Some results on the class of σ-unbounded Dunford-Pettis operators

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Abstract. We introduce and study the class of unbounded Dunford–Pettis operators. As consequences, we give basic properties and derive interesting results about the duality, domination problem and relationship with other known classes of operators.

Keywords: σ-un-Dunford–Pettis operator; unbounded norm convergence; order continuous Banach lattice; atomic Banach lattice; relatively sequentially un-compact set; Schur property

Classification: 46B42, 47B60, 47B65

1. Introduction

In this paper the term operator means a bounded linear mapping.

The notion of unbounded order convergence was appeared in [10] and was studied in many several papers [6], [5], [12]. Our interest focus on unbounded norm convergence in Banach lattices, which is a recent extension of unbounded order convergence investigated in many works such as [4], [8], [7].

The class of Dunford–Pettis operators is among the most extensively studied in literature about operators acting on Banach lattices. In this note we present a generalization of the Dunford–Pettis operators by introducing and studying a new classe of operators. Indeed, using the unbounded norm convergence and unbounded norm topology in Banach lattices, we introduce the concept of sequentailly unbounded Dunford–Pettis operators (for short σ-un-Dunford–Pettis) and we investigate the following facts:

- Properties of this class of operators.
- Domination property.
- Duality property.
- Relationships with other known classes of operators.

DOI 10.14712/1213-7243.2021.035

2. Preliminaries

Let us recall that a linear mapping $T\colon X \longrightarrow Y$ between two Banach spaces X and Y is said to be Dunford–Pettis, whenever $x_n \xrightarrow{\text{w}} 0$ in X implies $\|T(x_n)\| \to 0$. Alternatively, T is Dunford–Pettis if and only if T carries relatively weakly compact sets into compact sets. A linear mapping $T\colon X \longrightarrow Y$ between two Banach spaces X and Y is said to be weakly compact, whenever T carries the closed unit ball of X to a weakly relatively compact subset of Y. An operator $T\colon E \longrightarrow X$ from a Banach lattice E to a Banach space X is said to be M-weakly compact if $\|T(x_n)\| \to 0$ for each norm bounded disjoint sequence (x_n) of E. An operator $T\colon E \longrightarrow X$ from a Banach lattice E to a Banach space X is order weakly compact whenever $T[0,x]$ is a relatively weakly compact subset of X for each $x \in E^+$, equivalently, $\|T(x_n)\| \to 0$ for each order bounded disjoint sequence (x_n) of E.

To state our results, we need to fix some notations and recall some definitions. A Banach lattice is a Banach space $(E, \|\cdot\|)$ such that E is a Riesz space and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \le |y|$, we have $\|x\| \le \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. A Banach lattice E is order continuous, if for each net (x_α) such that $x_\alpha \downarrow 0$ in E, the sequence (x_α) converges to 0 for the norm $\|\cdot\|$, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing, its infimum exists and $\inf(x_\alpha) = 0$.

A Riesz space E is Dedekind σ-complete if every majorized countable nonempty subset of E has a supremum.

A Banach lattice E is said to be KB-space, if every increasing norm bounded sequence of E^+ is norm convergent.

A nonzero element x of a Riesz space E is discrete if the order ideal generated by x equals the vector subspace generated by x. The Riesz space E is atomic, if it admits a complete disjoint system of discrete elements.

The lattice operations in E are weakly sequentially continuous, if the sequence $(|x_n|)$ converges to 0 in the weak topology, whenever the sequence (x_n) converges weakly to 0 in E.

A vector subspace G of a Riesz space X is majorizing the Riesz space E, whenever for each $x \in E$ there exists some $y \in G$ with $x \le y$, equivalently, if for each $x \in E$ there exists some $y \in G$ with $y \le x$.

A band B in a Riesz space X that satisfies $X = B \oplus B^d$ is called a projection band, where B^d stands for the disjoint complement of B.

A vector $e > 0$ in a Riesz space X is said to be order unit whenever for each $x \in X$ there exists some $\lambda > 0$ satisfying $x \le \lambda e$.

A positive element u of a normed Riesz space is called quasi-interior point if the ideal E_u generated by u is norm dense.

