On skew derivations as homomorphisms or anti-homomorphisms

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Abstract. Let R be a prime ring with center Z and I be a nonzero ideal of R. In this manuscript, we investigate the action of skew derivation (δ,φ) of R which acts as a homomorphism or an anti-homomorphism on I. Moreover, we provide an example for semiprime case.

Keywords: skew derivation; generalized polynomial identity (GPI); prime ring; ideal

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1. Introduction

Throughout this paper, let R be a prime ring with center Z and Q be the Martindale quotient ring of R. Note that Q is also prime and the center C of Q, which is called the extended centroid of R, is a field (we refer the reader to [2] for the definitions and related properties of these objects).

Given any automorphism φ of R, an additive mapping $\delta : R \to R$ satisfying $\delta(xy) = \delta(x)y + \varphi(x)\delta(y)$ for all $x, y \in R$ is called a φ-derivation of R, or a skew derivation of R with respect to φ, denoted by (δ,φ). It is easy to see if $\varphi = 1_R$, the identity map of R, then a φ-derivation is merely an ordinary derivation, and if $\varphi \neq 1_R$, then $\varphi - 1_R$ is a skew derivation, i.e., the basic example of skew derivation are usual derivation and the map $\varphi - I_R$. Therefore, the concept of skew derivations can be regarded as a generalization of both derivations and automorphisms. Moreover, any skew derivation (δ,φ) extends uniquely to a skew derivation of Q [12] via extensions of each map to Q. Thus, we may assume that any skew derivation of R is the restriction of a skew derivation of Q. When $\delta(x) = \varphi(x)b - bx$, for some $b \in Q$, then (δ,φ) is called an inner skew derivation, otherwise it is outer. Recall that φ is an inner automorphism if, when acting on Q, $\varphi(q) = uqu^{-1}$, for some invertible $u \in Q$, otherwise φ is an outer automorphism (see [17, 18] and the references therein). For any nonempty subset S of R, if $\delta(xy) = \delta(x)\delta(y)$ or $\delta(xy) = \delta(y)\delta(x)$, for all $x, y \in S$, then (δ,φ) is called

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a skew derivation which acts as a homomorphism or an anti-homomorphism on S, respectively.

Let $Q_{*C}C\{X\}$ be the free product of Q and the free algebra $C\{X\}$ over C on an infinite set X of indeterminates. Elements of $Q_{*C}C\{X\}$ are called generalized polynomials and a typical element in $Q_{*C}C\{X\}$ is a finite sum of monomials of the form $\alpha a_{i_0}x_{j_1}a_{i_1}x_{j_2}\cdots x_{j_n}a_{i_n}$ where $\alpha \in C$, $a_{ik} \in Q$ and $x_{jk} \in X$. We say that R satisfies a nontrivial generalized polynomial identity (abbreviated as GPI), if there exists a nonzero polynomial $\phi(x_i) \in Q_{*C}C\{X\}$ such that $\phi(r_i) = 0$ for all $r_i \in R$. By a generalized polynomial identity with automorphisms and skew derivations, we mean an identity of R expressed as the form $\phi(\varphi_j(x_i), \delta_k(x_i))$, where each φ_j is an automorphism, each δ_k is a skew derivation of R and $\phi(y_{ij}, z_{ik})$ is a generalized polynomial in distinct indeterminates y_{ij}, z_{ik}.

We need some well-known facts which will be used in the sequel.

Fact 1.1 ([5])**.** Let R be a prime ring and I an ideal of R, then I, R and Q satisfy the same generalized polynomial identities with coefficients in Q.

Fact 1.2 ([6, Theorem 1])**.** Let R be a prime ring and I an ideal of R, then I, R and Q satisfy the same generalized polynomial identities with automorphisms.

Fact 1.3 ([13])**.** Let R be a prime ring with an automorphism φ. Suppose that φ is Q-outer (in the sense that it is not Q-inner). If $\phi(x_i, \varphi(x_i)) = 0$ is a generalized polynomial identity for R, then R also satisfies the non-trivial generalized polynomial identity $\phi(x_i, y_i)$, where x_i, y_i are distinct indeterminates.

Fact 1.4 ([7, Theorem 1])**.** Let R be a prime ring and δ is a Q-outer skew derivation of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i) = 0$ of R, where x_i, y_i are distinct indeterminates.

