Some remarks on the interpolation spaces A^θ, A_θ

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Abstract. Let (A_0, A_1) be a regular interpolation couple. Under several different assumptions on a fixed A^β, we show that $A^\theta = A_\theta$ for every $\theta \in (0,1)$. We also deal with assumptions on $\overline{A}^\beta$, the closure of A^β in the dual of $(A_0^*, A_1^*)_\beta$.

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Introduction

We are looking for sufficient conditions on a regular interpolation couple (A_0, A_1) implying that $A^\theta = A_\theta$ for every $\theta \in (0,1)$. We already considered such questions in [Da1] and [Da2]. Unhappily, there was a mistake in a crucial lemma at the beginning of [Da2]. A corrected version of this paper was put on arXiv as [Da3]. The present paper uses the same machinery, which we essentially reproduce in part 2, with simplifications.

In the first part we recall the definitions and some known properties of A^θ and A_θ. In the second part, we collect results about the mapping $\tau \in \mathbb{R} \to g'(\theta + i\tau)$, where $g \in \mathcal{G}(A_0, A_1)$, and give in Theorem 5 a key abstract condition on a fixed A^β, stronger than $A^\beta = A_\beta$, implying that $A^\theta = A_\theta$ for every $\theta \in (0,1)$. We also define and study the maps $R^\theta : A^\theta \to [(A_0^*, A_1^*)_\theta]^*$.

In the third part we deduce that $A^\theta = A_\theta$ for every $\theta \in (0,1)$ under geometric conditions on a fixed A^β, or on $\overline{A}^\beta$, defined as the norm closure of $R^\beta(A^\beta)$ in the dual space of $(A_0^*, A_1^*)_\beta$.

1. Notation, definitions and properties of interpolation spaces

We denote by X^* the dual of a Banach space X, by $\mathcal{C}_0(\mathbb{R}, X)$ the space of X-valued continuous functions on $\mathbb{R}$ that tend to 0 at infinity. We denote by $\mathcal{B}_1(\mathbb{R}, \mathbb{C})$ the space of first Baire class functions $f : \mathbb{R} \to \mathbb{C}$. Let $\mathcal{B}$ be the σ-algebra of Borel subsets of $\mathbb{R}$, completed by sets with Lebesgue measure zero. An a.s. defined map $f : \mathbb{R} \to X$ is strongly measurable if there exists a sequence $(f_n)_n$ of finitely valued maps $f_n : \mathbb{R} \to X$ such that, for every open ball B in X and $n \in \mathbb{N}$, $f_n^{-1}(B) \in \mathcal{B}$, and a.s. $\|f - f_n\|_X \to_{n\to\infty} 0$.

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Let $S = \{z \in \mathbb{C} \mid 0 \le \mathrm{Re}(z) \le 1\}$ and S^0 its interior. Given a map $f : S \to X$, we denote by $f(\theta + i\cdot) : \mathbb{R} \to X$ the restriction of f to the line $\mathrm{Re}\, z = \theta$, $\theta \in [0,1]$ and by f_τ the translated map $f_\tau(z) = f(z + i\tau)$, $\tau \in \mathbb{R}$.

Let $\overline{C} = (C_0, C_1)$ be a complex interpolation couple in the sense of [BL]. We first recall the definition of the interpolation space C_θ, $\theta \in (0,1)$ [BL, Chapter 4]. Let $\mathcal{F}(\overline{C})$ be the space of functions f with values in $C_0 + C_1$, which are bounded and continuous on S, holomorphic on S^0, such that, for $j \in \{0,1\}$, the maps $f(j + i\cdot)$ lie in $\mathcal{C}_0(\mathbb{R}, C_j)$. We equip $\mathcal{F}(\overline{C})$ with the norm

$$\|f\|_{\mathcal{F}(\overline{C})} = \max\bigl[\sup_{\tau\in\mathbb{R}} \|f(i\tau)\|_{C_0}, \sup_{\tau\in\mathbb{R}} \|f(1+i\tau)\|_{C_1}\bigr].$$

The space $C_\theta = (C_0, C_1)_\theta = \bigl\{f(\theta) \mid f \in \mathcal{F}(\overline{C})\bigr\}$, $0 < \theta < 1$, is a Banach space [BL, Theorem 4.1.2] for the norm defined by

$$\|a\|_{C_\theta} = \inf\bigl\{\|f\|_{\mathcal{F}(\overline{C})} \mid f(\theta) = a\bigr\}.$$

We now recall the definition of the complex interpolation space C^θ [BL, Chapter 4]. Let $\mathcal{G}(\overline{C})$ be the space of functions g with values in $C_0 + C_1$, which are continuous on S, holomorphic on S^0, such that the map $z \to (1 + |z|)^{-1}\|g(z)\|_{C_0+C_1}$ is bounded on S (this condition will be denoted by (C)), such that $g(j + i\tau) - g(j + i\tau') \in C_j$ for every $\tau, \tau' \in \mathbb{R}$, $j \in \{0,1\}$, and such that the following quantity is finite:

$$\|g^\cdot\|_{Q\mathcal{G}(\overline{C})} = \max\left[\sup_{\tau\neq\tau'\in\mathbb{R}} \left\|\frac{g(i\tau) - g(i\tau')}{\tau - \tau'}\right\|_{C_0}, \sup_{\tau\neq\tau'\in\mathbb{R}} \left\|\frac{g(1+i\tau) - g(1+i\tau')}{\tau - \tau'}\right\|_{C_1}\right].$$

This defines a norm on the space $Q\mathcal{G}(\overline{C})$, quotient of $\mathcal{G}(\overline{C})$ by the subspace of constant functions with values in $C_0 + C_1$, and $Q\mathcal{G}(\overline{C})$ is complete with respect to this norm [BL, Lemma 4.1.3]. We recall [BL, proof of Lemma 4.1.3] that every $g \in \mathcal{G}(\overline{C})$ satisfies

$$\|g'(z)\|_{C_0+C_1} \le \|g^\cdot\|_{Q\mathcal{G}(\overline{C})}, \quad z \in S. \tag{1}$$

The space $C^\theta = \bigl\{a \in C_0 + C_1 \mid \exists\, g \in \mathcal{G}(\overline{C}), a = g'(\theta)\bigr\}$ is a Banach space [BL, Theorem 4.1.4] with respect to the norm defined by:

$$\|a\|_{C^\theta} = \inf\bigl\{\|g^\cdot\|_{Q\mathcal{G}(\overline{C})} \mid g'(\theta) = a\bigr\}.$$

By (1), the canonical map $C^\theta \to C_0 + C_1$ is a one to one contraction. By [B], C_θ is isometrically identified with a subspace of C^θ, and by [BL, Theorem 4.2.2], $C_0 \cap C_1$ is dense in C_θ, $0 < \theta < 1$.

