Further generalized versions of Ilmanen's lemma on insertion of $C^{1,\omega}$ or $C^{1,\omega}_{\rm loc}$ functions

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Abstract. The author proved in 2018 that if G is an open subset of a Hilbert space, $f_1, f_2 \colon G \to \mathbb{R}$ continuous functions and ω a nontrivial modulus such that $f_1 \leq f_2$, f_1 is locally semiconvex with modulus ω and f_2 is locally semiconcave with modulus ω, then there exists $f \in C^{1,\omega}_{\rm loc}(G)$ such that $f_1 \leq f \leq f_2$. This is a generalization of Ilmanen's lemma (which deals with linear modulus and functions on an open subset of $\mathbb{R}^n$). Here we extend the mentioned result from Hilbert spaces to some superreflexive spaces, in particular to L^p spaces, $p \in [2, \infty)$. We also prove a "global" version of Ilmanen's lemma (where a $C^{1,\omega}$ function is inserted between functions on an interval $I \subset \mathbb{R}$).

Keywords: Ilmanen's lemma; $C^{1,\omega}$ function; semiconvex function with general modulus

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1. Introduction

Let $A \subset \mathbb{R}^n$ be a convex set. We say that $f \colon A \to \mathbb{R}$ is classically semiconvex if there exists $C > 0$ such that the function $x \mapsto f(x) + C|x|^2$, $x \in A$, is convex. We say that $f \colon A \to \mathbb{R}$ is classically semiconcave if $-f$ is classically semiconvex. T. Ilmanen proved the following result [8, proof of 4F from 4G, page 199]:

Ilmanen's lemma. *Let $G \subset \mathbb{R}^n$ be an open set and $f_1, f_2 \colon G \to \mathbb{R}$. Suppose that $f_1 \leq f_2$, f_1 is locally classically semiconvex and f_2 is locally classically semiconcave. Then there exists $f \in C^{1,1}_{\rm loc}(G)$ such that $f_1 \leq f \leq f_2$.*

We will work with semiconvex (or semiconcave) functions with general modulus, see Definition 2.2 below and cf. [3, Definition 2.1.1]. Note that the classically semiconvex functions coincide with semiconvex functions with modulus $\omega(t) = Ct$ where $C > 0$, cf. [3, Proposition 1.1.3].

The following generalization of Ilmanen's lemma was proved in [9]:

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Theorem ([9, Theorem 4.5])**.** *Let X be a Hilbert space, $G \subset X$ an open set, $f_1, f_2 \colon G \to \mathbb{R}$ continuous functions and ω a modulus. Suppose that*

$$\liminf_{t \to 0+} \frac{\omega(t)}{t} > 0, \qquad f_1 \leq f_2$$

and that for every $x \in G$ there exist $C, r > 0$ such that $B(x,r) \subset G$, $f_1 \restriction_{B(x,r)}$ is semiconvex with modulus $C\omega$ and $f_2 \restriction_{B(x,r)}$ is semiconcave with modulus $C\omega$. Then there exists $f \in C^{1,\omega}_{\mathrm{loc}}(G)$ such that $f_1 \leq f \leq f_2$.

We generalize this result to some superreflexive spaces, see Theorem 3.9 below, in particular to Lebesgue spaces L^p, $p \in (2,\infty)$, see Corollary 3.10 below.

In [9], we also proved a result ([9, Corollary 3.2]) on insertion of $C^{1,\omega}$ functions which generalizes [2, Theorem 2] (which works with linear modulus and functions on Hilbert space). [9, Corollary 3.2] can be reformulated as follows:

Theorem. *Let X be a normed linear space, $f_1, f_2 \colon X \to \mathbb{R}$ continuous functions and ω a modulus. Suppose that f_1 is semiconvex with modulus ω, f_2 is semiconcave with modulus ω and $f_1 \leq f_2$. Then there exists $f \in C^{1,\omega}(X)$ such that $f_1 \leq f \leq f_2$.*

In the last section we prove Corollary 4.4 which is a precise analogue of the previous theorem for functions on an interval $I \subset \mathbb{R}$.