Fact 1.5 ([7, Theorem 1])**.** Let R be a prime ring with an outer automorphism φ. Suppose that (δ, φ) is a Q-outer skew derivation of R. Then any generalized polynomial identity of R in the form $\phi(x_i, \varphi(x_i), \delta(x_i)) = 0$ yields the generalized polynomial identity $\phi(x_i, y_i, z_i) = 0$ of R, where x_i, y_i, z_i are distinct indeterminates.

Fact 1.6 ([15, Proposition])**.** Let R be a prime algebra over an infinite field k and let K be a field extension over k. Then R and $R \otimes_k K$ satisfy the same generalized polynomial identities with coefficients in R.

The next fact can be obtained directly by the proof of [14, Lemma 2] and Fact 1.6.

Fact 1.7. Let R be a non-commutative simple algebra, finite dimensional over its center Z. Then $R \subseteq M_n(F)$ with $n > 1$ for some field F, R and $M_n(F)$ satisfy the same generalized polynomial identities with coefficients in R.

In [3], Bell and Kappe proved that if d is a derivation of a prime ring R which acts as a homomorphism or as anti-homomorphism on a nonzero right ideal of R, then $d = 0$ on R. In [1], Ali et al. obtained a similar result in the setting of Lie ideals. To be more specific, they proved the following. Let R be a 2-torsion free prime ring and L be a nonzero Lie ideal of R such that $l^2 \in L$ for all $l \in L$. If d is a derivation of R which acts as a homomorphism or an anti-homomorphism on L, then $d = 0$ or $L \subseteq Z$. In [20], Wang and You discussed the same result, by eliminating the hypothesis $l^2 \in L$ for all $l \in L$. On the other hand, the first author [16] extended Bell and Kappe's result replacing the derivation d by a generalized derivation F proving the following. Let R be a 2-torsion free prime ring, I be a nonzero ideal and (F, d) be a nonzero generalized derivation of R. If (F, d) acts as a homomorphism or an anti-homomorphism of I and $d \neq 0$, then R is commutative. Later, Gusic [10] obtained similar results when $F, d : R \to R$ are any functions. For more related results we refer the reader to [4], [8], [19].

Here we will continue the study of analogous problems on ideals of a prime ring by using the theory of generalized polynomial identities with automorphisms and skew derivations. Our main result is

Theorem 1.1. *Let R be a prime ring with center Z and I be a nonzero ideal of R. If (δ, φ) is a skew derivation of R which acts as a homomorphism or an anti-homomorphism on I, then either $\delta = 0$ or $I \subseteq Z$.*

When $\delta = \varphi - 1_R$, we obtain the following

Corollary 1.1. *Let R be a prime ring and I be a nonzero ideal of R. If φ is a nonidentity automorphism of R which acts as a homomorphism or an anti-homomorphism on I, then R is commutative.*

Let R be a unital ring. For a unit $u \in R$, the map $\varphi_u : x \to uxu^{-1}$ defines an automorphism of R. If d is a derivation of R, then it is easy to see that the map $ud : x \to ud(x)$ defines a φ_u-derivation of R. So we have

Corollary 1.2. *Let R be a prime unital ring, u be a unit in R and I be a nonzero ideal of R. Suppose that φ_u is a derivation of R which acts as a homomorphism or an anti-homomorphism on I, then R is commutative.*

2. Proof of Theorem 1.1

Assume to the contrary that both $\delta \neq 0$ and $I \nsubseteq Z$. We divide the proof into two cases:

Case 1. If (δ, φ) acts as a homomorphism on I, then we have $\delta(xy) = \delta(x)\delta(y)$, for all $x, y \in I$, i.e.,

$$\delta(x)y + \varphi(x)\delta(y) = \delta(x)\delta(y), \quad \text{for all } x, y \in I. \tag{2.1}$$

In the light of Kharchenko's theory [13], we split the proof into two cases.

Let δ is Q-outer, by Fact 1.4 and (2.1), I satisfies the polynomial identities

$$sy + \varphi(x)t = st \text{ for all } x, y, s, t \in I. \tag{2.2}$$

Now, if we take φ being not Q-inner, by Fact 1.5, I satisfies

$$sy + wt = st, \text{ for all } x, y, s, t, w \in I$$

and for $t = 0$, we have $sy = 0$, for all $s, y \in I$. In other words $I^2 = 0$ which implies that $I = 0$, a contradiction.

