supremum of $\{d(u, v) : u, v \in G\}$. If u is a vertex of a graph G, then *eccentricity* of u, denoted by $\operatorname{ecc}(u)$, is defined $\max\{d(u, v) : v \in G\}$. The set of all vertices with the smallest eccentricity is called *center* of G and $\min\{\operatorname{ecc}(u) : u \in G\}$ is called the *radius* of G and is denoted by $\operatorname{Rad}(G)$. The minimum length of cycles in a graph G is called the *girth* of G and is denoted by girth (G). For every $u, v \in G$, let us denote by $\operatorname{gi}(u, v)$ the length of the shortest cycle containing u and v. It is clear that girth $(G) = \min\{\operatorname{gi}(u, v) : u, v \in G\}$. G is called *triangulated* (*hypertriangulated*) if each vertex (edge) of G is a vertex (edge) of a triangle. A subset A of G is called a *dominating set* if for each $u \in G \setminus A$, there exists v in A such that u is adjacent to v. The *dominating number* of G, denoted by $\operatorname{dt}(G)$, is the smallest cardinal number of the form $|A|$, where A is a dominating set of G. It is said that two vertices u and v of G are *orthogonal*, written $u \perp v$, if u and v are adjacent and there is no a vertex w of G which is adjacent to both u and v. A graph G is called *complemented* if for each vertex u of G, there is a vertex v of G such that $u \perp v$. A clique of a graph G is defined as

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a maximal complete subgraph of G and the supremum of $|A|$, where A is clique of G, is called the clique number of G, and is denoted by clique G.

Let R be a commutative ring with unity. R is called an almost ring if each non-unit element of R is a zero-divisor element of R. *Comaximal graph* $\Gamma(R)$ is defined as a graph with vertices of elements of R, where two distinct vertices a and b are adjacent if and only if $Ra + Rb = R$. Also consider a subgraph $\Gamma_2(R)$ of $\Gamma(R)$ which consists of all non-unit elements of R. If $J(R)$ is Jacobson radical of R, then $\Gamma_2(R) \setminus J(R)$ is denoted by $\Gamma'_2(R)$.

We assume throughout the paper that $C(X)$ is the ring of all real valued continuous functions on a Tychonoff space X. The *density* (*weight*) of X, denoted by $d(X)$ ($w(X)$), is the infimum of the cardinalities of dense subsets (bases) of X. The character of X at a point p, denoted by $\chi(p, X)$, is the infimum of the cardinalities of neighborhood bases at x and the *character* of space X, denoted by $\chi(X)$, is the supremum of $\chi(p, X)$, where $p \in X$. A space X is called *first* (*second*) *countable* if $w(X)$ ($\chi(X)$) is countable. The *cellularity* of X, denoted by $c(X)$, is defined by

$$\sup\{|\mathcal{U}| : \mathcal{U} \text{ is a family of mutually disjoint nonempty open subsets of } X\}.$$

For any $f \in C(X)$, we denote $f^{-1}\{0\}$ and $X \setminus f^{-1}\{0\}$ by $Z(f)$ and $Coz(f)$, respectively. Every set of the form $Z(f)$ ($Coz(f)$) is called *zeroset* (*cozeroset*). A subset S of X is C-embedded in X if for every f in $C(S)$, there exists g in $C(X)$ such that $g|_S = f$. It is clear that every clopen subset of X is C-embedded in X. Suppose $p \in \beta X$, then by M^p we mean the set $\{f \in C(X) : p \in \mathrm{cl}_{\beta X} Z(f)\}$. By [18, Theorem 7.3 (Gelfand-Kolmogoroff)], $\{M^p : p \in \beta X\}$ is the family of all maximal ideal of $C(X)$. X is a *P-space* if every prime ideal of X is maximal and we say that X is an *almost P-space* if the interior of every nonempty zeroset of X is nonempty. It is easy to check that X is an almost P-space if and only if $C(X)$ is an almost regular ring. By [18, Theorem 14.28], X is a P-space if and only if every zeroset of X is open. For more details we refer the reader to [15], [18], [11] and [29].

The study of translating graph properties to algebraic properties is an interesting subject for mathematicians. In [14], linear algebra and some properties of polynomials were used to describe properties of graphs. In [13], the studying of zero-divisor graph of commutative rings has been started. The investigation on zero-divisor graph of commutative rings was then continued in [7], [10], [20], [25], [4], [9], [6], and [8].

In [27], comaximal graph of a commutative ring was defined. On later, in [21], [26], [16], [28], [24], [22], [19], [2], [3], [1], [30], and [23], this investigation was continued.