2. Preliminaries

Throughout this article, all normed linear spaces (Banach spaces, respectively) are real. By $B(x,r)$ we denote the open ball with center x and radius r. If P is a metric space, then we denote by $C(P)$ the set of all continuous functions $f \colon P \to \mathbb{R}$.

Notation 2.1. We denote by $\mathcal{M}$ the set of all $\omega \colon [0,\infty) \to [0,\infty)$ which are non-decreasing and satisfy $\lim_{t \to 0+} \omega(t) = 0$.

Definition 2.2. Let X be a normed linear space, $A \subset X$ a convex set and $\omega \in \mathcal{M}$.

- We say that $f \colon A \to \mathbb{R}$ is semiconvex with modulus ω if

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y) + \lambda(1-\lambda)\|x-y\|\omega(\|x-y\|)$$

  for every $x, y \in A$ and $\lambda \in [0,1]$.
- We say that $f \colon A \to \mathbb{R}$ is semiconcave with modulus ω if $-f$ is semiconvex with modulus ω.

- We denote by $SC^{\omega}(A)$ the set of all $f\colon A\to\mathbb{R}$ which are semiconvex with modulus $C\omega$ for some $C>0$. We denote by $-SC^{\omega}(A)$ the set of all $f\colon A\to\mathbb{R}$ such that $-f\in SC^{\omega}(A)$.

Let G be an open subset of a normed linear space, $\alpha\in(0,1]$ and $\omega\in\mathcal{M}$. We denote by $C^{1,\omega}(G)$ the set of all Fréchet differentiable $f\colon G\to\mathbb{R}$ such that f' is uniformly continuous with modulus $C\omega$ for some $C>0$, and we denote by $C^{1,\omega}_{\mathrm{loc}}(G)$ the set of all $f\colon G\to\mathbb{R}$ which are locally $C^{1,\omega}$. If $\omega(t)=t^{\alpha}$ for every $t\in[0,\infty)$, then we sometimes write $C^{1,\alpha}(G)$ instead of $C^{1,\omega}(G)$.

Let X be a normed linear space, $G\subset X$ an open convex set and $\omega\in\mathcal{M}$. Then (cf. [5, Corollary 3.6])

$$\Big(f\in C(G)\cap SC^{\omega}(G), \liminf_{t\to 0+}\frac{\omega(t)}{t}=0\Big)\Rightarrow f \text{ is convex}$$

and, see [9, Proposition 2.5],

$$(f\in C(G)\cap SC^{\omega}(G))\Rightarrow f \text{ is locally Lipschitz} \tag{1}$$

and (see [9, Theorem 2.6] or for the case $X=\mathbb{R}^d$ see [3, Proposition 2.1.2])

$$C^{1,\omega}(G)\subset C(G)\cap SC^{\omega}(G)\cap(-SC^{\omega}(G)). \tag{2}$$

If, moreover, G is bounded or $G=X$, then, see [9, Theorem 2.6],

$$C^{1,\omega}(G)=C(G)\cap SC^{\omega}(G)\cap(-SC^{\omega}(G)).$$

Let $\alpha\in(0,1]$. We say that a Banach space X admits an equivalent norm with modulus of smoothness of power type $1+\alpha$ if there exists an equivalent norm $|||\cdot|||$ on X and $C>0$ such that

$$\frac{|||x+ty|||+|||x-ty|||}{2}-1\le Ct^{1+\alpha},\qquad t>0,\ x,y\in X,\ |||x|||=|||y|||=1.$$

For a Banach space X we have the following (Pisier's) result, see [1, Theorem A.6]:

$$\begin{gathered}X \text{ is superreflexive if and only if } X \text{ admits an equivalent norm}\\ \text{with modulus of smoothness of power type } 1+\alpha \text{ for some } \alpha\in(0,1].\end{gathered} \tag{3}$$

3. Insertion of a $C^{1,\omega}_{\mathrm{loc}}$ function

In this section we prove a generalization of Ilmanen's lemma which improves [9, Theorem 4.5] (which works with functions on Hilbert spaces) to some superreflexive spaces, in particular to L^p spaces, $p\in[2,\infty)$ (see Theorem 3.9 or

Corollary 3.10 below). We use essentially the same methods as in [9]. However, besides these methods, we need Proposition 3.7 below. This result is implicitly contained in [7] for $G = X$. We reduce the case of an arbitrary G to the case $G = X$ using the notion of a partition ring and [7, Lemma 7.49], see the proof of Lemma 3.6 below. So, we need to recall some definitions from [7].

