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Rings whose nonsingular right modules are R-projective

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Abstract. A right R-module M is called R-projective provided that it is projective relative to the right R-module R_R. This paper deals with the rings whose all nonsingular right modules are R-projective. For a right nonsingular ring R, we prove that R_R is of finite Goldie rank and all nonsingular right R-modules are R-projective if and only if R is right finitely Σ-CS and flat right R-modules are R-projective. Then, R-projectivity of the class of nonsingular injective right modules is also considered. Over right nonsingular rings of finite right Goldie rank, it is shown that R-projectivity of nonsingular injective right modules is equivalent to R-projectivity of the injective hull $E(R_R)$. In this case, the injective hull $E(R_R)$ has the decomposition $E(R_R) = U_R \oplus V_R$, where U is projective and $\mathrm{Hom}(V, R/I) = 0$ for each right ideal I of R. Finally, we focus on the right orthogonal class $\mathcal{N}^{\perp}$ of the class $\mathcal{N}$ of nonsingular right modules.

Keywords: nonsingular module; R-projective module; flat module; perfect ring

Classification: 16D10, 16D40, 16D80, 16E30

1. Introduction and preliminaries

In this paper, R stands for an associative ring with identity. As usual, we denote by Mod-R the category of all right R-modules. For a right module M, $E(M)$, $Z(M)$ and $\mathrm{Rad}(M)$ denote the injective hull, singular submodule and Jacobson radical of M, respectively. The notation $K \ll M$ means that K is a small submodule of M in the sense that $K + L \neq M$ for any proper submodule L of M. Moreover, $N \trianglelefteq M$ is used for denoting an essential submodule of M and this means that $N \cap S \neq 0$ for any nonzero submodule S of M.

Baer's Criterion for injectivity asserts that a right R-module M is injective if and only if each homomorphism from any right ideal I of R into M extends to R. Dually, a right R-module M is called R-projective provided that each homomorphism $f\colon M \to R/I$, where I is any right ideal, factors through the canonical projection $\pi\colon R \to R/I$. However, R-projective modules need not be projective. In [14], C. Faith asked when R-projectivity implies projectivity for all right R-modules. This problem has been considered by several authors, see [25], [18], [21], [8], [2], [31], [32].

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Characterizing rings by projectivity of some classes of their modules is a classical problem in ring and module theory. A result of Bass, see [5, Theorem 28.4], states that a ring R is right perfect if and only if each flat right R-module is projective. Also, a ring R is QF if and only if each injective right R-module is projective [14, Theorem 24.20]. On the other hand, K. R. Goodearl proved the following remarkable theorem which is an inspiration source for our result, see Theorem 3.5.

Theorem 1.1 ([17, Theorem 5.21]). *If R is a right nonsingular ring, then the following conditions are equivalent:*

(1) *All nonsingular right R-modules are projective.*
(2) *Ring R is right perfect, left semihereditary and $E(R_R)$ is flat.*

Recently, the notion of R-projectivity and rings characterized by R-projectivity of some classes of their modules were considered in [4], [3], [1]. The rings whose flat right R-modules are R-projective were characterized in [3], [4] and these rings are termed as right A-perfect, and the rings whose injective right R-modules are R-projective were characterized in [1].

At this point, it is natural to ask "What are the rings over which each nonsingular right R-module is R-projective?"

The main purpose of this paper is to derive necessary and sufficient conditions on a right nonsingular ring R under which all nonsingular right R-modules are R-projective and to describe the structure of such rings.

The paper is organized as follows.

Along the way, in Section 2, some properties of nonsingular right modules are investigated. We first recall the following result due to D. R. Turnidge, see [33, Theorem 2.1].

Theorem 1.2 ([17, Proposition 5.16]). *Let R be a right nonsingular ring. Then, all nonsingular right R-modules are flat if and only if R is left semihereditary and $E(R_R)$ is flat.*

Related to the above result, in Section 2, we obtain that every flat right R-module is nonsingular if and only if R is right nonsingular and pure submodules of free right R-modules are closed. Over a right nonsingular ring R, we prove that pure submodules of nonsingular right R-modules are closed if and only if R_R is of finite Goldie rank. We also show that every flat right R-module is nonsingular over a right semihereditary semiperfect ring, over a right nonsingular right perfect ring, over a right semihereditary right A-perfect ring, and over a right nonsingular ring of finite right Goldie rank.

In Section 3, we call a ring R *right NR* in case all nonsingular right R-modules are R-projective. We show that for a right nonsingular right NR ring, all nonsingular right modules are flat. If R is of finite right Goldie rank and right nonsingular, then every right NR-ring is right A-perfect. We prove that a right nonsingular ring R is of finite right Goldie rank and right NR if and only if R is semihereditary, right A-perfect, right CS and $E(R_R)$ is flat if and only if R is right finitely Σ-CS and right A-perfect, see Theorem 3.5. As a consequence, we obtain that if R is a semiprime right and left Goldie ring, then R is a right NR-ring if and only if R is semihereditary and right A-perfect. For a right nonsingular ring R of finite right Goldie rank, we prove that every nonsingular injective right R-module is R-projective if and only if $E(R_R)$ is R-projective if and only if $E(R_R) = U_R \oplus V_R$ where U is projective and $\mathrm{Hom}(V, R/I) = 0$ for each cyclic right R-module R/I. In particular, over a right nonsingular right Noetherian ring, nonsingular injective right R-modules are R-projective if and only if $E(R_R) = U_R \oplus V_R$, where U is projective and $\mathrm{Rad}(V) = V$.