Every function $f \in \mathcal{F}(\overline{C})$ admits an integral representation involving the harmonic measure

$$(2)\qquad f(z) = \int_{\mathbb{R}} f(it) Q_0(z,t)\,dt + \int_{\mathbb{R}} f(1+it) Q_1(z,t)\,dt, \quad z \in S^0,$$

where $t \to \frac{Q_0(z,t)}{1-\operatorname{Re} z}$ and $\frac{Q_1(z,t)}{\operatorname{Re} z}$, $z \in S^0$, $t \in \mathbb{R}$ are probability densities. By [BL, Lemma 4.3.2], every $f \in \mathcal{F}(\overline{C})$ satisfies

$$(3)\quad \|f(\theta)\|_{C_\theta} \le \Big(\int_{\mathbb{R}} \|f(it)\|_{C_0} \frac{Q_0(\theta,t)}{1-\theta}\,dt\Big)^{1-\theta} \Big(\int_{\mathbb{R}} \|f(1+it)\|_{C_1} \frac{Q_1(\theta,t)}{\theta}\,dt\Big)^{\theta}.$$

For $x \in C_0 \cap C_1$, taking $f = \varphi \otimes x$ for a suitable φ, (3) implies

$$(4)\qquad \|x\|_{C_\theta} \le \|x\|_{C_0}^{1-\theta} \|x\|_{C_1}^{\theta}.$$

Let $\overline{A} = (A_0, A_1)$ be an interpolation couple. If $A_0 \cap A_1$ is dense in A_0 and A_1, $\overline{A}$ is called a *regular* interpolation couple. Then we have [BL, Theorem 2.7.1]

$$(5)\qquad (A_0 \cap A_1)^* = A_0^* + A_1^*, \qquad A_0^* \cap A_1^* = (A_0 + A_1)^*$$

(in general, there is only a canonical contraction $A_0^* + A_1^* \to (A_0 \cap A_1)^*$). Moreover we may apply the reiteration theorem [BL, Theorem 4.6.1] and the dual of A_θ is the space $(A_0^*, A_1^*)^\theta$, $0 < \theta < 1$ [BL, Theorem 4.5.1].

When $\overline{A}$ is a regular interpolation couple, let B_j be the closure of $A_0^* \cap A_1^*$ in A_j^*, $j = 0,1$. It is clear that

$$(6)\qquad B_0 \cap B_1 = A_0^* \cap A_1^*$$

isometrically and the couple $\overline{B} = (B_0, B_1)$ is regular. By (5) and (6), isometrically,

$$(7)\qquad B_0^* + B_1^* = (B_0 \cap B_1)^* = (A_0^* \cap A_1^*)^* = (A_0 + A_1)^{**}.$$

By [BL, Theorem 4.2.2 b] we have isometrically, for $0 < \theta < 1$,

$$(8)\qquad B_\theta = (A_0^*, A_1^*)_\theta.$$

Since $\overline{B}$ is regular, for $0 < \theta < 1$,

$$(9)\qquad (B_\theta)^* = (B_0^*, B_1^*)^\theta.$$

We now define maps $\widetilde{\rho} : \mathcal{G}(A_0, A_1) \to \mathcal{G}(B_0^*, B_1^*)$ and $R : Q\mathcal{G}(A_0, A_1) \to Q\mathcal{G}(B_0^*, B_1^*)$. Let ρ be the canonical isometry $A_0 + A_1 \to (A_0 + A_1)^{**}$. By (7), ρ is also an isometry $A_0 + A_1 \to B_0^* + B_1^*$. Since A_j, $j \in \{0,1\}$, embeds in $A_0 + A_1$, for $a_j \in A_j$, $\rho(a_j)$ is well defined as a continuous linear form on $B_0 \cap B_1 = A_0^* \cap A_1^*$.

Let $i_j : B_j \to A_j^*$ be the canonical isometry and let $i_j^* : A_j^{**} \to B_j^*$ be the conjugate onto contraction (which is not one to one in general). Note that B_j^* embeds in $B_0^* + B_1^*$. If $a_j \in A_j$, $i_j^*(a_j) = \rho(a_j)$ is in $B_0^* + B_1^*$ (in particular i_j^* is one to one on A_j), hence ρ is also a one to one contraction $A_j \to B_j^*$. Consequently the map $g(z) \to \rho(g(z))$ defines a one to one map $\widetilde{\rho} : \mathcal{G}(A_0, A_1) \to \mathcal{G}(B_0^*, B_1^*)$ and a one to one contraction $R : Q\mathcal{G}(A_0, A_1) \to Q\mathcal{G}(B_0^*, B_1^*)$. We shall see in Lemma 6 below that R induces a one to one contraction $R^\theta : A^\theta \to (B_0^*, B_1^*)^\theta$, $0 < \theta < 1$.

2. Properties of $g'(\theta + i\cdot)$, $g \in \mathcal{G}(C_0, C_1)$; the map R^θ

We first collect some basic properties.

Lemma 1. *Let $\overline{C} = (C_0, C_1)$ be an interpolation couple.*

a) *Let $f \in \mathcal{F}(\overline{C})$. Then, for every $\theta \in (0,1)$, $\tau \in \mathbb{R}$, we have that $\|f(\theta + i\tau)\|_{C_\theta} \leq \|f\|_{\mathcal{F}(\overline{C})}$ and $f(\theta + i\cdot) : \mathbb{R} \to C_\theta$ is continuous.*

b) *If moreover $f(\beta + i\cdot)$ lies in $\mathcal{C}_0(\mathbb{R}, C_\beta)$ and $f(\gamma + i\cdot)$ in $\mathcal{C}_0(\mathbb{R}, C_\gamma)$ for some $\beta, \gamma \in [0,1]$, then the map $F : z \to f((\gamma - \beta)z + \beta)$ belongs to $\mathcal{F}(C_\beta, C_\gamma)$, with norm less than $\|f\|_{\mathcal{F}(\overline{C})}$.*

c) *Let $G \in \mathcal{G}(\overline{C})$ be such that $G(j + i\cdot)$ is valued in C_j, $j \in \{0,1\}$. Let $\delta \in (0,1]$. Then the map $f_\delta(z) = e^{\delta z^2} G(z)$, $z \in S$, lies in $\mathcal{F}(\overline{C})$. In particular, for every $\theta \in (0,1)$, $G(\theta + i\cdot) : \mathbb{R} \to C_\theta$ is continuous.*

PROOF: a) Since $\|f\|_{\mathcal{F}(\overline{C})} = \|f_\tau\|_{\mathcal{F}(\overline{C})}$ for every $\tau \in \mathbb{R}$, the first assertion follows from the definition of C_θ. By (3), for $\tau, \tau' \in \mathbb{R}$,

$$\|f_\tau(\theta) - f_{\tau'}(\theta)\|_{C_\theta} \leq \Big(\int_{\mathbb{R}} \|f_\tau(it) - f_{\tau'}(it)\|_{C_0} \frac{Q_0(\theta,t)}{1-\theta}\, dt\Big)^{1-\theta} (2\|f\|_{\mathcal{F}(\overline{C})})^\theta.$$

Since functions in $\mathcal{C}_0(\mathbb{R}, C_0)$ are uniformly continuous, this implies the (uniform) continuity of $f(\theta + i\cdot) : \mathbb{R} \to C_\theta$.

b) The function F has on S^0 the integral representation, with values in $C_0 + C_1$:

$$F(z) = \int_{\mathbb{R}} F(i\tau) Q_0(z,\tau)\, d\tau + \int_{\mathbb{R}} F(1 + i\tau) Q_1(z,\tau)\, d\tau. \tag{10}$$

Indeed, since $F(j + i\cdot)$ lies in $\mathcal{C}_0(\mathbb{R}, C_0 + C_1)$, the RHS of (10) is well defined, harmonic, bounded: $S^0 \to C_0 + C_1$ and extends as a continuous function: $S \to C_0 + C_1$ (by conformal mapping this follows from the well known analogous result on the unit disk). It coincides with F on the boundary of S, hence on S^0 since $F : S^0 \to C_0 + C_1$ is holomorphic (harmonic). Since $F(i\cdot)$ lies in $\mathcal{C}_0(\mathbb{R}, C_\beta)$ and $F(1 + i\cdot)$ in $\mathcal{C}_0(\mathbb{R}, C_\gamma)$, with norm less than $\|f\|_{\mathcal{F}(\overline{C})}$, the RHS of (10) lies in $C_\beta + C_\gamma$, with norm less than $\|f\|_{\mathcal{F}(\overline{C})}$ and, as before, extends as a bounded continuous function: $S \to C_\beta + C_\gamma$.