Definition 3.1 (cf. [7, page 411])**.** Let P be a metric space. We say that a family $\{A_\gamma\}_{\gamma\in\Gamma}$ of subsets of P is

- locally finite if for every $x \in P$ there exists a neighbourhood U of x such that $\{\gamma \in \Gamma\colon A_\gamma \cap U \neq \emptyset\}$ is finite;
- uniformly discrete if there exists $\delta > 0$ such that $\operatorname{dist}(A_{\gamma_1}, A_{\gamma_2}) \geq \delta$ whenever $\gamma_1, \gamma_2 \in \Gamma$, $\gamma_1 \neq \gamma_2$;
- σ-uniformly discrete if there are $\Gamma_n \subset \Gamma$, $n \in \mathbb{N}$, such that $\Gamma = \bigcup_{n\in\mathbb{N}} \Gamma_n$ and $\{A_\gamma\}_{\gamma\in\Gamma_n}$ is uniformly discrete for every $n \in \mathbb{N}$.

Remark 3.2. If $\mathcal{A}$ is a (nonindexed) family of subsets of a metric space, then we can regard it as an (indexed) family $\{A\}_{A\in\mathcal{A}}$. Hence the previous definition can be applied to $\mathcal{A}$ too.

Notation 3.3. Let P be a metric space and $f\colon P \to \mathbb{R}$. Then we set $\operatorname{supp}_o f := \{x \in P\colon f(x) \neq 0\}$.

Definition 3.4 (see [7, page 411])**.** Let P be a metric space and $S \subset C(P)$. We say that P admits locally finite and σ-uniformly discrete S-partitions of unity if for every open cover $\mathcal{U}$ of P there exists a system $\{\psi_\gamma\}_{\gamma\in\Gamma}$ of functions from S such that the following hold:

- $\{\operatorname{supp}_o \psi_\gamma\}_{\gamma\in\Gamma}$ is locally finite and σ-uniformly discrete.
- For every $\gamma \in \Gamma$ there exists $U \in \mathcal{U}$ such that $\operatorname{supp}_o \psi_\gamma \subset U$.
- $0 \leq \psi_\gamma \leq 1$ for every $\gamma \in \Gamma$, and $\sum_{\gamma\in\Gamma} \psi_\gamma(x) = 1$ for every $x \in P$.

Definition 3.5 (see [7, page 411])**.** Let P be a metric space and $S \subset C(P)$ a ring of functions. We say that S is a partition ring on P if the following hold:

- For every $S_0 \subset S$ such that $\operatorname{supp}_o f$ is bounded for every $f \in S_0$ and $\{\operatorname{supp}_o f\colon f \in S_0\}$ is uniformly discrete, there exists $g \in S$ such that
$$\operatorname{supp}_o g = \bigcup_{f\in S_0} \operatorname{supp}_o f.$$
- For every $f \in S$ and every open $U_1, U_2 \subset P$ such that $\operatorname{dist}(U_1, U_2) > 0$ and $\operatorname{supp}_o f = U_1 \cup U_2$, we have $\chi_{U_1} \cdot f \in S$.
- For every $f \in S$ bounded below and every $\varepsilon > 0$ there exists $g \in S$ such that $0 \leq g \leq 1$, $f^{-1}((-\infty, \varepsilon]) \subset g^{-1}(\{0\})$ and $f^{-1}([2\varepsilon, \infty)) \subset g^{-1}(\{1\})$.