Now consider the case when φ is Q-inner. Then $\varphi(x) = gxg^{-1}$, for some $g \in Q$. Thus from (2.2), we have $sy + gxg^{-1}t = st$, for all $x, y, s, t \in I$. If $t = 0$, then as above we get a contradiction.

Let δ is Q-inner, then $\delta(x) = \varphi(x)q - qx$, for all $x \in R$, $q \in Q$. From (2.1), we have

$$(2.3)\quad (\varphi(x)q - qx)y + \varphi(x)(\varphi(y)q - qy) = (\varphi(x)q - qx)(\varphi(y)q - qy), \text{ for all } x, y \in I.$$

Since I and Q satisfy the same generalized polynomial identities with automorphisms (Fact 1.2), therefore Q also satisfies (2.3), i.e.,

$$(2.4)\quad (\varphi(x)q - qx)y + \varphi(x)(\varphi(y)q - qy) = (\varphi(x)q - qx)(\varphi(y)q - qy), \text{ for all } x, y \in Q.$$

If φ is not Q-inner, then Q satisfies

$$(2.5)\qquad (wq - qx)y + w(vq - qy) = (wq - qx)(vq - qy), \text{ for all } x, y, w, u \in Q.$$

In particular, by (2.5), one can see that

$$w(vq) - (wq - qx)(vq) = 0, \text{ for all } x, w, v \in Q.$$

By Chuang [5], this generalized polynomial identity is also satisfied by R. Note that this is a generalized polynomial identity and by Fact 1.7, there exists a field $\mathbb{F}$ such that $R \subseteq M_k(\mathbb{F})$, the ring of $k \times k$ matrices over a field $\mathbb{F}$, where $k \geq 1$. Moreover, R and $M_k(\mathbb{F})$ satisfy the same polynomial identity [5], i.e.,

$$w(vq) - (wq - qx)(vq) = 0, \text{ for all } x, w, v \in M_k(\mathbb{F}).$$

Let e_{ij} be the usual matrix unit with 1 in (i, j)-entry and zero elsewhere. By choosing $x = e_{11}$, $v = e_{12}$, $w = 0$, $q = e_{21}$, we see that

$$0 = w(vq) - (wq - qx)(vq) = e_{21} \neq 0, \text{ which is a contradiction.}$$

If φ is Q-inner, then $\varphi(x) = gxg^{-1}$. From (2.3) we can write,

$$(gxg^{-1}q - qx)y + gxg^{-1}(gyg^{-1}q - qy) = (gxg^{-1}q - qx)(gyg^{-1}q - qy), \text{ for all } x, y \in I.$$

We see that, if $g^{-1}q \in C$, then $\delta(x) = gxg^{-1}q - qx = g(xg^{-1}q - g^{-1}qx) = g[x, g^{-1}q] = 0$, a contradiction. So we may assume that $g^{-1}q \notin C$. Let

$$(2.6)\quad \phi(x, y) = (gxg^{-1}q - qx)y + gxg^{-1}(gyg^{-1}q - qy) - (gxg^{-1}q - qx)(gyg^{-1}q - qy).$$

Since by [5] or [2, Theorem 6.4.4], I and Q satisfy the same generalized polynomial identities, we can easily see that $\phi(x, y) = 0$ is a nontrivial generalized polynomial identity of Q. Let $\mathcal{F}$ be the algebraic closure of C, when C is infinite and $\mathcal{F} = C$, otherwise. By Fact 1.6, $\phi(x, y)$ is also a generalized polynomial identity of $Q \otimes_C \mathcal{F}$. Moreover, in view of [9, Theorem 3.5], both Q and $Q \otimes_C \mathcal{F}$ are prime and centrally closed, we may replace R by Q or $Q \otimes_C \mathcal{F}$. Thus, R is centrally closed over Z which is either algebraically closed or finite, and R satisfies generalized polynomial identity (2.6). By Martindale's theorem [2, Corollary 6.1.7], R is a primitive ring having nonzero socle and the commuting division ring D which is finite-dimensional central division algebra over Z. Since Z is either finite or algebraically closed, D must coincide with Z. Therefore, in view of Jacobson theorem [11, p. 75], R is isomorphic to a dense subring of the ring of linear transformations on a vector space V over Z (or $End(V_Z)$ in brief), containing nonzero linear transformations of finite rank.