A Banach lattice E has the Schur (or positive Schur) property whenever weak-null sequences (x_n) of E (of E^+, respectively) are norm-null.

A net (x_α) in a Riesz space X is *unbounded order convergent to* x (abbreviation: uo-convergent) if $|x_\alpha - x| \wedge u$ converges to 0 in order (abbreviation: $x_\alpha \xrightarrow{\text{uo}} x$) for each $u \in X^+$.

A net (x_α) in a Banach lattice E is *unbounded norm convergent to* x (abbreviation: un-norm convergent) if $\||x_\alpha - x| \wedge u\| \longrightarrow 0$ (abbreviation: $x_\alpha \xrightarrow{\text{un}} x$) for each $u \in E^+$. Note that the norm convergence implies the un-norm convergence. We can easily check, that un-convergence coincides with norm convergence on a Banach lattice with order unit, in particular for order bounded nets.

If T is a linear mapping from a Banach space X into a Banach space Y then, its adjoint operator T' is defined from Y' into X' by $T'(f)(x) = f(T(x))$ for each $f \in Y'$ and for each $x \in X$. We refer the reader to [1] for unexplained terminology on Banach lattice theory.

3. Main results

We start by the following definitions:

Definition 3.1. An operator T from a Banach space X into a Banach lattice F is said to be *sequentially unbounded Dunford–Pettis* (abbreviation: σ-un-Dunford–Pettis) whenever $(T(x_n))$ is un-norm null for each weak-null sequence (x_n) in E.

The class of σ-un-Dunford–Pettis operators will be denoted by $DP_{\sigma-un}(X,F)$.

As an immediate consequence of Proposition 6.2 in [4], we have the following result:

Proposition 3.1. *Let E be a Banach lattice. Then the identity operator* Id_E: $E \longrightarrow E$ *is σ-un-Dunford–Pettis if and only if E is order continuous and atomic.*

We recall from [7] that a subset A of a Banach lattice E is said to be *un-compact* (or *sequentially un-compact*), if every net (x_α) (or every sequence (x_n)) in A has a subnet (a subsequence, respectively) which is un-norm convergent.

In the following result, we give an important characterization of σ-un-Dunford–Pettis operators.

Proposition 3.2. *Let X be a Banach space, F be a Banach lattice and T be an operator from X into F. Then the following statements are equivalent:*

1. *T is σ-un-Dunford–Pettis.*
2. *$T(A)$ is relatively sequentially un-compact for each weakly compact set A of X.*

Proof: (1) $\Longrightarrow$ (2) Let $A \subset X$ be a weakly compact set, we will show that $T(A)$ is relatively sequentially un-compact. To this end, let $y_n \in T(A)$, so that

$y_n = T(x_n)$ for some $(x_n) \in A$. Now, as A is weakly compact, by the Eberlian–Smulian theorem there exists a subsequence (x_{n_k}) of (x_n) such that $(x_{n_k}) \xrightarrow{\text{w}} y$ for some $y \in X$, we may assume that $y = 0$. As T is σ-un-Dunford–Pettis, then $T(x_{n_k}) \xrightarrow{\text{un}} 0$, this shows that $T(A)$ is a relatively sequentially un-compact set of F.

(2) $\Longrightarrow$ (1) Let (x_n) be a weak-null sequence in X and $(n_k)_k$ is any increasing sequence of positive integers. Then, notice that $(x_{n_k})_k$ is still a weak-null sequence. Hence if we set $A = \{x_{n_k} : k \in \mathbb{N}\} \cup \{0\}$ clearly A is a weakly compact set of X and by assumption $T(A)$ is a relatively sequentially un-compact set in F. Hence, there exists a subsequence $(T(x_{n_{k'}}))$ of $(T(x_{n_k}))$ such that $T(x_{n_{k'}}) \xrightarrow{\text{un}} y$ for some $y \in X$. Uniqueness of limit implies that $y = 0$, so that $T(x_{n_{k'}}) \xrightarrow{\text{un}} 0$. Thus, given any subsequence $(T(x_{n_k}))$ of $(T(x_n))$, there exists a subsequence of this subsequence which is un-norm null. This implies that $T(x_n) \xrightarrow{\text{un}} 0$ and shows that T is σ-un-Dunford–Pettis. □

Remark 3.1. Each Dunford–Pettis operator is σ-un-Dunford–Pettis, but the converse does not hold in general. Indeed, the identity operator Id_{c_0} of the Banach lattice c_0 is σ-un-Dunford–Pettis (since c_0 is order continuous and atomic) but fails to be Dunford–Pettis.