In Section 4, nonsingular covers will be considered. Let $\mathcal{N}$ be the class of all nonsingular right R-modules. Following E. E. Enochs and O. M. G. Jenda, see [12], an *$\mathcal{N}$-precover* (or a *nonsingular precover*) of a right R-module M is a homomorphism $\varphi\colon N \to M$ with $N \in \mathcal{N}$ such that for any homomorphism $\psi\colon N' \to M$ with $N' \in \mathcal{N}$, there exists $\lambda\colon N' \to N$ such that $\varphi\lambda = \psi$. An $\mathcal{N}$-precover $\varphi\colon N \to M$ is said to be an *$\mathcal{N}$-cover* (or a *nonsingular cover*) if every endomorphism λ of N with $\varphi\lambda = \varphi$ is an isomorphism. Works on the torsion-free covers date back to 1960s and some of the results about the existence of torsion-free covers for abstract torsion theories were given in [29], [16], [30]. As a particular corollary, M. L. Teply in [29] proved that nonsingular covers exist for all right modules over a right nonsingular ring of finite right Goldie rank. This result was further discussed and a sort of converse of this result was given in [7]. Then, in 2003, L. Bican extended the aforementioned result for Goldie's torsion theory. More precisely, in [6, Thoerem 2], among the other things, L. Bican proved the following noticable theorem.

Theorem 1.3 ([6, Theorem 2]). *Let $(\mathcal{T}, \mathcal{F})$ be Goldie's torsion theory. The following conditions are equivalent:*

(1) *$\mathcal{F}$ is a covering class.*
(2) *$(\mathcal{T}, \mathcal{F})$ is of finite type.*

If moreover, the ring R is right nonsingular, then these conditions are equivalent to the following condition:

(3) *Every nonzero right ideal of R contains a finitely generated essential right ideal.*

Since the class $\mathcal{N}$ contains the class of projective modules over a right nonsingular ring, every right module has an epic nonsingular cover over a right nonsingular ring of finite right Goldie rank. We will denote by

$$\mathcal{N}^{\perp} = \{X \in \text{Mod}-R\colon\ \text{Ext}^1_R(N, X) = 0 \ \text{ for all } \ N \in \mathcal{N}\}$$

the right orthogonal class of the class $\mathcal{N}$ of nonsingular right modules. In Section 4, several properties of the class $\mathcal{N}^{\perp}$ of right modules are obtained. Particularly, we obtain that a right nonsingular ring R having finite right Goldie rank is right NR if and only if nonsingular covers of finitely generated right R-modules are (finitely generated) projective. In the same section, we also need the following fact of [20] which is needed for the deduction of Lemma 4.4.

Theorem 1.4 ([20, Theorem 3.4]). *If a class $\mathcal{F}$ contains the ground ring R and is closed under extensions, direct sums, pure submodules, and pure quotient modules, then $\mathcal{F}$ is covering and $\mathcal{F}^{\perp}$ is enveloping.*

2. Nonsingular modules

Recall that the singular submodule $Z(M)$ of a right R-module M is the set of elements $m \in M$ such that $mI = 0$ for some essential right ideal I of R. A right module M is called *singular* if $Z(M) = M$, and *nonsingular* if $Z(M) = 0$. Thus, R is called a right nonsingular ring if $Z(R_R) = 0$. Moreover, for any right module M, the submodule $Z_2(M)$ is defined by $Z_2(M)/Z(M) = Z(M/Z(M))$.

We begin with the following lemma that we use frequently in this paper.

Lemma 2.1 ([26, Lemma 2.3]). *Let N be a submodule of a right module M.*

(1) *If $Z(M/N) = 0$, then N is closed in M.*
(2) *If N is closed in M and $Z(M) = 0$, then $Z(M/N) = 0$.*