Lemma 3.6. *Let P be a metric space and S a partition ring on P. Then the following are equivalent:*

(i) *Space P admits locally finite and σ-uniformly discrete S-partitions of unity.*

(ii) *Every open $G \subset P$ admits locally finite and σ-uniformly discrete S_G-partitions of unity (where $S_G := \{f \restriction_G : f \in S,\ \mathrm{supp}_o f \subset G\}$).*

(iii) *$\{\mathrm{supp}_o f : f \in S\}$ contains a σ-uniformly discrete basis for the topology of P.*

Proof: (ii) trivially implies (i) and (i) is equivalent to (iii) by [7, Lemma 7.49].

(iii) $\Rightarrow$ (ii): Let $G \subset P$ be an open set. It follows easily from the definitions that S_G is a partition ring on G. By the assumption, there exists $S^* \subset S$ such that $\{\mathrm{supp}_o f : f \in S^*\}$ is a σ-uniformly discrete basis for the topology of P. Set $S^*_G := \{f \restriction_G : f \in S^*,\ \mathrm{supp}_o f \subset G\}$. Then $S^*_G \subset S_G$ and $\{\mathrm{supp}_o f : f \in S^*_G\}$ is a σ-uniformly discrete basis for the topology of G. Thus $\{\mathrm{supp}_o f : f \in S_G\}$ contains a σ-uniformly discrete basis for the topology of G. If we apply the implication (iii) $\Rightarrow$ (i) to the case $P := G$ and $S := S_G$, we obtain that G admits locally finite and σ-uniformly discrete S_G-partitions of unity. $\square$

Proposition 3.7. *Let X be a Banach space, $G \subset X$ an open set and $\alpha \in (0, 1]$. Suppose that X admits an equivalent norm with modulus of smoothness of power type $1 + \alpha$. Then G admits locally finite and σ-uniformly discrete $C^{1,\alpha}(G)$-partitions of unity.*

Proof: Firstly, X is superreflexive, see (3). Further, X admits an equivalent norm which is uniformly rotund and has modulus of smoothness of power type $1 + \alpha$ (this fact is contained in the proof of [4, Proposition IV.5.2], see also the note after that proposition). Note also that the Fréchet derivative of such a norm is α-Hölder on the corresponding unit sphere, see [4, Proposition IV.5.1] or [5, Lemma 2.6] for more details. So, all the statements of [7, Theorem 7.56] and [7, Proposition 7.58] holds with our α (for this, see the beginning of the proof of [7, Proposition 7.58]).

Denote by S the set of all $f \in C^{1,\alpha}(X)$ which are bounded and have bounded derivative, and set $S_G := \{f \restriction_G : f \in S,\ \mathrm{supp}_o f \subset G\}$. It is proved in the proof of [7, Theorem 7.56] that S is a partition ring on X and that the condition (ii) of [7, Lemma 7.49] holds. Thus, by [7, Lemma 7.49], X admits locally finite and σ-uniformly discrete S-partitions of unity. Hence G admits locally finite and σ-uniformly discrete S_G-partitions of unity by Lemma 3.6. Now the assertion of the proposition follows. $\square$

Note that the converse of the previous proposition also holds as is shown in the following remark:

Remark 3.8. Let X be a Banach space and $\alpha \in (0,1]$. Suppose that some nonempty open subset G of X admits locally finite and σ-uniformly discrete $C^{1,\alpha}(G)$-partitions of unity. Then we can easily find $f \in C^{1,\alpha}(X)$ such that $\operatorname{supp}_o f$ is nonempty and bounded (i.e. X admits a $C^{1,\alpha}(X)$-smooth bump). Hence X admits an equivalent norm with modulus of smoothness of power type $1+\alpha$ by [7, Theorem 5.50].