Let us verify that $F : S^0 \to C_\beta + C_\gamma$ is holomorphic. More generally, if a function $F : S^0 \to X$ is holomorphic, bounded by K as mapping: $S^0 \to Y$ where Y continuously embeds in X, then $F : S^0 \to Y$ is holomorphic. Indeed let $\overline{D}(z_0, r) \subset S^0$ be a closed disk, with $0 < r < 1$. Since F is holomorphic with values in X, we have $F(z) = \sum_{k\geq 0} c_k(z-z_0)^k$ in X for $z \in D(z_0, r)$. Since

$$\|c_k\|_Y = \Big\| \int_0^{2\pi} F(z_0 + re^{it}) e^{-ikt} \, \frac{dt}{2\pi} \Big\|_Y \leq K,$$

the series converges normally in Y on $\overline{D}(z_0, r)$, hence its sum $F : D(z_0, r) \to Y$ is holomorphic. Taking $Y = C_\beta + C_\gamma$, $X = C_0 + C_1$, $K = \|f\|_{\mathcal{F}(\overline{C})}$ ends the verification.

c) In order to show that f_δ lies in $\mathcal{F}(\overline{C})$ we only have to verify that $f_\delta(j+i\cdot)$ lies in $\mathcal{C}_0(\mathbb{R}, C_j)$, $j \in \{0,1\}$, and that $f_\delta : S \to C_0 + C_1$ is bounded. By assumption $G(j+i\cdot)$ is valued and Lipschitz in C_j, hence continuous: $\mathbb{R} \to C_j$. Moreover

$$\begin{aligned} \|f_\delta(j+i\tau)\|_{C_j} &\leq e^{1-\tau^2}(\|G(j+i\tau) - G(j)\|_{C_j} + \|G(j)\|_{C_j}) \\ &\leq e^{1-\tau^2}(|\tau| \|G^{\cdot}\|_{Q\mathcal{G}(\overline{C})} + \|G(j)\|_{C_j}), \end{aligned}$$

which proves the first assertion. Condition (C) gives the desired boundedness since, for $z = \theta + i\tau \in S$,

$$\|f_\delta(\theta + i\tau)\|_{C_0+C_1} \leq K(G) e^{1-\tau^2}(1 + \sqrt{1+\tau^2}).$$

By a), $f_\delta(\theta + i\cdot) : \mathbb{R} \to C_\theta$ is continuous, hence so is $G(\theta + i\cdot)$. □

Lemma 2. *Let $\overline{C} = (C_0, C_1)$ be an interpolation couple and let $g \in \mathcal{G}(\overline{C})$. Let $F_h(z) = \frac{1}{h}[g(z+ih) - g(z)]$, $z \in S^0$ and $h \neq 0$. Then, for every $0 < \theta < 1$, for every $\tau \in \mathbb{R}$,*

i) *in $C_0 + C_1$, one has that*

$$hF_h(\theta + i\tau) = g(\theta + i\tau + ih) - g(\theta + i\tau) = i \int_\tau^{\tau+h} g'(\theta + it) \, dt, \tag{11}$$

and letting n be in $\mathbb{N}^*$,

$$F_{\frac{1}{n}}(\theta + i\tau) \to_n ig'(\theta + i\tau). \tag{12}$$

ii) *$F_h(\theta + i\cdot) : \mathbb{R} \to C_\theta$ is continuous (hence (11) holds in C_θ) and is bounded by $\|g^{\cdot}\|_{Q\mathcal{G}(\overline{C})}$.*

iii) *$\|g'(\theta + i\tau)\|_{C^\theta} \leq \|g^{\cdot}\|_{Q\mathcal{G}(\overline{C})}$.*

Note that in general the map $g'(\theta + i\cdot) : \mathbb{R} \to C^\theta$ is not strongly measurable.

Proof: i) The function $g : S^0 \to C_0 + C_1$ is holomorphic, which implies (11) and the continuity of $t \to g'(\theta + it) : \mathbb{R} \to C_0 + C_1$, hence (12).

ii) The map F_h lies in $\mathcal{G}(\overline{C})$; on $\operatorname{Re} z = j$ its values in C_j are bounded by $\|g^{\cdot}\|_{Q\mathcal{G}(\overline{C})}$, $j \in \{0,1\}$. Lemma 1 c) applied to $G = F_h$ gives the first assertion.

Let $f_{h,\delta}(z) = e^{\delta z^2} F_h(z)$, $z \in S$, $\delta > 0$. By Lemma 1 c) again

$$
\begin{aligned}
\|F_h(\theta)\|_{C_\theta} &= \|e^{-\delta\theta^2} f_{h,\delta}(\theta)\|_{C_\theta} \leq \|f_{h,\delta}\|_{\mathcal{F}(\overline{C})} \\
&\leq \max\bigl(\sup_{\tau\in\mathbb{R}} \|F_h(it)\|_{C_0}, e^{\delta} \sup_{\tau\in\mathbb{R}} \|F_h(1+it)\|_{C_1}\bigr) \qquad (13)\\
&\leq e^{\delta} \|g^{\cdot}\|_{Q\mathcal{G}(\overline{C})}.
\end{aligned}
$$

Let $g_\tau(z) = g(z+i\tau)$, so that $\|g_\tau^{\cdot}\|_{Q\mathcal{G}(\overline{C})} = \|g^{\cdot}\|_{Q\mathcal{G}(\overline{C})}$, and $(g_\tau(z+ih)-g_\tau(z))/h = F_h(z+i\tau)$. By (13) applied to g_τ we get

$$\|F_h(\theta+i\tau)\|_{C_\theta} \leq e^{\delta} \|g^{\cdot}\|_{Q\mathcal{G}(\overline{C})}.$$

Taking $\delta \to 0$ ends the proof.

iii) Keeping the notation of ii), by definition,

$$\|g'(\theta+it)\|_{C^\theta} \leq \|g_t^{\cdot}\|_{Q\mathcal{G}(\overline{C})} = \|g^{\cdot}\|_{Q\mathcal{G}(\overline{C})}. \qquad \square$$

Lemma 3. *Let $\overline{A}$ be a regular interpolation couple.*

a) *Every x^* in the unit ball of $(A_\theta)^*$, $0 < \theta < 1$, is w^*-limit of a sequence in the unit ball of $(A_0^*, A_1^*)_\theta$.*

b) *Let $g \in \mathcal{G}(\overline{A})$ and assume that, for some $\beta \in (0,1)$, for every $t \in \mathbb{R}$, $g'(\beta+it) \in A_\beta$. Then, for every $x^* \in (A_\beta)^*$, $\langle g'(\beta+i\cdot), x^*\rangle$ lies in $\mathcal{B}_1(\mathbb{R},\mathbb{C})$. In particular the function $g'(\beta+i\cdot) : \mathbb{R} \to A_\beta$ is weakly measurable.*

Proof: a) Let x^* be in the open unit ball of $(A_\theta)^* = (A_0^*, A_1^*)^\theta$ and let $h \in \mathcal{G}(A_0^*, A_1^*)$ be such that $h'(\theta) = x^*$ and $\|h^{\cdot}\|_{Q\mathcal{G}(A_0^*,A_1^*)} \leq 1$. Let $H_{1/n}$ be associated to h as in Lemma 2. By Lemma 2 ii), i), the sequence $(H_{1/n}(\theta))_n$ lies in the closed unit ball of $(A_0^*, A_1^*)_\theta$, hence of $(A_0^*, A_1^*)^\theta$ and converges to $h'(\theta)$ in $A_0^* + A_1^*$, hence w^* on $A_0 \cap A_1$. Since $A_0 \cap A_1$ is dense in A_θ, $(H_{1/n}(\theta))_n$ converges w^* in $(A_\theta)^*$ to $h'(\theta) = x^*$.