In [12] and in a section of [5] the zero-divisor graph and the comaximal ideal graph of $C(X)$ were studied, respectively. These investigations tried to associate the ring properties of $C(X)$, the graph properties of graphs on $C(X)$ and the topological properties of X.

In this article we study the $\Gamma'_2 C(X)$. Since $J(C(X)) = 0$, so $\Gamma'_2 C(X) = \Gamma_2(C(X)) - \{0\}$. If X is singleton, then $\Gamma'_2 C(X)$ is empty. Thus, subsequently we assume $|X| > 1$.

By [21, Theorem 3.1, Lemma 3.2 and Proposition 3.3] and [24, Corollary 3.4], we can conclude the following.

Proposition 1.1. *For each Tychonoff space X,*

(a) $\Gamma'_2 C(X)$ *is connected;*
(b) $\operatorname{diam}\ \Gamma'_2 C(X) = 3$*;*
(c) *if X is infinite, then* $\operatorname{girth}\ \Gamma'_2 C(X) = 3$.

Lemma 1.2. *Suppose $f, g \in \Gamma'_2 C(X)$. Then f is adjacent to g if and only if $Z(f) \cap Z(g) = \emptyset$.*

Proof: f is not adjacent to g if and only if both f and g are contained in a maximal ideal, that is

$$\begin{aligned} \exists p \in \beta X \quad f, g \in M^p &\Leftrightarrow \exists p \in \beta X \quad p \in \mathrm{cl}_{\beta X} Z(f) \ \wedge\ p \in \mathrm{cl}_{\beta X} Z(g) \\ &\Leftrightarrow \exists p \in \mathrm{cl}_{\beta X} Z(f) \cap \mathrm{cl}_{\beta X} Z(g) = \mathrm{cl}_{\beta X} \big(Z(f) \cap Z(g)\big) \\ &\Leftrightarrow Z(f) \cap Z(g) \neq \emptyset. \end{aligned}$$ □

In Section 2 we investigate the radius of $\Gamma'_2 C(X)$ and show that $2 \leq \operatorname{Rad} \Gamma'_2 C(X) \leq 3$. The girth of this graph is investigated in Section 3 and we show that if $|X| > 2$, then $\operatorname{girth}\ \Gamma'_2 C(X) = 3$. In Section 4 we study the dominating number and the clique number of the graph $\Gamma'_2 C(X)$. We prove that $d(X) \leq \operatorname{dt}\ \Gamma'_2 C(X) \leq w(X)$, introduce zeroset cellularity of X and show that it is equal to clique $\Gamma'_2 C(X)$. In Section 5 we use the notions of the previous sections to associate the topological properties of X, the ring properties $C(X)$ and the graph properties of $\Gamma'_2 C(X)$. In this section we observe that $\Gamma'_2 C(X)$ is triangulated (hypertriangulated, complemented) if and only if X does not have any isolated points (X is connected, X is a P-space), and finally we conclude that X is an almost P-space which does not have isolated points if and only if $C(X)$ is regular ring which does not have any principal maximal ideals if and only if $\operatorname{Rad} \Gamma'_2 C(X) = 3$.

Similar results to Theorem 4.4, Proposition 4.7 and Corollary 4.8 devoted to zero divisor graphs may be found in [12]. Here we prove them for comaximal graphs.

2. Radius of the graph

Lemma 2.1. *For any f and g in $\Gamma'_2 C(X)$*

(a) $d(f, g) = 1$ *if and only if* $Z(f) \cap Z(g) = \emptyset$*;*
(b) $d(f, g) = 2$ *if and only if* $Z(f) \cap Z(g) \neq \emptyset$ *and* $Z(f) \cup Z(g) \neq X$*;*
(c) $d(f, g) = 3$ *if and only if* $Z(f) \cap Z(g) \neq \emptyset$ *and* $Z(f) \cup Z(g) = X$.

Proof: (a) By Lemma 1.2, it is clear.

(b) $\Rightarrow$ Since $d(f,g)=2$, f is not adjacent to g, thus $Z(f)\cap Z(g)\neq\emptyset$, by Lemma 1.2. We now show that $Z(f)\cup Z(g)\neq X$. Suppose that, on the contrary $Z(f)\cup Z(g)=X$. From $d(f,g)=2$ it follows that h in $\Gamma'_2C(X)$ exists such that h is adjacent to both f and g, hence by Lemma 1.2,

$$\begin{cases} Z(h)\cap Z(f)=\emptyset \\ Z(h)\cap Z(g)=\emptyset \end{cases} \quad\Rightarrow\quad Z(h)=Z(h)\cap X=Z(h)\cap\big[Z(f)\cup Z(g)\big]=\emptyset$$

which is a contradiction.