Remark 3.2. We can check easily that the set of all σ-un-Dunford–Pettis operators between a Banach space X and a Banach lattice F is a closed linear subspace of $L(X, F)$.

For further results, we need to focus on other concepts around unbounded norm convergence and unbounded norm topology. Namely, it seems reasonable to introduce the notion of continuity related to these concepts and which will be used in the next results.

Definition 3.2 ([11])**.** Let E and F be two Banach lattices.

(1) An operator $T\colon E \longrightarrow F$ is called unbounded norm continuous (abbreviation: un-continuous), if $x_\alpha \xrightarrow{\text{un}} 0$ in E implies $T(x_\alpha) \xrightarrow{\text{un}} 0$ in F.
(2) An operator $T\colon E \longrightarrow F$ is called σ-unbounded norm continuous (abbreviation: σ-un-continuous), if $x_n \xrightarrow{\text{un}} 0$ in E implies $T(x_n) \xrightarrow{\text{un}} 0$ in F.

Remark 3.3. We should mention that unbounded continuity is different from the norm continuity. Indeed, we consider the canonical injection $i\colon c_0 \longrightarrow l^\infty$ which is a norm continuous operator, but by considering Example 2.6 in [4] we can check that the canonical injection $i\colon c_0 \longrightarrow l^\infty$ fails to be un-continuous.

Proposition 3.3. *Let E and F be two Banach lattices. If $T\colon E \longrightarrow F$ is a lattice homomorphism such that its range is norm-dense or majorizing or projection* *band, then T is an un-continuous operator. In particular, if $T\colon E \longrightarrow F$ is an onto lattice homomorphism, then T is an un-continuous operator.*

Proof: Let (x_α) be a net in E such that $x_\alpha \xrightarrow{\text{un}} 0$ and $v \in (T(E))^+ = T(E^+)$ (since T is lattice homomorphism), then there exists $u \in E^+$ such that $T(u) = v$, in particular we have $|x_\alpha| \wedge u \xrightarrow{\|\cdot\|} 0$. On the other hand, we have the equality

$$|T(x_\alpha)| \wedge v = T(|x_\alpha|) \wedge T(u) = T(|x_\alpha| \wedge u) \xrightarrow{\|\cdot\|} 0 \qquad \text{for all } u \in E^+.$$

Thus, $T(x_\alpha) \xrightarrow{\text{un}} 0$ in $T(E)$. As T is a lattice homomorphism, we have that $T(E)$ is a sublattice of F and hence the result follows from Theorem 4.3 in [8]. □

Our next interest is to determine whether the set of σ-un-Dunford–Pettis operators forms a two-sided ideal in the class of all operators. Note that the class of $DP_{\sigma-un}(X,F)$ is a right ideal, but unfortunately it is not a left ideal. Indeed, if we consider the composed operator;

$$c_0 \xrightarrow{\text{Id}_{c_0}} c_0 \xrightarrow{i} l^\infty$$

where $i\colon c_0 \longrightarrow l^\infty$ is the canonical injection, we check easily that $i \circ \text{Id}_{c_0} = i$ is not σ-un-Dunford–Pettis. However, the identity operator Id_{c_0} of c_0 is σ-un-Dunford–Pettis.

Nevertheless, we have the following result;

Proposition 3.4. *Let X be a Banach space, and G, F be Banach lattices. If $T\colon G \longrightarrow F$ is a σ-un-continuous operator, then for each σ-un-Dunford–Pettis operator $S\colon X \longrightarrow G$ the composed operator $T \circ S$ is σ-un-Dunford–Pettis.*

Proof: Let (x_n) be weak-null sequence in E. Since S is a σ-un-Dunford–Pettis operator, we have that $S(x_n) \xrightarrow{\text{un}} 0$. Using the fact that T is σ-un continuous, it follows that $T \circ S(x_n) \xrightarrow{\text{un}} 0$, and therefore $T \circ S$ is σ-un-Dunford–Pettis. □

Note that an order weakly compact operator is not in general σ-un-Dunford–Pettis. Indeed, the canonical injection $i\colon c_0 \longrightarrow l^\infty$ is order weakly compact but not σ-un-Dunford–Pettis. Indeed, by Example 2.6 in [4] $i(e_n) = e_n \overset{un}{\nrightarrow} 0$ in l^∞.