Assume that $dim(V_Z) = 1$, then $R = Z$ so $I \subseteq Z$, which is a contradiction. Therefore $dim(V_Z) \geq 2$. In this case, our aim is to show that, for any $v \in V$, v and $g^{-1}qv$ are Z-dependent. Suppose to the contrary that v and $g^{-1}qv$ are Z-independent, by the density of R in $End(V_Z)$, there exist $x_0, y_0 \in R$, such that

$$\begin{aligned} x_0v = 0, &\quad x_0g^{-1}qv = g^{-1}v; \\ y_0v = v, &\quad y_0g^{-1}qv = g^{-1}qv. \end{aligned}$$

With all these, we obtain from the assumption that

$$\begin{aligned} 0 &= \big((gx_0g^{-1}q - qx_0)y_0 + gx_0g^{-1}(gy_0g^{-1}q - qy_0) \\ &\quad -(gx_0g^{-1}q - qx_0)(gy_0g^{-1}q - qy_0)\big)\, v \\ &= (gx_0g^{-1}q - qx_0)v + gx_0g^{-1}(gg^{-1}qv - qv) - (gx_0g^{-1}q - qx_0)(gg^{-1}qv - qv) \\ &= (gx_0g^{-1}q - qx_0)v \\ &= v, \text{ a contradiction.} \end{aligned}$$

Thus, v and $g^{-1}qv$ are Z-dependent as claimed. From above we have prove that $g^{-1}qv = v\mu(v)$, for all $v \in V$, where $\mu(v) \in Z$ depends on $v \in V$. We claim that $\mu(v)$ is independent of the choice of $v \in V$. Indeed, for any $v, w \in V$, if v and w are Z-independent, then there exist $\mu(v), \mu(w), \mu(v + w) \in Z$ such that

$$g^{-1}qv = v\mu(v),\ g^{-1}qw = w\mu(w),\ \text{and } g^{-1}q(v + w) = (v + w)\mu(v + w).$$

Moreover, $v\mu(v) + w\mu(w) = g^{-1}q(v + w) = (v + w)\mu(v + w)$. Hence

$$v(\mu(v) - \mu(v + w)) + w(\mu(w) - \mu(v + w)) = 0.$$

Since v and w are Z-independent, we have $\mu(x) = \mu(v + w) = \mu(w)$. If v and w are Z-dependent, say $v = w\beta$, where $\beta \in Z$, then $v\mu(v) = g^{-1}qv = g^{-1}qw\beta = w\mu(w)\beta = v\mu(w)$ and so $\mu(v) = \mu(w)$ as claimed. Therefore, there exist $\gamma \in Z$ such that $g^{-1}qv = v\gamma$, for all $v \in V$. Hence $g^{-1}q \in Z$ and $\delta = 0$, a contradiction.

Case 2. If (δ, φ) acts as an anti-homomorphism on I, then we have $\delta(xy) = \delta(y)\delta(x)$, for all $x, y \in I$, i.e.,

$$\delta(x)y + \varphi(x)\delta(y) = \delta(y)\delta(x), \quad \text{for all } x, y \in I. \tag{2.7}$$

We apply the same technique as Case 1. If δ is not inner on Q, by Fact 1.4 and (2.7) we get

$$sy + \varphi(x)t = ts, \text{ for all } x, y, s, t \in I.$$

If φ is not Q-inner, by Fact 1.5 one can have

$$sy + wt = ts, \text{ for all } x, y, s, t, w \in I.$$

We obtain a contradiction, as already discuses in case 1. Now we assume that φ is Q-inner, then $\varphi(x) = gxg^{-1}$, for some $g \in Q$. From (2.7), we have

$$sy + gxg^{-1}t = ts, \text{ for all } x, y, s, t \in I.$$

In particular $t = 0$, I satisfied the blended component $sy = 0$, for all $s, y \in I$, again we get a contradiction.