Theorem 3.9. *Let X be a Banach space, $G \subset X$ an open set, $f_1, f_2 \colon G \to \mathbb{R}$, $\omega \in \mathcal{M}$ and $\alpha \in (0,1]$. Suppose that the following hold:*

- (a) *Space X admits an equivalent norm with modulus of smoothness of power type $1+\alpha$.*
- (b) $\liminf_{t\to 0+} \omega(t)/t^\alpha > 0$.
- (c) *Functions f_1 and f_2 are continuous, $f_1 \le f_2$ and for every $x \in G$ there exist $C, r > 0$ such that $B(x,r) \subset G$, $f_1 \restriction_{B(x,r)}$ is semiconvex with modulus $C\omega$ and $f_2 \restriction_{B(x,r)}$ is semiconcave with modulus $C\omega$.*

Then there exists $f \in C^{1,\omega}_{\mathrm{loc}}(G)$ such that $f_1 \le f \le f_2$.

Proof: By (1) and condition (c), for every $x \in G$ we can find $r_x \in (0,1)$ such that $B(x, 2r_x) \subset G$,

$$f_1 \restriction_{B(x,2r_x)} \in SC^\omega(B(x,2r_x)), \qquad f_2 \restriction_{B(x,2r_x)} \in -SC^\omega(B(x,2r_x))$$

and f_1 and f_2 are Lipschitz on $B(x,2r_x)$. Since $\{B(x,r_x) \colon x \in G\}$ is an open cover of G, we can find, by Proposition 3.7, a system $\{\psi_\gamma\}_{\gamma\in\Gamma}$ of functions from $C^{1,\alpha}(G)$ such that the following hold:

- ∘ $\{\operatorname{supp}_o \psi_\gamma\}_{\gamma\in\Gamma}$ is locally finite;
- ∘ for every $\gamma \in \Gamma$ there exists $x_\gamma \in G$ such that $\operatorname{supp}_o \psi_\gamma \subset B(x_\gamma, r_{x_\gamma})$;
- ∘ $0 \le \psi_\gamma \le 1$ for every $\gamma \in \Gamma$, and $\sum_{\gamma\in\Gamma} \psi_\gamma(x) = 1$ for every $x \in G$.

We will show that for every $\gamma \in \Gamma$ there exists $F_\gamma \in C^{1,\omega}(G)$ such that

$$\psi_\gamma(x) f_1(x) \le F_\gamma(x) \le \psi_\gamma(x) f_2(x), \qquad x \in G. \tag{4}$$

Let $\gamma \in \Gamma$. We put $B := B(x_\gamma, 2r_{x_\gamma})$ and for $i \in \{1,2\}$ define a function h_i by

$$h_i(x) := \begin{cases} \psi_\gamma(x) f_i(x), & x \in G, \\ 0, & x \in X \setminus G. \end{cases}$$

Then clearly f_1 and f_2 are Lipschitz on B, $f_1 \restriction_B \in SC^\omega(B)$, $f_2 \restriction_B \in -SC^\omega(B)$, $h_1 \le h_2$ and

$$\operatorname{supp} h_i := \overline{\operatorname{supp}_o h_i} \subset \overline{B(x_\gamma, r_{x_\gamma})} \subset B \subset G, \qquad i \in \{1,2\}. \tag{5}$$

It follows from condition (b) that there exists $c > 0$ such that

$$t^\alpha \le c\,\omega(t), \qquad t \in [0,4]. \tag{6}$$

Noting that $\operatorname{diam} B = 4r_{x_\gamma} < 4$, (6) implies $\psi_\gamma \restriction_B \in C^{1,\omega}(B)$. Further, it also follows from condition (b) that $\liminf_{t\to 0+} \omega(t)/t > 0$. Thus

$$h_1 \restriction_B \in SC^\omega(B) \qquad \text{and} \qquad h_2 \restriction_B \in -SC^\omega(B)$$

by [9, Lemma 4.2] (applied with $A := B$, $g_1 := \psi_\gamma \restriction_B$ and $g_2 := f_1 \restriction_B$ or $g_2 := -f_2 \restriction_B$). Hence $h_1 \in SC^\omega(X)$ and $h_2 \in -SC^\omega(X)$ by [9, Lemma 4.3]. Note that it follows from (5) (and the continuity of ψ_γ, f_1 and f_2) that h_1 and h_2 are continuous. Thus, by [9, Corollary 3.2], there exists $h \in C^{1,\omega}(X)$ such that $h_1 \le h \le h_2$. Then (4) holds with $F_\gamma := h \restriction_G$ and we are done.