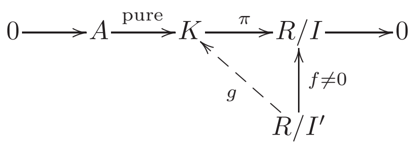
A right R-module M is said to be *of finite Goldie rank* provided that M contains no infinite independent families of nonzero submodules. For example, all Noetherian modules are of finite Goldie rank. A ring R is said to be *of finite right Goldie rank* if the right R-module R_R is of finite Goldie rank, equivalently, every right ideal has a finitely generated essential submodule, see [17, Proposition 3.13 (a)]. Next, we recall the notion of torsion theories. A *torsion theory* $(\mathcal{T}, \mathcal{F})$ for the category Mod-R consists of two classes of right R-modules $\mathcal{T}$ and $\mathcal{F}$ satisfying the properties $\mathcal{T} = \{M \in \text{Mod-}R\colon \text{Hom}(M, F) = 0$ for every $F \in \mathcal{F}\}$ and $\mathcal{F} = \{M \in \text{Mod-}R\colon \text{Hom}(T, M) = 0$ for every $T \in \mathcal{T}\}$. If we take the classes $\mathcal{T} = \{M \in \text{Mod-}R\colon Z(M) \trianglelefteq M\} = \{M \in \text{Mod-}R\colon Z_2(M) = M\}$ and $\mathcal{F} = \{M \in \text{Mod-}R\colon Z(M) = 0\}$, then the pair $(\mathcal{T}, \mathcal{F})$ becomes the torsion theory known as Goldie's torsion theory, we refer the reader to [28, page 139 and page 148]. It should be pointed out that in Goldie's torsion theory when $Z(R_R) = 0$, the class $\mathcal{T}$ will be exactly the class of singular modules. Finally, a torsion theory $(\mathcal{T}, \mathcal{F})$ is said to be *of finite type* if each right ideal I for which R/I is in $\mathcal{T}$ contains a finitely generated right ideal J for which R/J is in $\mathcal{T}$, see [6, page 396], and said to be *hereditary* if the class $\mathcal{T}$ is closed under submodules, see [28, page 141]).

In [11], it was proved that a ring R is right Noetherian if and only if pure submodules of right R-modules are closed. Considering only nonsingular right modules we have the following corresponding result over right nonsingular rings.

Proposition 2.2. *Let R be a right nonsingular ring. Then, the following are equivalent:*

(1) *R_R is of finite Goldie rank.*
(2) *Pure submodules of nonsingular right R-modules are closed.*

Proof: (1) $\Rightarrow$ (2) Let A be a pure submodule of a nonsingular right R-module M. Suppose for the contrary that A is not closed in M. In that case, there exists a proper essential extension B of A in M. For $b \in B \setminus A$, if we set $K = A + bR$, then K/A becomes singular by [17, Proposition 1.20 (b)]. Moreover, K/A is cyclic and so $K/A \cong R/I$ for some right ideal I of R. The right ideal I is essential in R by [17, Proposition 1.21]. Additionally, there exists a finitely generated essential submodule I' of I with the help of finiteness condition on R_R. Now, consider the following diagram:



where f is just the projection of R/I' modulo I/I'. By the fact that R/I' is finitely presented, we have $\pi g = f$ for some $g\colon R/I' \to K$. However, using $I' \trianglelefteq R$, we obtain that $g(Z(R/I')) = g(R/I') \le Z(K) = 0$, that is, $g = 0$ which is a contradiction.

(2) $\Rightarrow$ (1) Take a family $\{E_\gamma\}_{\gamma\in\Gamma}$ of nonsingular injective right modules. Because $\bigoplus E_\gamma$ is pure in $\prod E_\gamma$, by the assumption it is closed. Considering [17, Corollary 1.9], we see that $\bigoplus E_\gamma$ is injective. Consequently, [17, Theorem 3.17] yields that R_R is of finite Goldie rank. □

Before proving Corollary 2.4, we state the following lemma of [20].

Lemma 2.3 ([20, Lemma 4.7 (iii)])**.** *Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. If $(\mathcal{T}, \mathcal{F})$ is hereditary and of finite type, then $\mathcal{F}$ is closed under pure submodules and pure quotient modules.*

Corollary 2.4. *The following are equivalent for a right nonsingular ring R:*

(1) *R_R is of finite Goldie rank.*
(2) *Pure submodules of nonsingular right R-modules are closed.*
(3) *The torsion theory $(\mathcal{S}, \mathcal{N})$ is of finite type, where $\mathcal{S}$ is the class of all singular right R-modules and $\mathcal{N}$ is the class of all nonsingular right R-modules.*
(4) *Nonsingular right R-modules are closed under pure quotients.*

Proof: (1) $\Leftrightarrow$ (2) is shown in Proposition 2.2. (1) $\Leftrightarrow$ (3) is proved in Theorem 1.3. (3) $\Rightarrow$ (4) follows from Lemma 2.3 and (4) $\Rightarrow$ (2) can be easily seen from Lemma 2.1 (1). □

Proposition 2.5. *Every flat right R-module is nonsingular if and only if R is right nonsingular and pure submodules of free right R-modules are closed.*

Proof: We immediately obtain that $Z(R_R) = 0$ since R_R is flat. Now, let K be a pure submodule of a free right R-module F. Then, F/K is a flat right R-module by [22, Corollary 4.86 (1)]. By the assumption, F/K is nonsingular and Lemma 2.1 (1) implies that K is closed in F. Conversely, let M be a flat right R-module and consider the short exact sequence $0 \to K \hookrightarrow F \to M \to 0$ where F is a free right R-module. As M is flat, K is a pure submodule of F by [22, Corollary 4.86 (1)]. Hence, K is closed in F and so $F/K \cong M$ is nonsingular by Lemma 2.1 (2). □

Following [4], a ring R is called *right almost-perfect* (*A-perfect*, for short) if every flat right R-module is R-projective. These are exactly the rings over which flat covers of finitely generated right modules are projective. It was shown in [4] that, right A-perfect rings lies properly between right perfect rings and semiperfect rings.