Next, assume that δ be an inner derivation on Q, i.e., $\delta(x) = \varphi(x)q - qx$, for some $q \in Q$. From (2.7), we can write

$$(\varphi(x)q-qx)y+\varphi(x)(\varphi(y)q-qy) = (\varphi(y)q-qy)(\varphi(x)q-qx) \text{ for all } x, y \in I. \tag{2.8}$$

Since I and Q satisfy the same generalized polynomial identities with automorphisms [Fact 1.2], so Q satisfies (2.3), i.e.,

$$(\varphi(x)q-qx)y+\varphi(x)(\varphi(y)q-qy) = (\varphi(y)q-qy)(\varphi(x)q-qx), \text{ for all } x, y \in Q. \tag{2.9}$$

If φ is not Q-inner, then Q satisfies

$$(wq - qx)y + w(vq - qy) = (vq - qy)(wq - qx), \text{ for all } x, y, w, v \in Q.$$

In particular $y = 0$, we have

$$w(vq) - (vq)(-wq + qx) = 0, \text{ for all } x, w, v \in Q.$$

In view of the above situation as in Case 1, we assume that $M_k(\mathbb{F})$ satisfy the same polynomial identity, i.e.,

$$w(vq) - (vq)(-wq + qx) = 0, \text{ for all } x, w, v \in M_k(\mathbb{F}).$$

By choosing $x = e_{12}$, $v = e_{21}$, $w = 0$, $q = e_{11}$, we see that

$$0 = w(vq) - (vq)(-wq + qx) = e_{22} \neq 0, \text{ which is a contradiction.}$$

Finally, we consider φ is Q-inner, then $\varphi(x) = gxg^{-1}$, for some $g \in Q$. If $g^{-1}q \in C$, then we see that $\delta = 0$. So, we assume that $g^{-1}q \notin C$, and hence Q satisfy the generalized polynomial identity,

$$(2.10)\quad (gxg^{-1}q - qx)y + gxg^{-1}(gyg^{-1}q - qy) - (gyg^{-1}q - qy)(gxg^{-1}q - qx) = 0.$$

Using the same arguments as in the proof of Case 1, we assume that R is centrally closed over Z which is either finite or algebraically closed, and hence R satisfies the nontrivial generalized polynomial identity (2.10). Moreover, we know that R is isomorphic to a dense subring of $End(V_Z)$, for some vector space V over Z. Now, for any $v \in V$, we claim that v and $g^{-1}qv$ are Z-dependent. Suppose to the contrary that v and $g^{-1}qv$ are Z-independent, by the density of R in $End(V_Z)$ there exist elements $x_0, y_0 \in R$ such that

$$\begin{array}{ll} x_0 v = 0, & x_0 g^{-1}qv = g^{-1}v, \\ y_0 v = 0, & y_0 g^{-1}qv = v. \end{array}$$

It follows from (2.10) that

$$\begin{aligned} 0 = (gx_0g^{-1}q - qx_0)y_0 + gx_0g^{-1}(gy_0g^{-1}q - qy_0) \\ - (gy_0g^{-1}q - qy_0)(gx_0g^{-1}q - qx_0) = gv = v \end{aligned}$$

which is a contradiction. Thus, v and $g^{-1}qv$ are Z-dependent as claimed. In view of Case 1, we know that $g^{-1}q \in Z$ and so $\delta = 0$, a contradiction. This completes the proof.

The following example demonstrates that, we cannot expect the same conclusion holds in semiprime ring.

Example 2.1. Let $\mathbb{C}$ be the usual ring of complex numbers. Define an automorphism $\Psi : \mathbb{C} \to \mathbb{C}$ as $\Psi(z) = \overline{z}$ for all $z \in \mathbb{C}$. Now let (δ_1, Ψ) a nonzero skew derivation on $\mathbb{C}$ such that $\delta_1(z) = a(\overline{z} - z)$, where a is fixed complex number. Consider $R = \mathbb{C} \oplus \mathbb{M}_{2\times 2}(\mathbb{C})$. It is easy to see that R is non-commutative semiprime ring. Next we define a map $\delta : R \to R$ as follows $\delta(r_1, r_2) = (\delta_1(r_1), 0)$. This can be seen easily that δ is a skew derivation associated with automorphism φ, where $\varphi : R \to R$ such that $\varphi(r_1, r_2) = (\psi(r_1), I(r_2))$. Consider $\mathbb{I} = \{0\} \times \mathbb{M}_{2\times 2}(\mathbb{C})$. It is easy to check that $\mathbb{I}$ is a nonzero ideal of R and (δ, φ) is a skew derivation of R which acts as a homomorphism as well as an anti-homomorphism on $\mathbb{I}$.

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