Set

$$f(x) := \sum_{\gamma\in\Gamma} F_\gamma(x), \qquad x \in G.$$

It follows from (4) that $\{\operatorname{supp}_o F_\gamma\}_{\gamma\in\Gamma}$ is locally finite. Hence f is well defined and $f \in C^{1,\omega}_{\mathrm{loc}}(G)$. Summing (4) over $\gamma \in \Gamma$ we obtain that $f_1 \le f \le f_2$. □

Corollary 3.10. *Let μ be a nonnegative measure (on an arbitrary σ-algebra) and $p \in [2,\infty)$. Denote by X the Lebesgue space $L^p(\mu)$. Let $G \subset X$ be an open set, $f_1, f_2\colon G \to \mathbb{R}$ and $\omega \in \mathcal{M}$. Suppose further that $\liminf_{t\to 0+} \omega(t)/t > 0$ and that condition* (c) *of Theorem* 3.9 *holds. Then there exists $f \in C^{1,\omega}_{\mathrm{loc}}(G)$ such that $f_1 \le f \le f_2$.*

Proof: The canonical norm on X has modulus of smoothness of power type 2 by [4, Corollary V.1.2]. The rest now follows from Theorem 3.9. □

4. Insertion of a $C^{1,\omega}$ function on an interval $I \subset \mathbb{R}$

The main results of this section are Theorem 4.3 and Corollary 4.4 below. We begin with two facts concerning the set $SC^\omega(I) \cap (-SC^\omega(I))$.

Proposition 4.1. *Let $I \subset \mathbb{R}$ be an interval and $\omega \in \mathcal{M}$. Then the following hold:*

(i) *$SC^\omega(I) \cap (-SC^\omega(I)) \subset C(I)$.*
(ii) *If I is open, then $C^{1,\omega}(I) = SC^\omega(I) \cap (-SC^\omega(I))$.*

Proof: (i): Let $f \in SC^\omega(I) \cap (-SC^\omega(I))$. Then there exists $C > 0$ such that

$$|f(\lambda x + (1-\lambda)y) - \lambda f(x) - (1-\lambda) f(y)| \le \lambda(1-\lambda)|x-y| C\omega(|x-y|)$$

for every $x, y \in I$ and $\lambda \in [0,1]$. Thus

$$\begin{aligned}\limsup_{\lambda\to 1-}&|f(\lambda x+(1-\lambda)y)-f(x)|\\ &=\limsup_{\lambda\to 1-}|f(\lambda x+(1-\lambda)y)-\lambda f(x)-(1-\lambda)f(y)|\\ &\le\limsup_{\lambda\to 1-}\lambda(1-\lambda)|x-y|C\omega(|x-y|)=0,\qquad x,y\in I,\end{aligned}$$

and hence $\lim_{\lambda\to 1-}|f(\lambda x+(1-\lambda)y)-f(x)|=0$, $x,y\in I$. This implies that f is continuous.

(ii): $C^{1,\omega}(I)\subset SC^{\omega}(I)\cap(-SC^{\omega}(I))$ by (2). The reverse inclusion follows from part (i), [6, Lemma 2.5 (i)] and [6, Proposition 2.8 (i)]. □

Lemma 4.2. *Let $I\subset\mathbb{R}$ be an interval, $u,v\in I$, $u<v$, and $\omega\in\mathcal{M}$. Let $f\colon I\to\mathbb{R}$ be semiconvex with modulus ω and let $q\colon[u,v]\to\mathbb{R}$ be convex. Suppose that $q(u)\le f(u)$, $q(v)\le f(v)$, and define a function s by*

$$s(x):=\begin{cases}\max\{q(x),f(x)\}, & x\in[u,v],\\ f(x), & x\in I\setminus[u,v].\end{cases}$$

Then s is semiconvex with modulus ω.