In the following proposition, we give some examples of rings whose flat right modules are nonsingular.

Proposition 2.6. *Over the following rings R, all flat right R-modules are nonsingular.*

(1) *R is right nonsingular ring of finite right Goldie rank.*
(2) *R is right semihereditary and semiperfect.*
(3) *R is right nonsingular and right perfect.*
(4) *R is right semihereditary and right A-perfect.*

Proof: (1) This holds by [17, page 139, Exercise 12]. We include the proof for completeness. Let M be a flat right R-module and $f\colon F \to M$ be an epimorphism where F is a free right R-module. By the assumption and Corollary 2.4, we obtain that $M \cong F/\ker(f)$ is nonsingular.

(2) Let M be a flat right R-module. We show that every finitely generated submodule K of M is nonsingular which implies that M is nonsingular. For a finitely generated submodule K of M, we have that K is flat by [22, Theorem 4.67]. Then K is projective by the semiperfect assumption, see [22, page 161, Exercise 21], and so nonsingular as desired.

(3) It can be obtained from the fact that flat right modules are projective over right perfect rings.

(4) This follows from (2). □

3. NR-rings

The right nonsingular rings whose nonsingular right modules are projective were characterized in Theorem 1.1. These are exactly the right perfect, left semi-hereditary rings with $E(R_R)$ being flat. On the other hand, the rings whose flat right and injective right modules are R-projective were characterized in [4] and [1], respectively. Motivated by the aforementioned rings, in this section, we investigate the rings whose nonsingular right modules are R-projective.

Definition 3.1. A ring R is called *right NR* if every nonsingular right R-module is R-projective. Left NR-rings are defined similarly.

Clearly, the rings whose nonsingular right modules are projective are right NR. We shall see in Example 3.10 that the converse is not true in general.

Proposition 3.2. *Let R be a right NR-ring having finite right Goldie rank with $Z(R_R) = 0$ and M be a nonsingular right R-module with* $\mathrm{Rad}(M) \ll M$. *Then, M is projective.*

Proof: Since R is of finite right Goldie rank and right nonsingular, flat right R-modules are nonsingular by Proposition 2.6. Then, R becomes right A-perfect as R is right NR. Thus, R is semiperfect by [4] and so [21, Theorem 1] yields that every right nonsingular R-module with small radical is projective. □

The next result indicates that being NR-ring is preserved by Morita equivalence.

Proposition 3.3. *Let R and S be Morita equivalent rings. Then, R is a right NR-ring if and only if S is a right NR-ring.*

Proof: A right R-module M is R-projective if and only if M is N-projective for any finitely generated projective right R-module N, see [5, Proposition 16.12]. Now, by [22, page 501, Exercise 2], being nonsingular is a categorical property.

Moreover, by [5, Proposition 21.6 and Proposition 21.8] projectivity, relative projectivity and being finitely generated are preserved by Morita equivalence, hence the proof is clear. □

A right R-module M is called CS if every closed submodule of M is a direct summand of M and a ring R is called *right* CS if R_R is CS. A ring R is called *right* Σ-CS (*right finitely* Σ-CS) if every (finite) direct sum of copies of R_R is CS (the reader might consult [10]).

Proposition 3.4. *Let R be a right NR-ring. Then the following hold:*

(1) *Finitely generated nonsingular right R-modules are projective.*
(2) *All nonsingular right R-modules are flat.*
Moreover, if R is right nonsingular, then
(3) *R is right finitely Σ-CS,*
(4) *R is right and left semihereditary.*

Proof: (1) Assembling the right NR-ring assumption and [2, Lemma 2.1] which states that finitely generated R-projective right R-modules are projective, we are done.

(2) Let M be a nonsingular right R-module and N be a finitely generated submodule of M. Since N is nonsingular, it is projective by (1). We conclude that M is flat by the fact that every module is a direct limit of its finitely generated submodules and the direct limit of projective modules is flat.

(3) Let K be a closed submodule of $R^{(n)}$. Then, $R^{(n)}/K$ is nonsingular by Lemma 2.1 (2), and so projective by (1). Therefore, the sequence $0 \to K \to R^{(n)} \to R^{(n)}/K \to 0$ splits, i.e., K is a direct summand of $R^{(n)}$ which in turn yields that R is right finitely Σ-CS.

(4) This follows from (3) and [10, Chapter 4, 12.17]. □

Now, we are ready to give a characterization of right NR-rings.

Theorem 3.5. *Let R be a right nonsingular ring. The following statements are equivalent:*

(1) *R is a right NR-ring and R_R is of finite Goldie rank.*
(2) *R is semihereditary, right A-perfect, right CS and $E(R_R)$ is flat.*
(3) *R is right finitely Σ-CS and right A-perfect.*

If any of these statements is satisfied, then the classes of all flat right R-modules and all nonsingular right R-modules coincide.