(b) $\Leftarrow$ Since $Z(f)\cap Z(g)\neq\emptyset$, $d(f,g)>1$, by Lemma 1.2. Since $Z(fg)=Z(f)\cup Z(g)\neq X$, $p\in X\setminus Z(fg)$ exists, hence there is some h in $C(X)$ such that $p\in Z(h)$ and $Z(fg)\cap Z(h)=\emptyset$, thus $h\in\Gamma'_2C(X)$ and

$$\begin{aligned}\emptyset=\big[Z(f)\cup Z(g)\big]\cap Z(h)&=\big[Z(f)\cap Z(h)\big]\cup\big[Z(g)\cap Z(h)\big]\\ \Rightarrow\quad&\begin{cases} Z(f)\cap Z(h)=\emptyset\\ Z(g)\cap Z(h)=\emptyset.\end{cases}\end{aligned}$$

Hence h is adjacent to both f and g, thus $d(f,g)=2$.

(c) By Proposition 1.1(b), it is clear. $\square$

Lemma 2.2. *For every $f\in\Gamma'_2C(X)$, $\mathrm{ecc}(f)\geq 2$.*

Proof: Since $Z(f)\cap Z(2f)=Z(f)\neq\emptyset$ and $Z(f)\cup Z(2f)=Z(f)\neq X$, $d(f,2f)=2$, by Lemma 2.1. This implies that $\mathrm{ecc}(f)\geq 2$. $\square$

Proposition 2.3. *Suppose $f\in\Gamma'_2C(X)$. Then $\mathrm{ecc}(f)=2$ if and only if either $\mathrm{int}Z(f)=\emptyset$ or $Z(f)=\{p\}$, in which p is an isolated point.*

Proof: $\Rightarrow$ By Lemma 2.2, $d(f,g)\neq 3$, for every $g\in\Gamma'_2C(X)$. From Lemma 2.1, it follows that

$$\begin{aligned} &\forall g\in\Gamma'_2C(X)\quad Z(f)\cup Z(g)\neq X\ \vee\ Z(f)\cap Z(g)=\emptyset\\ \equiv\ &\forall g\in\Gamma'_2C(X)\quad Z(f)\cup Z(g)=X\ \Rightarrow\ Z(f)\cap Z(g)=\emptyset\\ \equiv\ &\forall g\in\Gamma'_2C(X)\quad Coz(g)\subseteq Z(f)\ \Rightarrow\ Z(f)=Coz(g)\quad (1)\end{aligned}$$

If $\mathrm{int}Z(f)\neq\emptyset$, then $Z(f)$ is open. It is sufficient to show that $Z(f)$ is singleton. Suppose, on the contrary, there are two distinct points p and q in $Z(f)$, thus there is a function $h: Z(f)\to\mathbb{R}$, such that $h(p)=0$ and $h(q)=1$. Since $Z(f)$ is clopen, $Z(f)$ is C-embedded in X, thus k in $C(X)$ exists such that $k|_{Z(f)}=h$.

Let $g: X\to\mathbb{R}$ be given by

$$g(x)=\begin{cases}1 & x\in Z(f)\\ 0 & x\notin Z(f).\end{cases}$$

Since $Z(f)$ is clopen, $g\in\Gamma'_2C(X)$ and therefore $gk\in\Gamma'_2C(X)$. $Coz(gk)=Coz(g)\cap Coz(k)\subseteq Z(f)$, but $Z(f)\neq Coz(gk)$ since $p\in Z(f)\setminus Coz(gk)$. This contradicts the fact (1).

$\Leftarrow$ By Proposition 1.1, it suffices to prove that

$$\forall g \in \Gamma'_2 C(X) \qquad d(f,g) \neq 3.$$

According to the first part of the proof, the above statement is equivalent to

$$\forall g \in \Gamma'_2 C(X) \quad Coz(g) \subseteq Z(f) \;\Rightarrow\; Z(f) = Coz(g).$$

By the assumption, the above statement is clear. □

An immediate conclusion of Proposition 1.1, and Lemma 2.2, is the following corollary.

Corollary 2.4. $2 \leq \operatorname{Rad} \Gamma'_2 C(X) \leq 3$.