Proposition 3.5. *Let E and F be two Banach lattices and $T\colon E \longrightarrow F$ an order bounded operator. If T is σ-un-Dunford–Pettis, then T is order weakly compact.*

Proof: Let $(x_n)_n$ be an order bounded weak-null sequence in E^+. Since T is σ-un-Dunford–Pettis it follows that $T(x_n) \xrightarrow{\text{un}} 0$. By order boundedness of T we have $(T(x_n))_n$ is an order bounded sequence and then $T(x_n) \xrightarrow{\|\cdot\|} 0$. Hence, it follows from Corollary 3.4.9 in [9] that T is an order weakly compact operator. □

Remark 3.4. The condition "$T\colon E \longrightarrow F$ is an order bounded operator" is essential in Proposition 3.5. Indeed, we choose a non-weakly compact operator $T\colon E \longrightarrow F$ such that F is a non-reflexive atomic order continuous Banach lattice (like c_0) and E is a Banach lattice with order unit. Since F is an atomic order continuous Banach lattice, then the operator T is σ-un-Dunford–Pettis. On the other hand, E has order unit and since T is a non-weakly compact operator, T is necessary not order weakly compact and it is also not order bounded operator. Otherwise, since B_E is order bounded, where B_E is the closed unit ball of E, then $T(B_E)$ is order bounded in F which is order continuous. It follows from Theore 4.9 in [1] that $T(B_E)$ is weakly compact, that is, T is a weakly compact operator and this is a contradiction.

Proposition 3.6. *Let E and F be Banach lattices such that E has quasi-interior point and the lattice operations of E are weakly sequentially continuous. If $T\colon E \longrightarrow F$ is a lattice homomorphism with norm-dense range, then the following statements are equivalent:*

(1) *T is order weakly compact;*
(2) *T is σ-un-Dunford–Pettis.*

Proof: (1) $\Longrightarrow$ (2) Let (x_n) be a weak-null sequence and e a quasi-interior point of E. Since the lattice operations of E are weakly sequentially continuous, we have $|x_n| \wedge e$ is weak-null and it is a positive order bounded sequence. Now, as T is order weakly compact it follows from Corollary 3.4.9 in [9] that $T(|x_n| \wedge e) \xrightarrow{\|\cdot\|} 0$, on the other hand, T is a lattice homomorphism, hence we have the following equality

$$|T(x_n)| \wedge T(e) = T(|x_n|) \wedge T(e) = T(|x_n| \wedge e).$$

So, $|T(x_n)| \wedge Te \xrightarrow{\|\cdot\|} 0$. Now, by Exercise 11 of Section 15 in [1] $T(e)$ is a quasi-interior point of F and from Lemma 2.11 in [4] we infer that $(T(x_n))$ is un-null, therefore T is σ-un-Dunford–Pettis.

(2) $\Longrightarrow$ (1) Since each lattice homomorphism $T\colon E \longrightarrow F$ is order bounded, the implication follows from Proposition 3.5. ☐

Now, we are in position to give some results about the modulus of a σ-un-Dunford–Pettis operator. Firstly, we note that the first example given in remarks in [3] is a σ-un-Dunford–Pettis operator (by Proposition 3.1 and the fact that $DP_{\sigma-un}(X, F)$ is a right ideal) without modulus. And the second example given in remarks in [3] is a σ-un-Dunford–Pettis operator (because it is a Dunford–Pettis operator) such that its modulus exists but it is not σ-un-Dunford–Pettis (by Proposition 3.5).