Proof: (1) $\Rightarrow$ (2) By Proposition 2.6, we have that all flat right R-modules are nonsingular. Therefore, all flat right R-modules become R-projective by the right NR-ring assumption and so R is right A-perfect. In addition, by Proposition 3.4 (4) R is both right and left semihereditary, and by Proposition 3.4 (2) $E(R_R)$ is flat. To show that R is right CS, let I be a closed right ideal of R. By Lemma 2.1 (2) R/I is nonsingular. Then, R/I is projective by Proposition 3.4 (1) which implies that I is a direct summand of R. Therefore, R is right CS.

(2) ⇒ (1) Let M be a nonsingular right R-module. The assumptions $E(R_R)$ is flat and R is left semihereditary imply that M is flat by Theorem 1.2. As R is right A-perfect, we see that M is R-projective, that is, R is a right NR-ring. Since every right A-perfect ring is semiperfect, there exist orthogonal idempotents $e_1, \ldots, e_n$ in R such that $R_R = e_1R \oplus \cdots \oplus e_nR$ and each e_iR is an indecomposable right R-module, see [23, Theorem 23.6]. From this, by applying the same arguments as in [9, Theorem 3.1], we conclude that R is of finite right Goldie rank.

(1) ⇒ (3) By Proposition 3.4 (3), R is right finitely Σ-CS. Moreover, since flat right R-modules are nonsingular, see Proposition 2.6, we obtain that all flat right R-modules are R-projective, that is, R is right A-perfect.

(3) ⇒ (1) Note that over right finitely Σ-CS rings, finitely generated nonsingular right R-modules are projective, see [10, Corollary 11.4]. By this fact, we obtain that nonsingular right R-modules are flat. Now, by the A-perfectness assumption, we have that R is right NR. For the remaining part, recall that right finitely Σ-CS rings are also right CS. Therefore, combining with being A-perfect, as in the proof of (2) ⇒ (1), we conclude that R is of finite right Goldie rank.

Now, for the last statement, suppose one of these conditions holds. Then, that nonsingular right R-modules are flat follows from Theorem 1.2 and the converse holds by Proposition 2.6. □

Remark 3.6. For the sake of simplicity, call a ring R right G-ring if all nonsingular right R-modules are flat. Clearly, if R is any right G-ring which is also right A-perfect, then R is a right NR-ring. By Theorem 3.5, the converse implication holds in the case when R is right nonsingular and of finite right Goldie rank. In other words, the NR-property of the ring in that particular case is just the conjunction of two known properties: of being a G-ring, and of being A-perfect.

As a consequence of Theorem 3.5, we have the following corollaries.

Corollary 3.7. *Let R be a right nonsingular ring. If R is a right A-perfect left semihereditary ring with $E(R_R)$ being flat, then R is a right NR-ring.*

The fact that $E(R_R)$ is flat over semiprime right and left Goldie rings, see [17, page 85, Exercise 23 and Corollary 3.32], together with [10, Corollary 12.18] give rise to the following corollary.

Corollary 3.8. *Let R be a semiprime right and left Goldie ring. Then, the following statements are equivalent:*

(1) *R is a right NR-ring.*
(2) *R is semihereditary and right A-perfect.*
(3) *R is finitely Σ-CS and right A-perfect.*

Semihereditary commutative local domains are valuation domains. Since A-perfect rings are semiperfect, and semiperfect domains are local, we have the following corollary.

Corollary 3.9. *Let R be a commutative domain. Then, the following statements are equivalent:*

(1) *R is NR.*
(2) *R is an A-perfect valuation domain.*

By the following example, we show that there are right NR-rings which are not right perfect.

Example 3.10. Let F be a field and $R = F[[x]]$ be the ring of formal power series in one indeterminate x. Then, R is a valuation domain and also, R is an A-perfect ring which is not perfect by [4, Example 3.11]. Thus, R is an NR-ring by Corollary 3.9.

In [1], the authors studied the rings whose injective right modules are R-projective. In the following proposition, we characterize when every nonsingular injective right R-module is R-projective over a right nonsingular ring of finite right Goldie rank.

Proposition 3.11. *Let R be a right nonsingular ring having finite right Goldie rank. Then, the following statements are equivalent:*

(1) *Every nonsingular injective right R-module is R-projective.*
(2) *$E(R_R)$ is R-projective.*
(3) *$E(R_R) = U_R \oplus V_R$, where U is projective and $\mathrm{Hom}(V, R/I) = 0$ for each right ideal I of R.*

Proof: (1) $\Rightarrow$ (2) It is clear since $E(R_R)$ is nonsingular and injective.