3. Girth of the graph

Lemma 3.1. *Let $f \in \Gamma'_2 C(X)$. Then $Coz(f)$ is not singleton if and only if f is a vertex of a triangle.*

Proof: $\Rightarrow$ Let p and q be distinct elements of $Coz(f)$. There are two disjoint zerosets Z_1 and Z_2 containing p and q, respectively. Since $p, q \notin Z(f)$, there are two zerosets Z_3 and Z_4 containing p and q, respectively, such that $Z_3 \cap Z(f) = Z_4 \cap Z(f) = \emptyset$. Put $Z(g) = Z_3 \cap Z_1$ and $Z(h) = Z_4 \cap Z_2$. Consequently, $g, h \in \Gamma'_2 C(X)$, $Z(f) \cap Z(g) = \emptyset$, $Z(g) \cap Z(h) = \emptyset$ and $Z(h) \cap Z(f) = \emptyset$. Lemma 1.2 now shows that f is adjacent to g, g is adjacent to h and h is adjacent to f, thus f is vertex of a triangle.

$\Leftarrow$ There are vertices g and h in $\Gamma'_2 C(X)$ such that f is adjacent to g, g is adjacent to h and h is adjacent to f. By Lemma 1.2

$$\begin{cases} Z(f) \cap Z(g) = \emptyset \\ Z(f) \cap Z(h) = \emptyset \\ Z(g) \cap Z(h) = \emptyset \end{cases} \qquad \Rightarrow \qquad \begin{cases} \emptyset \neq Z(g) \subseteq Coz(f) \\ \emptyset \neq Z(h) \subseteq Coz(f) \\ Z(g) \cap Z(h) = \emptyset. \end{cases}$$

Hence $Coz(f)$ is not singleton. □

Theorem 3.2. *If* $|X| > 2$, *then* girth $\Gamma'_2 C(X) = 3$.

Proof: Since X has some non-singleton cozeroset, girth $\Gamma'_2 C(X) = 3$, by Lemma 3.1. □

Example 3.3. If $|X| > 2$ and finite, then $C(X)$ has finitely many maximal ideal and girth $\Gamma'_2 C(X) = 3$, by Theorem 3.2. This is a counterexample to the converse of [24, Corollary 3.4].

4. Dominating and clique number

Lemma 4.1. *Let $f, g \in \Gamma'_2 C(X)$.*

(a) *If $Z(f) \cap Z(g) = \emptyset$ and $Z(f) \cup Z(g) = X$, then* $\operatorname{gi}(f,g) = 4$.

(b) *If $Z(f) \cap Z(g) = \emptyset$ and $Z(f) \cup Z(g) \neq X$, then* $\operatorname{gi}(f,g) = 3$.

(c) *If* $Z(f) \cap Z(g) \neq \emptyset$ *and* $Z(f) \cup Z(g) \neq X$, *then* $\mathrm{gi}(f,g) = 4$.
(d) *If* $Z(f) \cap Z(g) \neq \emptyset$ *and* $Z(f) \cup Z(g) = X$, *then* $\mathrm{gi}(f,g) = 6$.

PROOF: (a) $Z(f) \cap Z(g) = \emptyset$, $Z(g) \cap Z(2f) = \emptyset$, $Z(2f) \cap Z(2g) = \emptyset$ and $Z(2g) \cap Z(f) = \emptyset$. By Lemma 1.2, f is adjacent to g, g is adjacent to $2f$, $2f$ is adjacent to $2g$ and $2g$ is adjacent to f, it follows that $\mathrm{gi}(f,g) \leq 4$. We claim that $\mathrm{gi}(f,g) \neq 3$ and therefore $\mathrm{gi}(f,g) = 4$. On the contrary, suppose $\mathrm{gi}(f,g) = 3$, then h in $\Gamma'_2C(X)$ exists such that h is adjacent to both f and g, by Lemma 1.2

$$\begin{cases} Z(h) \cap Z(f) = \emptyset \\ Z(h) \cap Z(g) = \emptyset \end{cases} \quad \Rightarrow \quad Z(h) = Z(h) \cap X = Z(h) \cap \big[Z(f) \cup Z(g)\big] = \emptyset$$

which is impossible.

(b) Suppose $x \in X \setminus [Z(f) \cup Z(g)]$. There is some h in $\Gamma'_2C(X)$ such that $x \in Z(h)$ and

$$Z(h) \cap \big[Z(f) \cup Z(g)\big] = \emptyset \quad \Rightarrow \quad Z(h) \cap Z(f) = \emptyset \text{ and } Z(h) \cap Z(g) = \emptyset$$

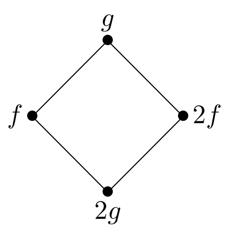
thus $h \in \Gamma'_2C(X)$ and h is adjacent to both f and g, by Lemma 1.2, hence $\mathrm{gi}(f,g) = 3$.