Proposition 3.7. *Let E and F be two Banach lattices and let $T\colon E \longrightarrow F$ be an order bounded σ-un-Dunford–Pettis operator. If T is a disjointness preserving operator, then T possesses a modulus which is σ-un-Dunford–Pettis.*

Proof: It follows from Theorem 2.40 in [1] that the modulus of T exists and we have $||T|(x_n)| \wedge u \leq ||T|(|x_n|)| \wedge u = |T(x_n)| \wedge u$. As T is σ-un-Dunford–Pettis, it follows that $|T|$ is σ-un-Dunford–Pettis. □

We should mention that the domination problem for the class of σ-un-Dunford–Pettis operators is not verified. Indeed, we consider the example mentioned in [1]. Let $S, T\colon L_1[0,1] \longrightarrow l^\infty$ be two positive operators defined by

$$\begin{array}{rccl} S\colon & L_1[0,1] & \longrightarrow & l^\infty \\ & f & \longmapsto & \left(\int_0^1 f(x) r_1^+(x)\,\mathrm{d}x, \int_0^1 f(x) r_2^+(x)\,\mathrm{d}x, \dots\right) \end{array}$$

and

$$\begin{array}{rccl} T\colon & L_1[0,1] & \longrightarrow & l^\infty \\ & f & \longmapsto & \left(\int_0^1 f(x)\,\mathrm{d}x, \int_0^1 f(x)\,\mathrm{d}x, \dots\right) \end{array}$$

where (r_n) is the sequence of Rademacher functions on $[0,1]$. We can easily check that $0 \leq S \leq T$ and that T is a compact operator, hence it is σ-un-Dunford–Pettis operator. On the other hand, we have $r_n \xrightarrow{\mathrm{w}} 0$ in $L_1[0,1]$. It is observed also that $\|S(r_n)\|_\infty \geq \int_0^1 r_n(x) r_n^+(x)\,\mathrm{d}x = 1/2$. Now, we remind that norm convergence and unbouded norm convergence coincide in Banach lattices with order unit. Therefore, $S(r_n) \overset{\mathrm{un}}{\nrightarrow} 0$ and this shows that S is not σ-un-Dunford–Pettis operator.

But under some conditions, we find positive solution as the next result shows.

Proposition 3.8. *Let E and F be two Banach lattices and $S, T\colon E \longrightarrow F$ be two positive operators such that $0 \leq S \leq T$. The class of positive σ-un-Dunford–Pettis operators satisfies the domination property if one of the following statements is valid:*

(1) *T is a lattice homomorphism.*
(2) *The lattice operations of E are weakly sequentially continuous.*

In particular, under the same sufficient conditions, if the modulus of the operator T exists and it is σ-un-Dunford–Pettis, then T is also σ-un-Dunford–Pettis.

Proof: (1) Let S and T be two operators from E into F such that $0 \leq S \leq T$ and T is σ-un-Dunford–Pettis and let (x_n) be a weak-null sequence, we have that $|S(x_n)| \wedge u \leq T(|x_n|) \wedge u$. As the lattice operations of E are weak sequentially continuous, it follows that $|x_n| \xrightarrow{\mathrm{w}} 0$ and as T is σ-un-Dunford–Pettis, we have

$T(|x_n|) \wedge u = |T(|x_n|)| \wedge u \xrightarrow{\|\cdot\|} 0$. So, we can conclude that S is a σ-un-Dunford–Pettis operator.

(2) Let S and T be two operators from E into F such that $0 \leq S \leq T$ and T is σ-un-Dunford–Pettis and let (x_n) be a weak-null sequence in E. Since T is σ-un-Dunford–Pettis, it follows that $|T(x_n)| \wedge u \xrightarrow{\|\cdot\|} 0$ for all $u \in F^+$. On the other hand, T is a lattice homomorphism, so $|T(x_n)| = T(|x_n|)$ and $|S(x_n)| \wedge u \leq S(|x_n|) \wedge u \leq T(|x_n|) \wedge u = |T(x_n)| \wedge u \xrightarrow{\|\cdot\|} 0$ for each $u \in F^+$. Therefore S is a σ-un-Dunford–Pettis operator. □

We are in position to give a new result about M-weakly compact operators in terms of sequentially un-compact sets.