Proof: Let $x_1,x_2\in I$, $x_1<x_2$, and $\lambda\in(0,1)$. We want to show that

$$s(\lambda x_1+(1-\lambda)x_2)\le\lambda s(x_1)+(1-\lambda)s(x_2)+\lambda(1-\lambda)(x_2-x_1)\omega(x_2-x_1).$$

Set $z:=\lambda x_1+(1-\lambda)x_2$. If $s(z)=f(z)$, then (using the semiconvexity of f)

$$\begin{aligned}s(z)=f(z)&\le\lambda f(x_1)+(1-\lambda)f(x_2)+\lambda(1-\lambda)(x_2-x_1)\omega(x_2-x_1)\\ &\le\lambda s(x_1)+(1-\lambda)s(x_2)+\lambda(1-\lambda)(x_2-x_1)\omega(x_2-x_1).\end{aligned}$$

So, we will further suppose that $s(z)=q(z)$ and $z\in(u,v)$. Define a function p by

$$\begin{aligned}p(x):=&\frac{x_2-x}{x_2-x_1}s(x_1)+\frac{x-x_1}{x_2-x_1}s(x_2)\\ &+\frac{x_2-x}{x_2-x_1}\frac{x-x_1}{x_2-x_1}(x_2-x_1)\omega(x_2-x_1),\qquad x\in[x_1,x_2].\end{aligned}$$

Then p is concave and

$$\begin{aligned}p(lx_1+(1-l)x_2)=&\,ls(x_1)+(1-l)s(x_2)\\ &+l(1-l)(x_2-x_1)\omega(x_2-x_1),\qquad l\in[0,1].\end{aligned}$$

In particular, $p(x_i) = s(x_i)$, $i = 1, 2$. Set $\overline{x_1} := \max\{u, x_1\}$ and $\overline{x_2} := \min\{v, x_2\}$. Then

$$u \le \overline{x_1} < z < \overline{x_2} \le v.$$

If $u \le x_1$, then $\overline{x_1} = x_1 \in [u, v)$ and

$$q(\overline{x_1}) = q(x_1) \le s(x_1) = p(x_1) = p(\overline{x_1}).$$

Otherwise $x_1 < u = \overline{x_1}$. Then we can find $l \in (0,1)$ such that $u = lx_1 + (1-l)x_2$. Thus (using the semiconvexity of f)

$$\begin{aligned} q(\overline{x_1}) &= q(u) \le f(u) = f(lx_1 + (1-l)x_2) \\ &\le lf(x_1) + (1-l)f(x_2) + l(1-l)(x_2 - x_1)\omega(x_2 - x_1) \\ &\le ls(x_1) + (1-l)s(x_2) + l(1-l)(x_2 - x_1)\omega(x_2 - x_1) \\ &= p(lx_1 + (1-l)x_2) = p(u) = p(\overline{x_1}). \end{aligned}$$

Hence we have shown that $q(\overline{x_1}) \le p(\overline{x_1})$, and we can analogously show that $q(\overline{x_2}) \le p(\overline{x_2})$. Thus (using the convexity of q and the concavity of p)

$$q(z) \le \frac{\overline{x_2} - z}{\overline{x_2} - \overline{x_1}}\, q(\overline{x_1}) + \frac{z - \overline{x_1}}{\overline{x_2} - \overline{x_1}}\, q(\overline{x_2}) \le \frac{\overline{x_2} - z}{\overline{x_2} - \overline{x_1}}\, p(\overline{x_1}) + \frac{z - \overline{x_1}}{\overline{x_2} - \overline{x_1}}\, p(\overline{x_2}) \le p(z)$$

and so

$$s(z) = q(z) \le p(z) = \lambda s(x_1) + (1-\lambda)s(x_2) + \lambda(1-\lambda)(x_2 - x_1)\omega(x_2 - x_1).$$