(c) Suppose $x \in X \setminus [Z(f) \cup Z(g)]$, then there is some $h \in \Gamma_2C(X)$, such that $x \in Z(h)$ and

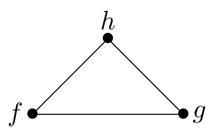
$$Z(h) \cap \big[Z(f) \cup Z(g)\big] = \emptyset \quad \Rightarrow \quad Z(h) \cap Z(f) = \emptyset \text{ and } Z(h) \cap Z(g) = \emptyset$$

thus $Z(f) \cap Z(2h) = \emptyset$ and $Z(2h) \cap Z(g) = \emptyset$. From Lemma 4.1, we deduce that f is adjacent to h, h is adjacent to g, g is adjacent to $2h$ and $2h$ is adjacent to f, this gives $\mathrm{gi}(f,g) \leq 4$. Since $Z(f) \cap Z(g) \neq \emptyset$, so f is not adjacent to g and therefore $\mathrm{gi}(f,g) \neq 3$, and so $\mathrm{gi}(f,g) = 4$.

(d) By Lemma 2.1, $d(f,g) = 3$, thus $\mathrm{gi}(f,g) \leq 6$ and there are h and k in $\Gamma'_2C(X)$ such that f is adjacent to h, h is adjacent to k and k is adjacent to g. It is easily seen that g is adjacent to $2k$, $2k$ is adjacent to $2h$ and $2h$ is adjacent to f. This clearly forces $\mathrm{gi}(f,g) = 6$. □



$Z(f) \cap Z(g) = \emptyset$
$Z(f) \cup Z(g) = X$
$\mathrm{gi}(f,g) = 4$



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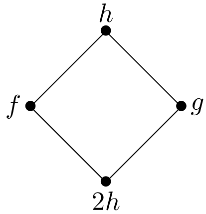

$Z(f) \cap Z(g) \neq \emptyset$
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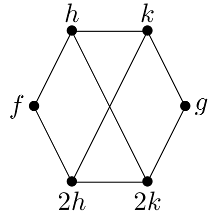
$\mathrm{gi}(f,g) = 4$



$Z(f) \cap Z(g) \neq \emptyset$
$Z(f) \cup Z(g) = X$
$\mathrm{gi}(f,g) = 6$

Corollary 4.2. (a) *Every cycle in $\Gamma'_2 C(X)$ has length 3 or 4.*
(b) *Every edge of $\Gamma'_2 C(X)$ is edge of a cycle with length 3 or 4.*
(c) *Every vertex of $\Gamma'_2 C(X)$ is vertex of a square.*

Proof: (a) and (b) are immediate conclusions of Lemma 4.1.
(c) For each $f \in \Gamma'_2 C(X)$, we have

$$\begin{cases} Z(f) \cap Z(2f) \neq \emptyset \\ Z(f) \cup Z(2f) \neq X. \end{cases}$$

By Lemma 4.1, $\mathrm{gi}(f, 2f) = 4$, and therefore f is vertex of a square. □

Lemma 4.3. *If X is an infinite space, then every dominating set of $\Gamma'_2 C(X)$ is infinite.*

Proof: We show that none of the finite subsets of $\Gamma'_2 C(X)$ is a dominating set. Suppose $A = \{f_1, f_2, \dots, f_n\}$ is a finite subset of $\Gamma'_2 C(X)$. Each $Z(f_i)$ is nonempty, thus p_i in $Z(f_i)$ exists. Since X is infinite, p_0 in X distinct from p_i's exists. Thus there are zerosets $Z_0, Z_1, \dots, Z_n$ in $\Gamma'_2 C(X)$ such that $p_i \in Z_i$, for every $0 \leq i \leq n$, and $i \neq j$ implies $Z_i \cap Z_j = \emptyset$. Set $Z(g) = Z_1 \cup Z_2 \cup \dots \cup Z_n$. Then $p_0 \notin Z(g) \neq X$ and $p_i \in Z(g) \cap Z(f_i) \neq \emptyset$, thus $g \in \Gamma'_2 C(X)$ and not adjacent to any f_i. This follows that A is not a dominating set. □

Theorem 4.4. *$d(X) \leq \mathrm{dt}\, \Gamma'_2 C(X) \leq w(X)$. In particular, whenever $d(X) = w(X)$, then $\mathrm{dt}\, \Gamma'_2 C(X) = w(X)$.*

Proof: If X is finite, then it is easy to check $d(X) = \mathrm{dt}\, \Gamma'_2 C(X) = w(X)$, thus we assume X is infinite. Let A be a dominating set in $\Gamma'_2 C(X)$. For each $f \in A$, we pick $x_f \in Z(f)$ and $y_f \in Coz(f)$. Set $D = \{x_f : f \in A\} \cup \{y_f : f \in A\}$. For every cozeroset $Coz(g)$, if $g \in A$, then $y_g \in D \cap Coz(g)$, if $g \notin A$, then $f \in A$ exists such that

$$Z(f) \cap Z(g) = \emptyset \quad \Rightarrow \quad Z(f) \subseteq Coz(g) \quad \Rightarrow \quad x_f \in Z(f) \cap D \subseteq Coz(g) \cap D.$$

Hence D is dense in X. Since D is infinite, $d(X) \leq \mathrm{dt}\, \Gamma'_2 C(X)$.