Proposition 3.9. *Let E be a Banach lattice and Y be a Banach space. If $T\colon E \longrightarrow Y$ is an M-weakly compact operator, then $T(A)$ is a relatively compact set in Y for each relatively sequentially un-compact set A in E.*

Proof: Let A be a relatively sequentially un-compact set in E and let $(y_n) \subset T(A)$, then $y_n = T(x_n)$ for some sequence (x_n) in A. Now, as A is relatively sequentially un-compact, it follows that (x_n) has a subsequence (x_{n_k}) which is un-convergent to some $x \in E$. Using Theorem 3.2 from [4] there exist a subsequence $(x_{n_{k''}})$ of (x_{n_k}) and a bounded disjoint sequence $(g_{k''})$ in E such that $x_{n_{k''}} - g_{k''} \xrightarrow{\|\cdot\|} 0$. On the other hand, since T is M-weakly compact, we have $T(g_{k''}) \xrightarrow{\|\cdot\|} 0$, we infer that $T(x_{n_{k''}}) \xrightarrow{\|\cdot\|} 0$, therefore $T(A)$ is a relatively compact set of Y. □

As a consequence of the above proposition, we have the following result.

Proposition 3.10. *Let X, Y be Banach spaces and F be a Banach lattice and let $S_1\colon X \longrightarrow F$ and $S_2\colon F \longrightarrow Y$ be operators, where S_1 is σ-un-Dunford–Pettis and S_2 is M-weakly compact. Then the product operator $S_2 \circ S_1$ is a Dunford–Pettis operator.*

Proof: Let A be a relatively weakly compact set in X. Since S_1 is σ-un-Dunford–Pettis it follows from Proposition 3.2 that $S_1(A)$ is a relatively sequentially un-compact set in F. Now, as S_2 is M-weakly compact it follows from Proposition 3.9 that $S_2 \circ S_1(A)$ is a relatively compact set in Y and hence $S_2 \circ S_1$ is a Dunford–Pettis operator. □

The following result is a simple consequence of the above proposition.

Corollary 3.1. *Let E and F be Banach lattices such that E is atomic and order continuous. Then, each M-weakly compact operator $T\colon E \longrightarrow F$ is Dunford–Pettis.*

Proof: It follows from Proposition 3.2 and Proposition 3.10. □

The class of σ-un-Dunford–Pettis operators does not satisfy the duality property. That is, there exists a σ-un-Dunford–Pettis operator whose adjoint is not σ-un-Dunford–Pettis. In fact, the identity operator Id_{l_1} of the Banach lattice l^1 is σ-un-Dunford–Pettis, but its adjoint which is the identity operator Id_{l^∞} of the Banach lattice l^∞ is not σ-un-Dunford–Pettis.

In the following result, we give sufficient and necessary conditions under which the direct duality of σ-un-Dunford–Pettis operators is satisfied.

Theorem 3.1. *Let E and F be two Banach lattices such that E' is atomic. Then, the following statements are equivalent:*

(1) *The adjoint of each σ-un-Dunford–Pettis operator $T\colon E \longrightarrow F$ is σ-un-Dunford–Pettis.*
(2) *E' is order continuous or F' has the Schur property.*

Proof: (2) $\Longrightarrow$ (1) If F' has the Schur property, then the operator $T'\colon F' \longrightarrow E'$ is Dunford–Pettis and hence it is σ-un-Dunford–Pettis. On the other hand, if E' is order continuous and atomic, by Proposition 2.6 in [4] and Proposition 3.4 we infer that $T'\colon F' \longrightarrow E'$ is a σ-un-Dunford–Pettis operator.