b) The map $\phi_\beta = g'(\beta+i\cdot) : \mathbb{R} \to A_0 + A_1$ is continuous, bounded: $\mathbb{R} \to A^\beta$ by Lemma 2 iii), hence by assumption it is bounded: $\mathbb{R} \to A_\beta$. Hence $\langle \phi_\beta(.), a^*\rangle$ is continuous on $\mathbb{R}$ for every $a^* \in A_0^* \cap A_1^*$ and even for every $a^* \in (A_0^*, A_1^*)_\beta$, since $(A_0^*, A_1^*)_\beta$ is the closure of $A_0^* \cap A_1^*$ in $(A_\beta)^* = (A_0^*, A_1^*)^\beta$. Let x^* be in the open unit ball of $(A_\beta)^*$. By a) there exists a sequence $(b_n^*)_n$ in the unit ball of $(A_0^*, A_1^*)_\beta$ such that

$$\forall t \in \mathbb{R} \qquad \langle \phi_\beta(t), b_n^*\rangle \underset{n}{\to} \langle \phi_\beta(t), x^*\rangle .$$

The functions $\langle \phi_\beta(.), b_n^*\rangle$ are continuous and uniformly bounded on $\mathbb{R}$, hence $\langle \phi_\beta(.), x^*\rangle$ belongs to $\mathcal{B}_1(\mathbb{R},\mathbb{C})$. $\square$

Lemma 4. *Let $\overline{C}$ be an interpolation couple, $g \in \mathcal{G}(\overline{C})$, let $F_{\frac{1}{n}}$ be associated to g as in Lemma 2, $0 < \beta < 1$. Let us consider the following properties:*

a) *for almost every* τ *the sequence* $(F_{\frac{1}{n}}(\beta+i\tau))$ *converges in* C_β*,*
b) $g'(\beta+i\cdot):\mathbb{R}\to C_\beta$ *is strongly measurable,*
c) *there is a closed separable subspace* E *of* C_β *such that* $g'(\beta+it)\in E$ *for every* $t\in\mathbb{R}$.

Then b) $\Leftrightarrow$ *a). If* $\overline{C}$ *is a regular couple, then c)* $\Rightarrow$ *b).*

Let a'), b') be analogous to a), b) with C^β *instead of* C_β*. Then we have that b')* $\Leftrightarrow$ *b)* $\Leftrightarrow$ *a')* $\Leftrightarrow$ *a).*

Comments. We shall prove in Theorem 5 that a) implies c) if $\overline{C}$ is regular.

The sequence $(F_{\frac{1}{n}}(\beta+i\tau))$ always lies in C_β by Lemma 2 ii). Condition b) obviously implies that $g'(\beta+i\cdot)$ is a.s. valued in a closed separable subspace E of C_β, but b) $\Rightarrow$ c) is less obvious. In the proof of c) $\Rightarrow$ b) we actually use that $g'(\beta+it)\in C_\beta$ for every $t\in\mathbb{R}$ and $g'(\beta+i\cdot)$ is a.s. valued in a closed separable subspace of C_β. In the appendix we shall remove the regularity assumption in c) $\Rightarrow$ b) and the same proof will give c') $\Rightarrow$ b'), where in c') F is a closed subspace of C^β.

Proof: b) $\Rightarrow$ b') and a) $\Rightarrow$ a') are obvious.

b') $\Rightarrow$ a): By Lemma 2 iii), $\phi_\beta=g'(\beta+i\cdot)$ is uniformly bounded in C^β. Hence, by assumption, $\phi_\beta:\mathbb{R}\to C^\beta$ is locally Bochner integrable. By the Lebesgue differentiation theorem [DU, Chapter II, Theorem 9, p. 49] in C^β,

$$\lim_n\, n\int_\tau^{\tau+\frac{1}{n}}\phi_\beta(t)\,dt=\phi_\beta(\tau),\quad \text{a.s. in }\ \tau.$$

By Lemma 2 i) and ii), the integral lies in C_β for every τ and coincides with $-\frac{i}{n}F_{\frac{1}{n}}(\beta+i\tau)$. Since C_β is closed in C^β, the limit holds in C_β, implying a).

a) $\Rightarrow$ b): The a.s. limit coincides a.s. with $ig'(\beta+i\cdot)$ by (12). By Lemma 2 ii), $F_{\frac{1}{n}}(\beta+i\cdot):\mathbb{R}\to C_\beta$ is continuous, hence the a.s. limit is strongly measurable: $\mathbb{R}\to C_\beta$. The same argument shows that a') $\Rightarrow$ b').

c) $\Rightarrow$ b): By assumption and Lemma 3 the map $g'(\beta+i\cdot):\mathbb{R}\to C_\beta$ is weakly measurable and a.s. valued in a closed separable subspace of C_β. By Pettis' theorem [DU, Chapter II, p. 42] it is strongly measurable. $\square$

By the equivalence a) $\Leftrightarrow$ b) in Lemma 4, the next theorem was proved in [Da3], in a more intricate way. The proof below closely follows the proof of [BL, Lemma 4.3.3].

Theorem 5. *Let* $\beta\in(0,1)$*. Let* $\overline{A}$ *be a regular interpolation couple.*

a) *Let* $g\in\mathcal{G}(\overline{A})$*, let* $F_{\frac{1}{n}}$ *be associated to* g *as in Lemma 2. Assume that for almost every* τ*, the sequence* $(F_{\frac{1}{n}}(\beta+i\tau))_n$*, which is valued in* A_β *by Lemma 2 ii), converges in* A_β *(necessarily to* $ig'(\beta+i\tau)$ *by Lemma 2 i)). Then, for every* $\theta\in(0,1)$ *and every* $\tau\in\mathbb{R}$*, the sequence* $(F_{\frac{1}{n}}(\theta+i\tau))_n$ *converges in* A_θ *(necessarily to* $ig'(\theta+i\tau)$*, which thus lies in* A_θ*). Moreover* $g'(\theta+i\cdot)$ *is valued in a closed separable subspace of* A_θ*.*

b) *If the assumption of a) holds for every $g \in \mathcal{G}(\overline{A})$, then $A_\theta = A^\theta$ for every $\theta \in (0,1)$.*

Proof: a) By Lemma 2 ii), the sequence $(F_{\frac{1}{n}}(\beta + i\cdot))_n$ is uniformly bounded by $\|g^\cdot\|_{Q\mathcal{G}(\overline{A})}$ and it is continuous: $\mathbb{R} \to A_\beta$. Let $f_{\frac{1}{n}}(z) = e^{z^2} F_{\frac{1}{n}}(z)$. Then $f_{\frac{1}{n}}(\beta + i\cdot) = e^{(\beta+i\cdot)^2} F_{\frac{1}{n}}(\beta + i\cdot)$ lies in $\mathcal{C}_0(\mathbb{R}, A_\beta)$. Let $\gamma \in \{0,1\}$. By Lemma 1, $f_{\frac{1}{n}}((\gamma-\beta)z+\beta)$ lies in $\mathcal{F}(A_\beta, A_\gamma)$, with norm less than $e\|g^\cdot\|_{Q\mathcal{G}(\overline{A})}$. By (3) applied in $\mathcal{F}(A_\beta, A_\gamma)$, for $\eta \in (0,1)$,

$$\begin{aligned}
&\|(f_{\frac{1}{n}} - f_{\frac{1}{m}})((\gamma-\beta)\eta+\beta)\|_{(A_\beta,A_\gamma)_\eta} \\
&\qquad \le \Big(\int_{\mathbb{R}} \|(f_{\frac{1}{n}} - f_{\frac{1}{m}})((\gamma-\beta)it+\beta)\|_{A_\beta} \frac{Q_0(\eta,t)}{1-\eta}\,dt\Big)^{1-\eta} (2e\|g^\cdot\|_{Q\mathcal{G}(\overline{A})})^\eta.
\end{aligned}$$