We now suppose that $\mathcal{B}$ is a base for X. Without loss of generality we can assume that $\mathcal{B}$ does not have any empty members. Then for every $B \in \mathcal{B}$, there is some f_B in $\Gamma'_2 C(X)$ such that $Z(f_B) \subseteq B$. For each f in $\Gamma'_2 C(X)$ there is some B in $\mathcal{B}$ such that

$$Z(f_B) \subseteq B \subseteq Coz(f) \quad \Rightarrow \quad Z(f_B) \cap Z(f) = \emptyset.$$

By Lemma 1.2, f is adjacent to f_B. Therefore $\{f_B : B \in \mathcal{B}\}$ is a dominating set and finally that $\operatorname{dt} \Gamma'_2 C(X) \leq w(X)$. $\square$

An immediate conclusion of the above theorem is the following corollary.

Corollary 4.5. *If X is an infinite second countable space, then* $\operatorname{dt} \Gamma'_2 C(X) = \omega$.

Example 4.6. Let X be Moore plane. For every $(x_\circ, y_\circ)$ in X, set $f_{x_\circ, y_\circ} : X \to \mathbb{R}$ as $f_{x_\circ, y_\circ}(x, y) = \sqrt{(x - x_\circ)^2 + (y - y_\circ)^2}$. It is clear that $f \in \Gamma'_2 C(X)$ and $Z(f_{x_\circ, y_\circ}) = \{(x_\circ, y_\circ)\}$. Suppose $A = \{f_{x,y} : x, y \in \mathbb{Q} \text{ and } y > 0\}$. If $Z(f) \cap Z(f_{x,y}) \neq \emptyset$, for each $f_{x,y} \in A$, then $\mathbb{Q} \times \mathbb{Q}^{>0} \subseteq Z(f)$ and therefore $XZ(f)$. This implies that A is a dominating set and therefore $\operatorname{dt} \Gamma'_2 C(X) = \omega \neq \mathbf{c} = \omega(X)$.

Proposition 4.7. *Suppose* $\Gamma C(X)$ *is the zero divisor graph of* $C(X)$*. If* $\chi(X) \leq d(X)$*, then* $\operatorname{dt} \Gamma'_2 C(X) = d(X) = \operatorname{dt} \Gamma C(X)$.

Proof: According to Theorem 4.4, we only need to show that $d(X) \geq \operatorname{dt} \Gamma'_2 C(X)$. Clearly, if X is finite, then $\operatorname{dt} \Gamma'_2 C(X) = d(X)$.

Now suppose X is infinite, then every dominating set is infinite, by Lemma 4.3. Let D be a dense subset of X and $\mathfrak{B}_x$ is a neighborhood base at x, for each x in D. For every $x \in D$ and $B \in \mathfrak{B}_x$, there is some $f_{x,B} \in \Gamma'_2 C(X)$ such that $x \in Z(f_{x,B}) \subseteq B$. Put $A = \{f_{x,B} : x \in D \text{ and } B \in \mathfrak{B}_x\}$. If $g \in \Gamma'_2 C(X)$, then $Coz(g) \neq \emptyset$, and it follows that $x \in D \cap Coz(g)$ exits. Hence there is a $B \in \mathfrak{B}_x$ such that $Z(f_{B,x}) \subseteq B \subseteq Coz(g)$, thus $Z(f_{B,x}) \cap Z(g) = \emptyset$, and, in consequence, $f_{B,x}$ is adjacent to g. This implies that A is dominating set. Since $|A| \leq \chi(X)|D| \leq d(X)d(X) = d(X)$, $d(X) \geq \operatorname{dt} \Gamma'_2 C(X)$. The equality $\operatorname{dt} \Gamma C(X) = d(X)$ was shown in [12, Proposition 3.4]. $\square$

By [17, Thorem 1.5.7], $w(X) \leq \exp d(X)$, hence the following corollary is immediate.

Corollary 4.8. $d(X) \leq \operatorname{dt} \Gamma'_2 C(X) \leq \exp d(X)$.

Definition 4.9. We define zero cellularity of X, denoted by $zc(X)$, by the supremum of $\{|\mathcal{Z}| : \mathcal{Z}$ is a family of pairwise disjoint nonempty zero subsets of $X\}$.