(1) $\Longrightarrow$ (2) Assume that E' is not order continuous and F' does not satisfy the Schur property, then there exists an order bounded disjoint sequence (g_n) in E' such that $\|g_n\| = 1$ for all $n \in \mathbb{N}$ and $|g_n| \le g$ for some $g \in (E')^+$. Also there exists $f_n \xrightarrow{w} 0$ in F' such $\|f_n\| = 1$ for each n, so we can find $(y_n) \in B_F$ such that $|f_n(y_n)| > 1/2$ for each $n \in \mathbb{N}$. Now, we consider the following operators,

$$\begin{array}{rccc} S_1\colon & l^1 & \longrightarrow & F \\ & (\lambda_n)_n & \longmapsto & \displaystyle\sum_{n=1}^{\infty} \lambda_n y_n \end{array}$$

and

$$\begin{array}{rccc} S_2\colon & E & \longrightarrow & l^1 \\ & x & \longmapsto & (g_n(x))_{n=1}^{\infty}. \end{array}$$

Then S_1 and S_2 are well-defined operators, and the product operator $T = S_1 \circ S_2$ is σ-un-Dunford–Pettis, but its adjoint operator is not σ-un-Dunford–Pettis. Indeed, let $\varphi = g/2 \in (E')^+$, we have

$$\begin{aligned} |T'(f_n)| \wedge \varphi &= \Big|\sum_{n=1}^{\infty} f_n(y_n) g_n\Big| \wedge \varphi = \Big|\bigvee_{n=1}^{\infty} f_n(y_n) g_n\Big| \wedge \varphi \\ &\ge |f_n(y_n)||g_n| \wedge \varphi \ge \frac{1}{2}|g_n| \wedge \frac{1}{2} g = \frac{1}{2}|g_n| \end{aligned}$$

then

$$\||T'(f_n)| \wedge \varphi\| \ge \frac{1}{2}\|g_n\| = \frac{1}{2},$$

which shows that $T'(f_n) \overset{\text{un}}{\nrightarrow} 0$ and hence T' is not σ-un-Dunford–Pettis operator, this makes a contradiction and ends the proof. □

Also, the reciprocal duality is not satisfied in the class of σ-un-Dunford–Pettis operators. Indeed, the canonical injection $i\colon c_0 \longrightarrow l^\infty$ is not σ-un-Dunford–Pettis, however its adjoint operator $i'\colon (l^\infty)' \longrightarrow l^1$ is σ-un-Dunford–Pettis (since it is a Dunford–Pettis operator).

Theorem 3.2. *Let E and F be Banach lattices such that the lattice operations of E are weakly sequentially continuous and F is atomic. Then the following statements are equivalent:*

(1) *Each operator $T\colon E \longrightarrow F$ is σ-un-Dunford–Pettis whenever its adjoint operator $T'\colon F' \longrightarrow E'$ is σ-un-Dunford–Pettis.*
(2) *One of the following conditions is valid:*
  (a) *E has the positive Schur property;*
  (b) *F is order continuous.*

Proof: (2) (a) $\Longrightarrow$ (1) If E has the positive Schur property and the lattice operations in E are weakly sequentially continuous, then E has the Schur property and hence $T\colon E \longrightarrow F$ is a σ-un-Dunford–Pettis operator.

(2) (b) $\Longrightarrow$ (1) Since F is atomic and order continuous, it follows from Proposition 2.6 in [4] and Proposition 3.4 that T is σ-un-Dunford–Pettis.

(1) $\Longrightarrow$ (2) Assume that F is not order continuous and E does not have the positive Schur property, then by Theorem 2.4.2 in [9] there exists an order bounded disjoint sequence (y_n) in F^+ such that $\|y_n\| = 1$ for all $n \in \mathbb{N}$ and there is an element $y \in F_+$ such that $0 \leq y_n \leq y$. We consider the operator

$$\begin{array}{rccc} R\colon & c_0 & \longrightarrow & F \\ & (\lambda_n)_n & \longmapsto & \displaystyle\sum_{n=1}^{\infty} \lambda_n y_n. \end{array}$$

Using some arguments from the proof of Theorem 117.1 in [13], we can check that the operator R is well defined. On the other hand, since E does not have the positive Schur property, there exists a disjoint weakly null sequence (x_n) in E^+ such that $\|x_n\| = 1$ for all n, hence by Lemma 3.4 in [2] there exists a positive disjoint sequence (g_n) in $(E')^+$ with $\|g_n\| \leq 1$ such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ if $n \neq m$. We consider the operator

$$\begin{array}{rccc} S\colon & E & \longrightarrow & c_0 \\ & x & \longmapsto & (g_n(x))_{n=0}^{\infty}, \end{array}$$

clearly S is well defined.