By the assumption and Lebesgue's convergence theorem the above integral tends to 0 as $n, m \to \infty$, hence so does the LHS. Let $\theta = (1-\eta)\beta + \eta\gamma \in (\beta,\gamma)$ (so θ runs through $(0,\beta) \cup (\beta,1)$). By the reiteration theorem [BL, Theorem 4.6.1] $(A_\beta, A_\gamma)_\eta = A_\theta$, and the LHS is $e^{\theta^2}\|(F_{\frac{1}{n}} - F_{\frac{1}{m}})(\theta)\|_{A_\theta}$. Hence $(F_{\frac{1}{n}}(\theta))_n$ is a Cauchy sequence in A_θ, so it converges in A_θ, to $ig'(\theta)$ by Lemma 2 i). Applying this to g_τ, $\tau \in \mathbb{R}$, instead of g, one gets $F_{\frac{1}{n}}(\theta+i\tau) \to ig'(\theta+i\tau)$ in A_θ. In particular the assumption of a) also holds at θ instead of β. Since $F_{\frac{1}{n}}(\theta + i\cdot) : \mathbb{R} \to A_\theta$ is continuous by Lemma 2 ii), it takes values in a closed separable subspace E_n of A_θ and $g'(\theta+i\cdot)$ is valued in the (separable) closure of $\cup_n E_n$ in A_θ. This proves a) for $\theta \neq \beta$. Since the assumption of a) holds at θ, the conclusion also holds at β.

b) is obvious from a). □

Lemma 6. *Let $\overline{A}$ be a regular interpolation couple. Then the mapping $R : Q\mathcal{G}(A_0, A_1) \to Q\mathcal{G}(B_0^*, B_1^*)$ (defined in part 1) induces a one to one contraction $R^\theta : A^\theta \to (B_0^*, B_1^*)^\theta$, for $\theta \in (0,1)$.*

Proof: We identify A^θ and $(B_0^*, B_1^*)^\theta$ with quotients of

$$Q\mathcal{G}(A_0, A_1) \textit{ and } Q\mathcal{G}(B_0^*, B_1^*)$$

respectively. We define R^θ by $R^\theta(g'(\theta)) = (R(g^\cdot))'(\theta)$. Since R is a contraction: $Q\mathcal{G}(A_0, A_1) \to Q\mathcal{G}(B_0^*, B_1^*)$, R^θ is a contraction: $A^\theta \to (B_0^*, B_1^*)^\theta$. Let us verify that it is one to one. For $a \in A^\theta$ and $b \in B_0 \cap B_1 = A_0^* \cap A_1^* = (A_0 + A_1)^*$, we have

$$\langle R^\theta(a), b\rangle = \langle a, b\rangle\,.$$

If $R^\theta(a) = 0$ in $(B_0^*, B_1^*)^\theta = (B_\theta)^*$, then $\langle a, b\rangle = 0$ for every b as above, thus $a = 0$ in $A_0 + A_1$, hence in A^θ. □

We denote by $\overline{A}^\theta$ the norm closure of $R^\theta(A^\theta)$ in $(B_0^*, B_1^*)^\theta$. Note that $\overline{A}^\theta$ embeds in A_0+A_1 since A^θ does, and $(B_0^*, B_1^*)^\theta$ embeds in $B_0^*+B_1^* = (A_0+A_1)^{**}$. Thus $A_0^* \cap A_1^*$ is a subspace of $(\overline{A}^\theta)^*$.

Let $\sigma_\theta : \overline{A}^\theta \to (B_0^*, B_1^*)^\theta = (B_\theta)^*$ be the isometric inclusion map. Its adjoint is onto, i.e. $(\overline{A}^\theta)^* = \sigma_\theta^*[(B_\theta)^{**}]$. Let U, respectively U_0, be the unit balls of $(\overline{A}^\theta)^*$, respectively B_θ. Since $B_0 \cap B_1$ is dense in B_θ, it follows that $\sigma_\theta^*(U_0 \cap (B_0 \cap B_1))$ is w^*-dense in U. Since σ_θ^* coincides with the identity on $B_0 \cap B_1 = A_0^* \cap A_1^*$, we get that

$$U_0 \cap (A_0^* \cap A_1^*) \text{ is } w^* \text{ dense in } U \subset (\overline{A}^\theta)^*. \tag{14}$$

Lemma 7. *Let $\overline{A}$ be a regular interpolation couple. For every $\theta \in (0,1)$, $R^\theta : A_\theta \to (B_0^*, B_1^*)^\theta = [(A_0^*, A_1^*)_\theta]^*$ is an isometry. In particular A_θ is closed in $\overline{A}^\theta$.*

Proof: By Lemma 3 the unit ball of $(A_0^*, A_1^*)_\theta = B_\theta$ is w^*-dense in the unit ball of $(A_\theta)^*$. Hence, for $a \in A_0 \cap A_1$,

$$\|a\|_{A_\theta} = \sup\{|\langle a, b\rangle| \mid \|b\|_{B_\theta} \le 1\} = \|R^\theta(a)\|_{(B_\theta)^*}. \qquad \square$$

Comment. Though we shall not use it, note that by Lemma 7, B_θ may be isometrically identified with a (closed) subspace of $(\overline{A}^\theta)^*$, hence, with the notation of (14), $U_0 \cap (A_0^* \cap A_1^*) = U \cap (A_0^* \cap A_1^*)$. Indeed, for $b \in B_0 \cap B_1$, by (8) for the first equality and Lemma 7 for the first inequality,

$$\|b\|_{B_\theta} = \|b\|_{(A_\theta)^*} \le \|b\|_{(\overline{A}^\theta)^*} \le \|b\|_{(B_\theta)^{**}} = \|b\|_{B_\theta}.$$

Remark 8. Let $g \in \mathcal{G}(\overline{A})$ and let $F_{\frac{1}{n}}$ be associated to g as in Lemma 2. Then, for every $t \in \mathbb{R}$ and $b \in (A_0^*, A_1^*)_\theta = B_\theta$

$$\left\langle F_{\frac{1}{n}}(\theta + it), b\right\rangle \to_n i \left\langle R^\theta \circ g'(\theta + it), b\right\rangle. \tag{15}$$

In particular the RHS of (15) lies in $\mathcal{B}_1(\mathbb{R}, \mathbb{C})$.

Indeed, by (12), (15) holds for every $t \in \mathbb{R}$, $a^* \in A_0^* \cap A_1^*$. By Lemma 2 ii) and Lemma 7, $\|F_{\frac{1}{n}}(\theta + it)\|_{(B_\theta)^*} \le \|g^\cdot\|_{Q\mathcal{G}(\overline{C})}$. By Lemma 6 and Lemma 2 iii)

$$\|R^\theta \circ g'(\theta + it)\|_{(B_\theta)^*} \le \|g'(\theta + it)\|_{A^\theta} \le \|g^\cdot\|_{Q\mathcal{G}(\overline{C})}.$$

Then a 3ε argument proves the first claim since $A_0^* \cap A_1^*$ is norm dense in B_θ. Lemma 2 ii) proves the second claim. $\square$

Lemma 9. *Let $\overline{A}$ be a regular interpolation couple and let $g \in \mathcal{G}(\overline{A})$. If, for some β, $R^\beta \circ \phi_\beta = R^\beta \circ g'(\beta + i\cdot) : \mathbb{R} \to \overline{A}^\beta$ is strongly measurable: , then $\phi_\beta : \mathbb{R} \to A_\beta$ is strongly measurable.*

Proof: It is similar to the proof of b') $\Rightarrow$ a) in Lemma 4, replacing A^β by $\overline{A}^\beta$, since A_β is closed in $\overline{A}^\beta$ by Lemma 7. $\square$

The following lemma completes Lemma 9.