Theorem 4.10. *We have* clique $\Gamma'_2 C(X) = zc(X) \leq |X|$*. In particular if X is first countable, then* clique $\Gamma'_2 C(X) = zc(X) = |X|$.

Proof: By Lemma 1.2, $A \subseteq \Gamma'_2 C(X)$ is a complete subgraph of $\Gamma'_2 C(X)$ if and only if $Z(A) = \{Z(f) : f \in A\}$ is a family of pairwise disjoint zerosets, thus clique number of $\Gamma'_2 C(X)$ is the supremum of

$$\{|\mathcal{Z}| : \mathcal{Z} \text{ is a family of pairwise disjoint zero sets of } X\},$$

hence clique $\Gamma'_2 C(X) = zc(X)$. It is clear that $zc(X) \leq |X|$.

If X is first countable, then for every p in X, $\{p\}$ is a zeroset and thus $zc(X) = |X|$. $\square$

5. Some applications

Theorem 5.1. *$\Gamma_2' C(X)$ is triangulated if and only if X does not have any isolated points.*

Proof: $\Rightarrow$ If p is an isolated point of X, then there is some f in $\Gamma_2' C(X)$ such that $Coz(f) = \{p\}$, thus f is not a vertex of a triangle, by Lemma 3.1. Consequently, $\Gamma_2' C(X)$ is not triangulated.

$\Leftarrow$ Suppose $\Gamma_2' C(X)$ is not triangulated, hence there is some f in $\Gamma_2' C(X)$ such that f is not a vertex of any triangle. By Lemma 3.1, $Coz(f) = \{p\}$ is singleton, hence p is an isolated point of X. $\square$

Theorem 5.2. *$\Gamma_2' C(X)$ is hypertriangulated if and only if X is connected.*

Proof: $\Rightarrow$ If X is disconnected, then there are zerosets $Z(f)$ and $Z(g)$ such that $Z(f) \cap Z(g) = \emptyset$ and $Z(f) \cup Z(g) = X$. From Lemma 4.1, $\mathrm{gi}(f, g) = 4$, which yields $\{f, g\}$ is not edge of any triangle, and therefore $\Gamma_2' C(X)$ is not hypertriangulated.

$\Leftarrow$ Suppose $\Gamma_2' C(X)$ is not hypertriangulated. Then there is an edge $\{f, g\}$ of $\Gamma_2' C(X)$, which is not an edge of any triangle, thus $\mathrm{gi}(f, g) = 4$. Lemma 4.1 now shows that $Z(f) \cup Z(g) = X$ and $Z(f) \cap Z(g) = \emptyset$, this implies X is disconnected. $\square$

Theorem 5.3. *$\Gamma_2' C(X)$ is complemented if and only if X is a P-space.*

Proof: $\Rightarrow$ For every zeroset $Z(f)$ there is a zeroset $Z(g)$ such that $f \perp g$, thus $\mathrm{gi}(f, g) = 3$. We conclude from Lemma 4.1 that $Coz(f) \cap Coz(g) = \emptyset$ and $Coz(f) \cup Coz(g) = X$, therefore $Z(f)$ is open. This follows that X is a P-space.

$\Leftarrow$ For every vertex f in $\Gamma_2' C(X)$, $Z(f)$ is open, thus g in $\Gamma_2' C(X)$ exists such that $Z(f) \cap Z(g) = \emptyset$ and $Z(f) \cup Z(g) = X$. Now Lemma 4.1 becomes $\mathrm{gi}(f, g) = 3$, thus $f \perp g$ and consequently $\Gamma_2' C(X)$ is complemented. $\square$

Lemma 5.4. *Suppose M^p is a maximal ideal of $C(X)$, for some $p \in \beta X$. Then M^p is principal if and only if p is an isolated point.*

Proof: $\Rightarrow$ Let $M^p = \langle f \rangle$, for some $f \in C(X)$. $Z(f) = \{p\}$, since $\bigcap_{Z \in Z(M^p)} Z = Z(f)$. If p is not an isolated point, then $p \in \overline{Coz(f)}$, and therefore there is a net (x_λ) in $Coz(f)$, which converges to p. We conclude from $Z(f^{\frac{1}{3}}) = Z(f)$ that there is some g in $C(X)$ such that $f^{\frac{1}{3}} = gf$, hence that $g(x) = 1/f^{\frac{2}{3}}(x)$, for each x in $Coz(f)$, therefore $g(p) = \lim g(x_\lambda) = \infty$, which is a contradiction.