□

Theorem 4.3. *Let $I \subset \mathbb{R}$ be an interval, $f_1, f_2 \colon I \to \mathbb{R}$ and $\omega_1, \omega_2 \in \mathcal{M}$. Suppose that f_1 is semiconvex with modulus ω_1, f_2 is semiconcave with modulus ω_2 and $f_1 \le f_2$. Denote by $\mathcal{S}$ the set of all $s \colon I \to \mathbb{R}$ which are semiconvex with modulus ω_1 and satisfy $s \le f_2$. Then the function f defined by*

$$f(x) = \sup\{s(x) \colon s \in \mathcal{S}\}, \qquad x \in I,$$

is semiconvex with modulus ω_1, semiconcave with modulus ω_2 and satisfies $f_1 \le f \le f_2$.

Proof: It is clear that $f_1 \le f \le f_2$. By [3, Proposition 2.1.5], f is semiconvex with modulus ω_1. Let $u, v \in I$, $u < v$. We want to show that

$$f(\lambda u + (1-\lambda)v) \ge \lambda f(u) + (1-\lambda)f(v) - \lambda(1-\lambda)(v-u)\omega_2(v-u), \quad \lambda \in [0,1].$$

Define a function q by

$$q(x) := \frac{v-x}{v-u} f(u) + \frac{x-u}{v-u} f(v) - \frac{v-x}{v-u}\frac{x-u}{v-u}(v-u)\omega_2(v-u), \qquad x \in [u,v],$$

and a function s by

$$s(x) := \begin{cases} \max\{q(x), f(x)\}, & x \in [u,v], \\ f(x), & x \in I \setminus [u,v]. \end{cases}$$

Then q is convex and

$$\begin{aligned} &q(\lambda u + (1-\lambda)v) \\ &\qquad = \lambda f(u) + (1-\lambda) f(v) - \lambda(1-\lambda)(v-u)\omega_2(v-u), \qquad \lambda \in [0,1]. \end{aligned}$$

Since f_2 is semiconcave with modulus ω_2, we have

$$\begin{aligned} q(\lambda u + (1-\lambda)v) &= \lambda f(u) + (1-\lambda) f(v) - \lambda(1-\lambda)(v-u)\omega_2(v-u) \\ &\leq \lambda f_2(u) + (1-\lambda) f_2(v) - \lambda(1-\lambda)(v-u)\omega_2(v-u) \\ &\leq f_2(\lambda u + (1-\lambda)v), \qquad \lambda \in [0,1]. \end{aligned}$$

This implies that $s \leq f_2$. And since $q(u) = f(u)$ and $q(v) = f(v)$, s is semiconvex with modulus ω_1 by Lemma 4.2. Consequently, $s \in \mathcal{S}$ and thus $s \leq f$. Hence

$$\begin{aligned} &f(\lambda u + (1-\lambda)v) \\ &\qquad \geq s(\lambda u + (1-\lambda)v) \geq q(\lambda u + (1-\lambda)v) \\ &\qquad = \lambda f(u) + (1-\lambda) f(v) - \lambda(1-\lambda)(v-u)\omega_2(v-u), \qquad \lambda \in [0,1]. \end{aligned}$$

□

Corollary 4.4. *Let $I \subset \mathbb{R}$ be an interval, $\omega \in \mathcal{M}$, $f_1 \in SC^{\omega}(I)$ and $f_2 \in -SC^{\omega}(I)$. Suppose that $f_1 \leq f_2$. Then there exists a continuous $f\colon I \to \mathbb{R}$ such that $f_1 \leq f \leq f_2$ and $f\restriction_{\operatorname{int} I} \in C^{1,\omega}(\operatorname{int} I)$.*

Proof: By Theorem 4.3 there exists $f \in SC^{\omega}(I) \cap (-SC^{\omega}(I))$ such that $f_1 \leq f \leq f_2$. Then f is continuous by Proposition 4.1 (i) and $f\restriction_{\operatorname{int} I} \in C^{1,\omega}(\operatorname{int} I)$ by Proposition 4.1 (ii). □

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