Lemma 10. a) *Let $\varphi : \mathbb{R} \to X^*$ be a strongly measurable function such that for every $x \in X$, $\langle \varphi(.), x\rangle = 0$ a.s.. Then $\varphi = 0$ a.s..*

b) *In particular, let $\varphi : \mathbb{R} \to \overline{A}^\beta$ be a strongly measurable function and $g \in \mathcal{G}(\overline{A})$. Then $R^\beta \circ \phi_\beta = \varphi$ a.s. as soon as, for every $a^* \in A_0^* \cap A_1^*$, $\langle \varphi(.), a^*\rangle = \left\langle R^\beta \circ \phi_\beta(.), a^*\right\rangle$ a.s.*

Proof: a) Since φ is strongly measurable, φ is a.s. valued in a closed separable subspace $E \subset X^*$. Then the closed unit ball of $E^* = X^{**}/E^\perp$, being compact and metrizable for its w^*-topology, is separable for this topology. Hence there exists a countable set (x_k) in the unit ball of X whose image is w^*-dense in X^*. By assumption, a.s. in t, $\langle \varphi(t), x_k\rangle = 0$ for every k. For such a t, $\varphi(t) = 0$.

b) Since R^β and the canonical map $(B_0^*, B_1^*)^\beta \to B_0^* + B_1^*$ are one to one, it is enough to show that $R^\beta \circ \phi_\beta = \varphi$ a.s. as functions with values in $B_0^* + B_1^*$. Note that $R^\beta \circ \phi_\beta = \phi_\beta$ is continuous: $\mathbb{R} \to B_0^* + B_1^* = (B_0 \cap B_1)^* = (A_0 + A_1)^{**}$ (see (7)). The claim follows from the assumption and from a) applied to $X = B_0 \cap B_1 = A_0^* \cap A_1^*$ and $R^\beta \circ \phi_\beta - \varphi$. □

3. Conditions implying $A^\theta = A_\theta$ for every θ

Proposition 11. *Let $\overline{A}$ be a regular interpolation couple. Assume that A_β has the Radon-Nikodym property* [DU] *for some $0 < \beta < 1$. Then $A^\theta = A_\theta$ for every $0 < \theta < 1$.*

Proof: Since A_β has the Radon-Nikodym property, Lipschitz maps: $\mathbb{R} \to A_\beta$ are a.s. differentiable [DU, Chapter IV, Theorem 2, p. 107]. Actually, the proof does not use the fact that the Lipschitz map f under consideration is valued in a Radon-Nikodym space, but only that the differences $f(b) - f(a)$ are, for every $a, b \in \mathbb{R}$. So, for $g \in \mathcal{G}(\overline{A})$, by Lemma 2 ii), we may apply this result to $g(\beta + i\cdot)$: it is a.s. differentiable: $\mathbb{R} \to A_\beta$. The conclusion follows from Theorem 5. □

Comment. Actually, for any interpolation couple $\overline{C}$ and $g \in \mathcal{G}(\overline{C})$, there exists $c \in C_0 + C_1$ such that $g(j + it) + c$ lies in C_j, $j \in \{0, 1\}$, $t \in \mathbb{R}$, which, by Lemma 1 c), implies that $(g + c)(\theta + i\cdot)$ is valued in C_θ. Indeed, let $g(1) - g(0) = c_0 + c_1$, where $c_j \in C_j$ and where $\|c_0\|_{C_0} + \|c_1\|_{A_1} \le \|g(1) - g(0)\|_{C_0+C_1} + \|g^\cdot\|_{Q\mathcal{G}(\overline{C})}$. By (1), $\|g(1) - g(0)\|_{C_0+C_1} \le \|g^\cdot\|_{Q\mathcal{G}(\overline{C})}$, so that $\|c_0\|_{C_0} + \|c_1\|_{C_1} \le 2\|g^\cdot\|_{Q\mathcal{G}(\overline{C})}$, and we then let

$$c = -g(0) - c_0 = c_1 - g(1).$$

Theorem 12. *Let $\overline{A}$ be a regular interpolation couple. Assume that, for some $\beta \in (0, 1)$,*

1) *A_β is weakly sequentially complete,*
2) *$(A_0^*, A_1^*)^\beta = (A_0^*, A_1^*)_\beta$.*

Then $A^\theta = A_\theta$, for every $\theta \in (0, 1)$.

Proof: Let $g \in \mathcal{G}(\overline{A})$. We claim that $g'(\beta + i\cdot)$ is valued in a closed separable subspace of A_β. Indeed by Lemma 2 ii), the associated function $F_{1/n}(\beta + i\cdot) : \mathbb{R} \to$

A_β is bounded and continuous, hence valued in a separable subspace E_n of A_β. By Remark 8, for every $t \in \mathbb{R}$ and $a^* \in (A_0^*, A_1^*)_\beta$, the sequence $((F_{1/n}(\beta + it), a^*))_n$ is Cauchy. By assumption 2), $(A_0^*, A_1^*)_\beta = (A_\beta)^*$. So, for every $t \in \mathbb{R}$, $(F_{1/n}(\beta + it))_n$ is weak Cauchy in A_β, hence in E, the norm closure of $\cup_n E_n$ in A_β. By assumption 1) it converges weakly in E. Since the canonical map $A_\beta \to A_0 + A_1$ is one to one, the limit point is $ig'(\beta + it)$, which thus lies in the separable space E. Then Lemma 4, c) $\Rightarrow$ a) and Theorem 5 end the proof. $\square$

In [Da1] we showed that if A^β is a weakly compactly generated Banach space (in short WCG, see [DU, Chapter VIII, p. 251]) for some $\beta \in (0,1)$, then $A^\theta = A_\theta$, for every $\theta \in (0,1)$. The next theorem weakens the assumption. Two properties of a WCG space X will be used:

(P_1) if a convex set Z is w^*-dense in the unit ball B_{X^*}, then every $x^* \in B_{X^*}$ is the w^*-limit of a sequence in Z (see e.g. [FHHMZ]),

(P_2) if $\phi : \mathbb{R} \to X$ is a weakly measurable function, then there exists a strongly measurable function $\varphi : \mathbb{R} \to X$ such that, for every $a^* \in X^*$, $\langle \phi(.), a^* \rangle = \langle \varphi(.), a^* \rangle$ a.s. [DU, p. 642].

For the convenience of the reader we give a direct proof of (P_1): Since X is WCG, there exists, by the Davis–Figiel–Johnson–Pelczynski theorem (see e.g. [FHHMZ, Corollary 13.24]), a reflexive space E and an injection with dense range $J : E \to X$. Let x^* be in the unit ball of X^*. By assumption there is a net (z_α) in Z such that $z_\alpha \to x^*$ in the w^*-topology of X^*. Then $J^*(z_\alpha) \to J^*(x^*)$ weakly in E. So there is a sequence (y_n) in Z such that $J^*(y_n) \to_{n\to\infty} J^*(x^*)$ in the norm of E^*. Then $y_n \to_{n\to\infty} x^*$ in the w^*-topology of X^* because $J(E)$ is dense in X.