(2) $\Rightarrow$ (1) Let M be a nonsingular injective right R-module. Then, by [17, page 84, Exercise 5], M can be written as a direct sum of indecomposable injective right R-modules N_γ, i.e., $M = \bigoplus_{\gamma\in\Gamma} N_\gamma$. Now, let K_γ be a nonzero cyclic submodule of N_γ. Since K_γ's are nonsingular for each $\gamma \in \Gamma$, we see that K_γ is isomorphic to a submodule of $E(R_R)$, see [13, Lemma 4]. However, N_γ's are uniform. This implies that N_γ's can be embedded in $E(R_R)$, too. So, N_γ's are

direct summands of $E(R_R)$. Therefore, N_γ's are R-projective by the assumption, and then, using [5, Proposition 16.10] we obtain that M is R-projective.

(2) $\Rightarrow$ (3) $E(R_R)$ is of finite Goldie rank since R_R is of finite Goldie rank. Therefore, $E(R_R) = U_1 \oplus \cdots \oplus U_n$ where U_i's are indecomposable and injective right R-modules for $i = 1, \ldots, n$. Clearly, every U_i is R-projective. Now, we divide the proof into two cases:

Case 1: Let $\mathcal{U} = \{i \in \{1, \ldots, n\}\colon \operatorname{Hom}(U_i, R/I) \neq 0$ for some right ideal I of $R\}$ and $i \in \mathcal{U}$. Then, there exists a nonzero homomorphism $f\colon U_i \to R/I$. By the R-projectivity property of U_i, we have a nonzero homomorphism $g\colon U_i \to R$. As $\ker(g)$ is a closed submodule of the injective module U_i, $\ker(g)$ becomes a direct summand, and so $U_i = \ker(g) \oplus S$ for some submodule S of U_i. However, U_i is indecomposable and g is nonzero. Thus, we conclude that g is monic which means $U_i \cong g(U_i)$ is a direct summand of R_R. Therefore, U_i is projective, and so is $\bigoplus_{i\in\mathcal{U}} U_i$.

Case 2: Let $\mathcal{V} = \{i \in \{1, \ldots, n\}\colon \operatorname{Hom}(U_i, R/I) = 0$ for every right ideal I of $R\}$. This gives that $\operatorname{Hom}\left(\bigoplus_{i\in\mathcal{V}} U_i, R/I\right) = 0$ for each cyclic right R-module R/I.

(3) $\Rightarrow$ (2) Clearly, such U_R and V_R are R-projective. The rest follows from [5, Proposition 16.10]. □

We deduce the following corollary by the fact that for a right module M over a right Noetherian ring, $\operatorname{Rad}(M) = M$ if and only if $\operatorname{Hom}(M, R/I) = 0$ for each right ideal I of R.

Corollary 3.12. *Let R be a right nonsingular right Noetherian ring. Then, the following statements are equivalent:*

(1) *Nonsingular injective right R-modules are R-projective.*
(2) *$E(R_R)$ is R-projective.*
(3) *$E(R_R) = U_R \oplus V_R$, where U is projective and* $\operatorname{Rad}(V) = V$.

4. Right orthogonal class of nonsingular modules

In this section, $\mathcal{N}$ will denote the class of all nonsingular right R-modules and $\mathcal{N}^\perp = \left\{X \in \text{Mod-}R\colon \operatorname{Ext}^1_R(N, X) = 0 \text{ for all } N \in \mathcal{N}\right\}$ will represent the right orthogonal class of $\mathcal{N}$.

A right R-module C is said to be *cotorsion* (in the sense of Enochs) if $\operatorname{Ext}^1_R(F, C) = 0$ for every flat right R-module F.

Example 4.1. (1) Any injective right R-module M is contained in $\mathcal{N}^\perp$.

(2) Nonsingular right R-modules need not be flat in general. If R is right nonsingular, left semihereditary and $E(R_R)$ is flat or if R is right nonsingular

and right NR, then nonsingular right R-modules are flat, see Theorem 1.2 and Proposition 3.4, respectively. So, in these cases every cotorsion right R-module is contained in $\mathcal{N}^{\perp}$.

(3) Let R be a ring which is mentioned in Proposition 2.6. Then, every right R-module $M \in \mathcal{N}^{\perp}$ is cotorsion.

Corollary 4.2. *Let R be a right nonsingular ring. Then, the following are equivalent:*

(1) *R is left semihereditary and $E(R_R)$ is flat.*
(2) *All nonsingular right R-modules are flat.*
(3) *All cotorsion right R-modules are contained in $\mathcal{N}^{\perp}$.*

Proof: (1) $\Leftrightarrow$ (2) comes from Theorem 1.2 and (2) $\Rightarrow$ (3) follows from Example 4.1 (2). For (3) $\Rightarrow$ (2), let M be a nonsingular right R-module. Then, $\mathrm{Ext}^1_R(M, C) = 0$ for every cotorsion right R-module C, which means that M is flat. □

Having reminded the notion of covers in the introductory section, we now recall the dual notion of envelopes. For a class $\mathcal{X}$ of right R-modules, an *$\mathcal{X}$-preenvelope* of a right R-module M is a homomorphism $\varphi\colon M \to X$ with $X \in \mathcal{X}$ such that for any homomorphism $\varphi'\colon M \to X'$ with $X' \in \mathcal{X}$, there exists $f\colon X \to X'$ such that $f\varphi = \varphi'$. An $\mathcal{X}$-preenvelope $\varphi\colon M \to X$ is said to be an *$\mathcal{X}$-envelope* if every endomorphism f of X with $f\varphi = \varphi$ is an isomorphism. For the notions of special $\mathcal{X}$-precovers, special $\mathcal{X}$-preenvelopes and more on the subject we direct the reader to [12].