$\Leftarrow$ It is straightforward. $\square$

Proposition 5.5. *Let p be a G_δ-point of X. If $Z(f) = \{p\}$, then*

$$M^p = \{g \in \Gamma_2' C(X) : d(f, g) = 2\} \cup \{0, f\}.$$

Proof: Set $I = \{g \in \Gamma'_2 C(X) : d(f,g) = 2\} \cup \{0, f\}$. If $g \in M^p$, then $p \in Z(g)$, hence $Z(f) \cup Z(g) \neq X$ and $Z(f) \cap Z(g) \neq \emptyset$. Lemma 2.1 shows that $d(f,g) = 2$, and therefore $M^p \subseteq I \quad (1)$.

Now suppose $g \in I$. Since $d(f,g) = 2$, $\{p\} \cap Z(g) = Z(f) \cap Z(g) \neq \emptyset$. This implies $p \in Z(g)$, and thus $g \in M^p$. Hence $I \subseteq M^p \quad (2)$. By (1) and (2), $M^p = I$. □

Theorem 5.6. *The following are equivalent.*

(a) *X is an almost P-space which does not have any isolated points.*
(b) *$C(X)$ is almost regular ring which does not have any principal maximal ideals.*
(c) $\mathrm{Rad}\,\Gamma'_2 C(X) = 3$.
(d) *For each $f \in \Gamma'_2 C(X)$, there is some $g \in \Gamma'_2 C(X)$ such that* $\mathrm{gi}(f,g) = 6$.

Proof: (a) $\Leftrightarrow$ (b) By Lemma 5.4 it is obvious.

(a) $\Rightarrow$ (c) For each $f \in \Gamma'_2 C(X)$, $\mathrm{int}Z(f) \neq \emptyset$, thus $\mathrm{ecc}(f) = 3$. Hence $\mathrm{Rad}\,\Gamma'_2 C(X) = 3$.

(c) $\Rightarrow$ (a) Since $\mathrm{Rad}\,\Gamma'_2 C(X) = 3$, for every $f \in \Gamma'_2 C(X)$ we have $\mathrm{ecc}(f) = 3$. We conclude from Proposition 2.3, that $\mathrm{int}Z(f) \neq \emptyset$ and X does not have any isolated points, hence that X is an almost P-space without any isolated points.

(d) $\Rightarrow$ (b) For each $f \in \Gamma'_2 C(X)$, there is some $g \in \Gamma'_2 C(X)$ such that $\mathrm{gi}(f,g) = 6$, thus $Z(f) \cap Z(g) \neq \emptyset$ and $Z(f) \cup Z(g) = X$. We conclude that f is a zero divisor and consequently $C(X)$ is an almost regular ring.

(c) $\Rightarrow$ (d) Since $\mathrm{Rad}\,\Gamma'_2 C(X) = 3$, for each $f \in \Gamma'_2 C(X)$ we have $\mathrm{ecc}(f) = 3$, thus g in $\Gamma'_2 C(X)$ exists such that $d(f,g) = 3$ and therefore $\mathrm{gi}(f,g) = 6$. □

Proposition 5.7. *The following statements are equivalent.*

(a) *$C(X)$ is almost regular ring which has some principal maximal ideal.*
(b) *X is an almost P-space which has some isolated point.*
(c) $\mathrm{Rad}\,\Gamma'_2 C(X) = 2$ *and for each f in the center of $\Gamma'_2 C(X)$ there is $g \in \Gamma'_2 C(X)$ such that $\{f,g\}$ is an edge of a square.*

Proof: (a) $\Leftrightarrow$ (b) By Lemma 5.4, it is evident.

(b) $\Rightarrow$ (c) By Theorem 5.6, $\mathrm{Rad}\,\Gamma'_2 C(X) = 2$, thus for every f in center of $\Gamma'_2 C(X)$, $\mathrm{ecc}(f) = 2$. Theorem 5.4 shows that $Z(f) = \{p\}$, for some isolated point p. Hence $g \in \Gamma'_2 C(X)$ exists such that $Coz(g) = \{p\}$. Since $Z(f) \cap Z(g) = \emptyset$ and $Z(f) \cup Z(g) = X$, by Lemma 4.1, $\{f,g\}$ is an edge of a square.

(c) $\Rightarrow$ (b) If $Z(f) = \emptyset$, for some $f \in \Gamma'_2 C(X)$, then f belongs to the center of $\Gamma'_2 C(X)$. This implies that there is a $g \in \Gamma'_2 C(X)$ such that $\{f,g\}$ is an edge of some square, thus $Z(f)$ is open, by Lemma 4.1, a contradiction. Since $\mathrm{Rad}\,\Gamma'_2 C(X) = 2$, from Theorem 5.6, X has some isolated point. □

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