Theorem 13. *Let $\overline{A}$ be a regular couple and let $\beta \in (0,1)$. Assume that $\overline{A}^\beta$ is WCG. Then $A^\theta = A_\theta$ for every $\theta \in (0,1)$.*

The proof needs the following lemma:

Lemma 14. *Let $\overline{A}$ be a regular couple, let $\beta \in (0,1)$ and assume that $\overline{A}^\beta$ is WCG. Let $g \in \mathcal{G}(\overline{A})$. Then the map $R^\beta \circ g'(\beta + i\cdot) = R^\beta \circ \phi_\beta : \mathbb{R} \to \overline{A}^\beta$ is strongly measurable. Moreover, for every $x^* \in (\overline{A}^\beta)^*$, $\left\langle R^\beta \circ \phi_\beta(.), x^* \right\rangle$ lies in $\mathcal{B}_1(\mathbb{R}, \mathbb{C})$.*

Proof: By assumption $\overline{A}^\beta$ satisfies (P_1) and (P_2). We first claim that $R^\beta \circ \phi_\beta : \mathbb{R} \to \overline{A}^\beta$ is weakly measurable. Let U be the closed unit ball of $(\overline{A}^\beta)^*$ and U_0 be the closed unit ball of B_β. Let $Z = U_0 \cap (A_0^* \cap A_1^*)$. By (14), Z is w^*-dense in U. Since $g'(\beta + i\cdot)$ is continuous: $\mathbb{R} \to A_0 + A_1$, for every $a^* \in A_0^* \cap A_1^* = B_0 \cap B_1$, $\left\langle R^\beta \circ \phi_\beta(.), a^* \right\rangle = \langle \phi_\beta(.), a^* \rangle$ is continuous. By (P_1), every $x^* \in U$ is the w^*-limit of a sequence in Z, hence $\left\langle R^\beta \circ \phi_\beta(.), x^* \right\rangle$ is in $\mathcal{B}_1(\mathbb{R}, \mathbb{C})$, which proves the claim and the last assertion of the lemma.

So, by (P_2), there exists a strongly measurable function $\varphi : \mathbb{R} \to \overline{A}^\beta$ such that, for every $x^* \in (\overline{A}^\beta)^*$, $\left\langle R^\beta \circ \phi_\beta(.), x^* \right\rangle = \langle \varphi(.), a^* \rangle$ a.s. In particular this holds for every $a^* \in B_0 \cap B_1 = A_0^* \cap A_1^*$. By Lemma 10 b), $R^\beta \circ \phi_\beta = \varphi$ a.s., which ends the proof. □

Proof of Theorem 13: Let $g \in \mathcal{G}(\overline{A})$. By Lemma 14 and Lemma 9, $g'(\beta + i\cdot) : \mathbb{R} \to A_\beta$ is strongly measurable. Lemma 4, b) $\Rightarrow$ a) and Theorem 5 end the proof. □

Definition 15. A Banach space X is weakly Lindelöf if every weakly open covering of X has a countable subcovering.

For example a WCG space is weakly Lindelöf [FHHMZ, Theorem 14.31]. We shall only use the fact that weakly Lindelöf spaces have Property (P_2) [E, Proposition 5.4 and (4), p. 671].

Proposition 16. *Let $\overline{A}$ be a regular couple. Assume that $A^\beta = A_\beta$ and that A_β is weakly Lindelöf for some $\beta \in (0,1)$. Then $A^\theta = A_\theta$ for every $\theta \in (0,1)$.*

Proof: The second assumption implies (P_2). Let $g \in \mathcal{G}(\overline{A})$. By the first assumption and Lemma 3 b), $\phi_\beta = g'(\beta + i\cdot) : \mathbb{R} \to A_\beta$ is weakly measurable. So, by (P_2), there exists a strongly measurable function $\varphi : \mathbb{R} \to A_\beta$ such that, for every $x^* \in (A_\beta)^*$, $\langle \phi_\beta(.), x^* \rangle = \langle \varphi(.), x^* \rangle$ a.s. This holds in particular for every $a^* \in A_0^* \cap A_1^* = (A_0 + A_1)^*$. By Lemma 7, $A_\beta = A^\beta$ implies $A_\beta = \overline{A}_\beta$. So, by Lemma 10, $\phi_\beta = \varphi$ a.s., i.e. $\phi_\beta : \mathbb{R} \to A_\beta$ is strongly measurable. Lemma 4, b) $\Rightarrow$ a) and Theorem 5 end the proof. □

The next theorem extends Proposition 16.

Theorem 17. *Let $\overline{A}$ be a regular couple such that A_β is weakly Lindelöf for some $\beta \in (0,1)$. Assume that*

1) *there exists a continuous projection $P : \overline{A}^\beta \to A_\beta$,*
2) *for every $g \in \mathcal{G}(\overline{A})$ and $y^* \in (\overline{A}^\beta)^*$, the map $\langle R^\beta \circ g'(\beta + i\cdot), y^* \rangle$ lies in $\mathcal{B}_1(\mathbb{R}, \mathbb{C})$.*

Then $A_\theta = A^\theta$ for every $\theta \in (0,1)$.

Comment. Assumption 1) is consistent by Lemma 7. The conclusion of 2) is always true for $y^* \in (A_0^*, A_1^*)_\beta$ by Remark 8. By the proof of Lemma 14, assumption 2) is verified if $(\overline{A}^\beta)^*$ satisfies (P_1).

Remark 18. Assume that A_β is a weakly Lindelöf space. Then assumptions 1) and 2) in Theorem 17 are equivalent to $A^\beta = A_\beta$.

Indeed Theorem 17 gives one implication. Conversely, if $A^\beta = A_\beta$, then $\overline{A}^\beta = A_\beta$ by Lemma 7, and 2) follows from Lemma 3 b).

Proof of Theorem 17: Let $g \in \mathcal{G}(\overline{A})$ and let us denote $g'(\beta + i\cdot) = \phi_\beta$.

Step 1: By both assumptions $P \circ R^\beta \circ \phi_\beta(.) : \mathbb{R} \to A_\beta$ is weakly measurable. Since A_β is weakly Lindelöf, there exists by (P_2) a strongly measurable function

$\varphi : \mathbb{R} \to A_\beta$ such that

$$\forall\, x^* \in (A_\beta)^* \quad \left\langle P[R^\beta \circ \phi_\beta(.)], x^* \right\rangle = \langle \varphi(.), x^* \rangle \qquad \text{a.s..} \tag{16}$$

We shall apply this only to $x^* = a^* \in A_0^* \cap A_1^*$. Note that $a^* \in (\overline{A}^\beta)^*$ (see (14)), but we do not know a priori whether $P^* a^* = a^*$. If we get

$$\forall\, a^* \in A_0^* \cap A_1^* = B_0 \cap B_1 \qquad \langle \phi_\beta(.), a^* \rangle = \langle \varphi(.), a^* \rangle \quad \text{a.s.,} \tag{17}$$

Lemma 10 implies $R^\beta \circ \phi_\beta = \varphi$ a.s., i.e. $\phi_\beta : \mathbb{R} \to A_\beta$ is strongly measurable. Then Lemma 4, b) $\Rightarrow$ a) and Theorem 5 will end the proof.