Remark 4.3. It is well known that the class $\mathcal{N}$ of all nonsingular right R-modules is closed under submodules, direct products, direct sums, essential extensions and module extensions. Over a right nonsingular ring of finite right Goldie rank, it is also closed under pure quotients by Corollary 2.4.

Lemma 4.4. *If R is a right nonsingular ring of finite right Goldie rank, then all right R-modules have an $\mathcal{N}$-cover and an $\mathcal{N}^{\perp}$-envelope. Besides, all right R-modules have a special $\mathcal{N}$-precover and a special $\mathcal{N}^{\perp}$-preenvelope.*

Proof: Since over a right nonsingular ring R of finite right Goldie rank all conditions of Theorem 1.4 are satisfied by Remark 4.3, we conclude that every right R-module has an $\mathcal{N}$-cover and $\mathcal{N}^{\perp}$-envelope. Observing the facts that the class $\mathcal{N}$ contains all projective right R-modules and the class $\mathcal{N}^{\perp}$ contains all injective right R-modules, we have that $\mathcal{N}$-covers are epic and $\mathcal{N}^{\perp}$-envelopes are monic. Also, it is clear that the class $\mathcal{N}^{\perp}$ is closed under module extensions. Thus, the remaining part follows from Wakamatsu's lemma, see, for example, [15, Lemma 2.1.13]. □

At this point, we emphasize that Lemma 4.4 does not extend to right nonsingular rings of infinite right Goldie rank which may be seen from the following example.

Example 4.5. Let R be the endomorphism ring of an infinite dimensional right vector space over a division ring, R is von Neumann regular, right self-injective, but not semisimple, see [17, Proposition 2.23]. Note that nonsingular right R-modules coincide with the flat Mittag–Leffler right R-modules by [19, Corollary 2.10 (i) and Example 6.8]. However, the class of all flat Mittag–Leffler right R-modules is not precovering by [27, Theorem 3.3].

Proposition 4.6. *Let R be a right nonsingular ring of finite right Goldie rank. Then, the following are equivalent:*

(1) *R is a right NR-ring.*
(2) *Nonsingular covers of finitely generated right R-modules are (finitely generated) projective.*
(3) *Nonsingular covers of cyclic right R-modules are (finitely generated) projective.*

Proof: (1) $\Rightarrow$ (2) Let R be a right NR-ring. Since R is a right nonsingular ring of finite right Goldie rank, R is right A-perfect, and flat right R-modules and nonsingular right R-modules coincide by Theorem 3.5. Now, (2) follows from [4, Theorem 3.7].

(2) $\Rightarrow$ (3) is clear.

(3) $\Rightarrow$ (1) This part of the proof is an analog of the proof of [4, Theorem 3.7, (e) $\Rightarrow$ (a)]. $\square$

Proposition 4.7. *Let R be a right nonsingular ring of finite right Goldie rank. Then, the following are equivalent:*

(1) *Every right R-module has an $\mathcal{N}^{\perp}$-envelope which is nonsingular.*
(2) *Every $M \in \mathcal{N}^{\perp}$ is injective.*
(3) *Every $M \in \mathcal{N}^{\perp}$ is nonsingular.*
(4) *R is semisimple.*

Proof: (4) $\Rightarrow$ (2), (4) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are clear.

(1) $\Rightarrow$ (4) Let M be a right R-module and $f\colon M \to L$ be its monic $\mathcal{N}^{\perp}$-envelope. Since L is nonsingular, M is also nonsingular. Hence, R is semisimple.

(2) $\Rightarrow$ (4) Let A be any right R-module. By Lemma 4.4, special $\mathcal{N}$-precovers exist and so there is a short exact sequence $0 \to M \to F \to A \to 0$ with $M \in \mathcal{N}^{\perp}$ and $F \in \mathcal{N}$. Then, by (2), M is injective, whence $A \in \mathcal{N}$. Therefore, R is semisimple. $\square$

In [24, Lemma 1.16], it was shown that for a projective right R-module M, if $M = P + K$, where P is a direct summand of M and K is a submodule of M, then there exists a submodule Q of K with $M = P \oplus Q$. Using the same method as in the proof of [3, Theorem 2.8], one can prove the following result.