Let $T = R \circ S$. we have

$$\begin{aligned} \||T(x_n)| \wedge y\| &= \||R \circ S(x_n)| \wedge y\| = \||R(e_n)| \wedge y\| \\ &= \||y_n| \wedge y\| = \|y_n\| = 1 \qquad \forall\, n \in \mathbb{N}. \end{aligned}$$

Therefore T fails to be σ-un-Dunford–Pettis. However, its adjoint operator T' is σ-un-Dunford–Pettis. □

For the next results, we need to recall the definition of sequentially un-compact operators.

Definition 3.3 ([8])**.** Let X be a Banach space and F be a Banach lattice. An operator $T\colon X \longrightarrow F$ is said to be *sequentially un-compact* if $T(B_X)$ is a relatively sequentially un-compact set of F, equivalently, the image of every bounded sequence (x_n) in X has a subsequence which is un-convergent.

Proposition 3.11. *Each sequentially un-compact operator $T\colon X \longrightarrow F$ from a Banach space X into a Banach lattice F is σ-un-Dunford–Pettis.*

Proof: Let $A \subset X$ be a weakly compact set, hence A is a bounded set of X. On the other hand, since T is sequentially un-compact we have $T(A)$ is a relatively sequentially un-compact set of F. Thus, T is σ-un-Dunford–Pettis operator. □

Note that the converse of the previous result does not hold. Indeed, if we consider the identity operator Id_{c_0} of c_0, it is clear that Id_{c_0} is σ-un-Dunford–Pettis (since c_0 is atomic and order continuous), but fails to be sequentially un-compact (since c_0 it is not KB-space, see Theorem 7.5 in [8].

From Proposition 3.2 we have the following result:

Corollary 3.2. *If X is a reflexive Banach lattice then for every Banach lattice F, each σ-un-Dunford–Pettis operator $T\colon X \longrightarrow F$ is sequentially un-compact.*

In the following result, we characterize Banach lattices under which each σ-un-Dunford–Pettis operator is sequentially un-compact.

Proposition 3.12. *Let E and F be Banach lattices such that E is a KB-space. The following statements are equivalent:*

(1) *Each σ-un-Dunford–Pettis operator $T\colon X \longrightarrow F$ is sequentially un-compact.*

(2) *One of the conditions holds:*
  - (a) *E reflexive;*
  - (b) *F is an atomic KB-space.*

Proof: (2) (b) $\Longrightarrow$ (1) Assume that F is an atomic KB-space, we infer by Proposition 9.1 from [8] that $T\colon X \longrightarrow F$ is a sequentially un-compact operator.

(2) (a) $\Longrightarrow$ (1) It follows from Corollary 3.2.

(1) $\Longrightarrow$ (2) Suppose that (2) does not hold, then neither F is an atomic KB-space nor E is reflexive. It follows from Theorem 7.5 in [8] that there is a bounded sequence (y_n) in F which does not have any un-convergent subsequence. We may consider the operator;

$$\begin{array}{rrcl} S\colon & l^1 & \longrightarrow & F \\ & (\lambda_n)_n & \longmapsto & \displaystyle\sum_{n=1}^{\infty}\lambda_n y_n \end{array}$$

which is clearly σ-un-Dunford–Pettis. On the other hand, by Theorem 2.4.15 in [9] E' is not order continuous and it follows from Theorem 2.4.14 in [9] that E contains a sublattice isomorphic copy of l^1 and there exist a positive projection $P\colon E \longrightarrow l^1$ and a canonical injection $i\colon l^1 \longrightarrow E$. Now, if we consider the composed operator $T = S \circ P\colon E \longrightarrow l^1 \longrightarrow F$, clearly T is σ-un-Dunford–Pettis and by our hypothesis T will be sequentially un-compact. So $T \circ i = S \circ P \circ i = S$ is sequentially un-compact, thus $S(e_n) = y_n$ has an un-convergent subsequence, this makes a contradiction and completes the proof. □

We end by the following question.

Question. If we limit ourselves in the previous definition (Definition 3.1) to linear mappings, is each σ-un-Dunford–Pettis operator bounded?

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(*Received* July 8, 2020, *revised* March 11, 2021)