Step 2: We now show that (16) implies (17). Let y^* be in the unit ball U of $(\overline{A}^\beta)^*$. By (14) there is a net $(a_\alpha^*)_\alpha$ in $U_0 \cap (A_0^* \cap A_1^*)$ such that $a_\alpha^* \to y^*$ in the w^*-topology of $(\overline{A}^\beta)^*$. Let $F_{\frac{1}{n}}(\beta + i\cdot)$ be associated to g as in Lemma 2 (and valued in A_β). By (11), for every $\tau \in \mathbb{R}$ and every integer n,

$$\int_\tau^{\tau+1/n} \langle \phi_\beta(t), a_\alpha^* \rangle \, dt = -\frac{i}{n} \left\langle F_{\frac{1}{n}}(\beta + i\tau), a_\alpha^* \right\rangle \to_\alpha -\frac{i}{n} \left\langle F_{\frac{1}{n}}(\beta + i\tau), y^* \right\rangle. \tag{18}$$

We shall prove in Step 3 that, for every τ, n, and $y^* \in (\overline{A}^\beta)^*$,

$$\int_\tau^{\tau+1/n} \left\langle \phi_\beta(t), a_\alpha^* \right\rangle dt \to_\alpha \int_\tau^{\tau+1/n} \left\langle R^\beta \circ \phi_\beta(t), y^* \right\rangle \, dt. \tag{19}$$

Note that $R^\beta \circ \phi_\beta(.)$ is bounded in $\overline{A}^\beta$ by Lemma 2 iii), weakly measurable by assumption 2, hence $\left\langle R^\beta \circ \phi_\beta(.), y^* \right\rangle$ is locally integrable). By (18) and (19),

$$\int_\tau^{\tau+1/n} \left\langle R^\beta \circ \phi_\beta(t), y^* \right\rangle \, dt = -\frac{i}{n} \left\langle F_{\frac{1}{n}}(\beta + i\tau), y^* \right\rangle. \tag{20}$$

By (16) and (20) applied to $y^* = P^* a^*$, for $a^* \in A_0^* \cap A_1^*$,

$$\begin{aligned} in \int_\tau^{\tau+1/n} \langle \varphi(t), a^* \rangle \, dt &= in \int_\tau^{\tau+1/n} \left\langle R^\beta \circ \phi_\beta(t), P^* a^* \right\rangle \, dt \\ &= \left\langle F_{\frac{1}{n}}(\beta + i\tau), P^* a^* \right\rangle = \left\langle F_{\frac{1}{n}}(\beta + i\tau), a^* \right\rangle. \end{aligned}$$

Note that $\langle \varphi(t), a^* \rangle$ is locally integrable since $\left\langle R^\beta \circ \phi_\beta(t), P^* a^* \right\rangle$ is. Taking limits when $n \to \infty$ (by Lebesgue's differentiation theorem on the LHS, by (12) on the RHS), we get (17), as desired.

Step 3: We prove the claim (19). Let U, U_0 be respectively the closed unit balls of $(\overline{A}^\beta)^*$ and $(A_0^*, A_1^*)_\beta$. By (14), $U_0 \cap (A_0^* \cap A_1^*)$ is w^*-dense in U. The map $y^* \to \left\langle R^\beta \circ \phi_\beta(.), y^* \right\rangle$ is continuous from (U, w^*) into the space of complex valued functions on $\mathbb{R}$ equipped with the topology of pointwise convergence. The image K of U is compact for this topology and the image K_0 of $U_0 \cap (A_0^* \cap A_1^*)$

is dense in K. Moreover K is bounded in $\ell^\infty(\mathbb{R})$ (see Step 2). By assumption 2), K actually lies in $\mathcal{B}_1(\mathbb{R},\mathbb{C})$. Hence (19) follows from [R, Main Theorem b)]. $\square$

Our last result does not deal with the equality between A_θ and A^θ, but uses some of the machinery from part 2.

Proposition 19. *Let (A_0, A_1) be a regular couple such that A_0 is a subspace of A_1, and let $0 < \theta < \beta < 1$. Assume that the embedding $i : A_0 \to A_1$ is compact. Then i extends as a compact embedding $A_\theta \to A_\beta$.*

Proof: *Step* 1: Since $A_0 = A_0 \cap A_1$ and $A_1 = A_0 + A_1$ we know that i factors through A_β. We claim that the embedding $i_\beta : A_0 \to A_\beta$ is compact. Indeed let $(x_n)_{n\geq 0}$ be a bounded sequence in A_0. Since $i : A_0 \to A_1$ is compact, there exists a subsequence $(x_{n_k})_{k\geq 0}$ such that $i(x_{n_k})$ has a limit in A_1, hence $(x_{n_k})_{k\geq 0}$ is a Cauchy sequence in A_1. By (4), for every $k, k' \in \mathbb{N}$, we have

$$\|x_{n_k} - x_{n_{k'}}\|_{A_\beta} \leq \|x_{n_k} - x_{n_{k'}}\|_{A_0}^{1-\beta} \|x_{n_k} - x_{n_{k'}}\|_{A_1}^{\beta},$$

so that the sequence $(i(x_{n_k}))_{k\geq 0}$ is Cauchy in A_β. (This step does not need the regularity of the couple (A_0, A_1)).

Step 2: By assumption A_0 is dense in A_1 and in A_β. Hence $i^* : A_1^* \to A_0^*$ is an injection which factors through $(A_\beta)^*$. Let B_j be the closure of $A_0^* \cap A_1^* = A_1^*$ in A_j^*, so that $i^* : B_1 = A_1^* \to B_0$. By the regularity of (A_0, A_1) and by Step 1, $i_\beta^* : (A_\beta)^* = (A_0^*, A_1^*)^\beta \to A_0^*$ is a compact embedding. Hence so is its restriction $(A_0^*, A_1^*)_\beta = B_\beta \to A_0^*$, which is actually an embedding $B_\beta \to B_0$.

Applying Step 1 to the regular couple (B_β, B_0), we get a compact embedding with dense range $j : B_\beta \to (B_\beta, B_0)_\eta$, $\eta \in (0,1)$. By [BL, Theorem 4.2.1] and the reiteration theorem [BL, Theorem 2.7.1], $(B_\beta, B_0)_\eta = (B_0, B_\beta)_{1-\eta} = B_\theta$ if $\theta = (1-\eta)\beta$.

Hence the adjoint $j^* : B_\theta^* \to B_\beta^*$ is a compact embedding. By Lemma 7, A_θ and A_β are respectively isometric subspaces of B_θ^* and B_β^*. The restriction of j^* to A_θ is a compact embedding which is identity on A_0, hence sends A_θ into A_β and coincides with i_β on A_0. $\square$

Appendix: We give a variant of Lemma 4, which does not need regularity for c) $\Rightarrow$ b) and proves c') $\Rightarrow$ b'). Lemma 3 is replaced by the following:

Lemma 20. *Let F be a separable Banach space which is a* (*non closed in general*) *subspace of a Banach space E, let $J : F \to E$ be the canonical map, and assume that J is continuous. Let $\varphi : \mathbb{R} \to F$ be a function such that $J \circ \varphi : \mathbb{R} \to E$ is continuous. Then $\varphi : \mathbb{R} \to F$ is strongly measurable.*

Proof: Since F is separable, F and $\overline{J(F)}$ (the closed subspace of E spanned by $J(F)$) are Polish spaces and $J : F \to \overline{J(F)}$ is one to one and continuous. By Souslin's theorem (see e.g. [A, Theorem 3.2.3 and its corollary]) the map $J^{-1} : J(F) \to F$ is Borel measurable. Since $\varphi = J^{-1} \circ J \circ \varphi$ and $J \circ \varphi : \mathbb{R} \to \overline{J(F)}$

is continuous, $\varphi : \mathbb{R} \to F$ is Borel measurable. Since F is separable, φ is strongly measurable by Pettis' theorem [DU, Chapter II, p. 42]. $\square$

Lemma 21. *Let* $\overline{C}$ *be an interpolation couple,* $g \in \mathcal{G}(\overline{C})$, $0 < \beta < 1$. *With the notation of Lemma* 4, c) $\Rightarrow$ b) *and* c') $\Rightarrow$ b').

Proof: This follows from Lemma 20 since $F = C_\beta$ or C^β embeds in $E = C_0 + C_1$ and $g'(\beta + i\cdot) : \mathbb{R} \to C_0 + C_1$ is continuous. $\square$

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