Proposition 4.8. *A ring R is right NR if and only if for every nonsingular right R-module N, if $N = P + L$, where P is a finitely generated projective direct summand of N and L is a submodule of N, then $N = P \oplus K$ for some K in L.*

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References

[1] Alagöz Y., Büyükaşık E., *Max-projective modules*, J. Algebra Appl. **20** (2021), no. 6, Paper No. 2150095, 25 pages.
[2] Alhilali H., Ibrahim Y., Puninski G., Yousif M., *When R is a testing module for projectivity?*, J. Algebra **484** (2017), 198–206.
[3] Amini B., Amini A., Ershad M., *Almost-perfect rings and modules*, Comm. Algebra **37** (2009), no. 12, 4227–4240.
[4] Amini A., Ershad M., Sharif H., *Rings over which flat covers of finitely generated modules are projective*, Comm. Algebra **36** (2008), no. 8, 2862–2871.
[5] Anderson F. W., Fuller K. R., *Rings and Categories of Modules*, Graduate Texts in Mathematics, 13, Springer, New York, 1992.
[6] Bican L., *Precovers and Goldie's torsion theory*, Math. Bohem. **128** (2003), no. 4, 395–400.
[7] Cheatham T. J., *Finite dimensional torsion-free rings*, Pacific J. Math. **39** (1971), 113–118.
[8] Dinh H. Q., Holston C. J., Huynh D. V., *Quasi-projective modules over prime hereditary Noetherian V-rings are projective or injective*, J. Algebra **360** (2012), 87–91.
[9] Dung N. V., *A note on hereditary rings or non-singular rings with chain condition*, Math. Scand. **66** (1990), no. 2, 301–306.
[10] Dung N. V., Huynh D. V., Smith P. F., Wisbauer R., *Extending Modules*, Pitman Research Notes in Mathematics Series, 313, Longman Scientific & Technical, Harlow, John Wiley & Sons, New York, 1994.
[11] Durğun Y., *A generalization of C-rings*, Ege Uni. J. of Faculty of Sci. **37** (2013), no. 2, 6–15.
[12] Enochs E. E., Jenda O. M. G., *Relative Homological Algebra*, De Gruyter Expositions in Mathematics, 30, Walter de Gruyter & Co., Berlin, 2000.
[13] Faith C., *When are proper cyclics injective?*, Pacific J. Math. **45** (1973), 97–112.
[14] Faith C., *Algebra. II. Ring Theory*, Grundlehren der Mathematischen Wissenschaften, 191, Springer, Berlin, 1976.
[15] Göbel R., Trlifaj J., *Approximations and endomorphism algebras of modules*, De Gruyter Expositions in Mathematics, 41, Walter de Gruyter GmbH & Co., Berlin, 2006.
[16] Golan J. S., Teply M. L., *Torsion-free covers*, Israel J. Math. **15** (1973), 237–256.
[17] Goodearl K. R., *Ring Theory*, Nonsingular Rings and Modules Pure and Applied Mathematics, 33, Marcel Dekker, New York, 1976.
[18] Hamsher R. M., *Commutative, noetherian rings over which every module has a maximal submodule*, Proc. Amer. Math. Soc. **17** (1966), 1471–1472.

[19] Herbera D., Trlifaj J., *Almost free modules and Mittag–Leffler conditions*, Adv. Math. **229** (2012), no. 6, 3436–3467.

[20] Holm H., Jørgensen P., *Covers, precovers, and purity*, Illinois J. Math. **52** (2008), no. 2, 691–703.

[21] Ketkar R. D., Vanaja N., *R-projective modules over a semiperfect ring*, Canad. Math. Bull. **24** (1981), no. 3, 365–367.

[22] Lam T. Y., *Lectures on Modules and Rings*, Graduate Texts in Mathematics, 189, Springer, New York, 1999.

[23] Lam T. Y., *A First Course in Noncommutative Rings*, Graduate Texts in Mathematics, 131, Springer, New York, 2001.

[24] Nicholson W. K., *Semiregular modules and rings*, Canadian J. Math. **28** (1976), no. 5, 1105–1120.

[25] Sandomierski F. L., *Relative Injectivity and Projectivity*, Ph.D. Thesis, The Pennsylvania State University, ProQuest LLC, Ann Arbor, 1964.

[26] Sandomierski F. L., *Nonsingular rings*, Proc. Amer. Math. Soc. **19** (1968), 225–230.

[27] Šaroch J., *Approximations and the Mittag-Leffler conditions the tools*, Israel J. of Math. **226** (2018), no. 2, 737–756.

[28] Stenström B., *Rings of Quotients, An introduction to Methods of Ring Theory*, Die Grundlehren der mathematischen Wissenschaften, 217, Springer, New York, 1975.

[29] Teply M. L., *Torsionfree injective modules*, Pacific J. Math. **28** (1969), no. 2, 441–453.

[30] Teply M. L., *Torsion-free covers. II*, Israel J. Math. **23** (1976), no. 2, 132–136.

[31] Trlifaj J., *Faith's problem on R-projectivity is undecidable*, Proc. Amer. Math. Soc. **147** (2019), no. 2, 497–504.

[32] Trlifaj J., *The dual Baer criterion for non-perfect rings*, Forum Math. **32** (2020), no. 3, 663–672.

[33] Turnidge D. R., *Torsion theories and semihereditary rings*, Proc. Amer. Math. Soc. **24** (1970), 137–143.

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