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On hereditary normality of ω^*, Kunen points and character ω_1

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Abstract. We show that $\omega^* \setminus \{p\}$ is not normal, if p is a limit point of some countable subset of ω^*, consisting of points of character ω_1. Moreover, such a point p is a Kunen point and a super Kunen point.

Keywords: non-normality point; butterfly point; Kunen point; super Kunen point

Classification: 54D15, 54D35, 54D40, 54D80, 54E35, 54G20

1. Introduction

We investigate properties of Čech–Stone compactification $\beta\omega$ of the countable discrete space $\omega = \{0, 1, 2, \dots\}$. One of the most intriguing problems in this area was stated probably around 1960 by L. Gillman in [4] or in [3]: Is $\omega^* \setminus \{p\}$ non-normal for any point p of the remainder $\omega^* = \beta\omega \setminus \omega$? If so, then p is called a non-normality point of ω^*.

Assuming that the continuum hypothesis of CH (axiom of choice) is valid, a positive answer was obtained independently of N. M. Warren in [8] and M. Rajagopalan in [6] in 1972. A. Bešlagić and E. K. van Douwen in [1] in 1990 weakened CH to Martin's axiom (MA).

But so far not much is known within ZFC (Zermelo–Fraenkel set theory with the axiom of choice). Thus $p \in \omega^*$ is called a Kunen point, if there exists a discrete subset P of ω^* of cardinality ω_1, that is no more than countable outside any open neighbourhood of p. Any Kunen point is a non-normality point of ω^* (E. van Douwen).

A. Szymański in [7] in 2012 proved the same, if p is a non-isolated point of some closed subset of ω^* of countable π-weight.

Some other more technical results were obtained in [5].

A. Błaszczyk and A. Szymański in [2] stated in 1980, that p is a non-normality point, if p is a limit point of some countable discrete subset P of ω^*. Now we

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omit the discrete requirement, assuming instead that P consists of points of the character ω_1.

Theorem 1. *Let P be a countable subset of ω^*, consisting of points of the character ω_1. Then $\omega^* \setminus \{p\}$ is not normal for any point p of ω^*, which is in the closure of P. Moreover, p is a Kunen point and a super Kunen point.*

2. Preliminaries

In our article $|N| = \omega$ and $|R| = \mathcal{C}$. Each ordinal number α can be represented in a unique way in the form $\alpha = \beta + n$, where β is a limit ordinal and $n \in \omega$. Then α is even if n is even and odd otherwise.

By $[\,]$ we always denote closure operator in ω^*, by Oa – a clopen neighbourhood of a, i.e. closed and open in ω^* set, containing a. A set A is strongly discrete if there is a cellular family $\{Oa\colon a \in A\}$. A family $\{O_\alpha a\}_{\alpha<\tau}$ is called a clopen local base in a, if each Oa contains some $O_\alpha a$. The minimal cardinality of the local base is called the character in a and denoted $\chi(a)$.

Definition. A point p of ω^* is called a super Kunen point, if there is a strongly discrete subset P of ω^* of cardinality ω_1, that is no more than countable outside any neighbourhood Op.

Of course, any super Kunen point is a Kunen point.

3. Proofs

Let from now on $P = \{p_i\colon i < \omega\}$ be a countable subset of ω^*, consisting of points of character ω_1 and let p be any point of $[P] \setminus P$. For every $i < \omega$ assume $\{O_{i\alpha}\colon \alpha < \omega_1\}$ to be a clopen local base of cardinality ω_1 in p_i. For any clopen neighbourhood O of p we denote

$$\mathcal{K}(O) = \min\{\lambda < \omega_1\colon \forall\, i < \omega(p_i \in O \to \exists\, \alpha \le \lambda(O_{i\alpha} \subset O))\}.$$

We define a filter $\mathcal{F}$ on ω as follows:

$$\mathcal{F} = \{\{i \in \omega\colon p_i \in O\}\colon O \text{ is a clopen neighbourhood of } p\}.$$

Some of the following facts are simple and sometimes well-known.

Lemma 1. *Every nonempty G_δ-subset of ω^* has nonempty interior in ω^*.*

Lemma 2. *Every point q of ω^* of character ω_1 is a super Kunen point.*

PROOF: Let $\{O_\alpha\colon \alpha<\omega_1\}$ be a local base in q. By the previous lemma for every $\alpha<\omega_1$ we can find a nonempty clopen set U_α so that $q\notin U_\alpha$ and

$$U_\alpha \subset \bigcap_{\beta<\alpha} O_\beta \setminus \bigcup_{\beta<\alpha} U_\beta.$$

For any points $x_\alpha\in U_\alpha$ the set $\{x_\alpha\colon \alpha<\omega_1\}$ witnesses that q is a super Kunen point. □

Lemma 3. *The family* $\{\bigcap_{\alpha<\lambda_i} O_{i\alpha}\colon i<\omega\}$ *is cellular for some* $\lambda_i<\omega_1$.

PROOF: We can choose every λ_i so that the set $\bigcap_{\alpha<\lambda_i} O_{i\alpha}$ is disjoint from both $\bigcap_{\beta<\lambda_j} O_{j\beta}$ for every $j<i$ and $\{p_j\colon j>i\}$. □

Lemma 4. *The family* $\{U_{i\alpha}\colon i<\omega$ *and* $\alpha<\omega_1\}$ *is cellular for some nonempty clopen sets* $U_{i\alpha}$ *such that* $U_{i\alpha}\subset\bigcap_{\beta<\alpha} O_{i\beta}$.

PROOF: By the previous lemma we can choose every $U_{i\alpha}$ so that

$$U_{i\alpha}\subset \bigcap\{O_{i\beta}\colon \beta<\max\{\alpha,\lambda_i\}\},$$

$p_i\notin U_{i\alpha}$ and $U_{i\beta}\cap U_{i\alpha}=\emptyset$ for every $\beta<\alpha$. □

Lemma 5. *For every* $\alpha<\omega_1$ *there is a point* $a_\alpha\in\omega^*$ *such that* $a_\alpha\notin[P]$ *and*

$$a_\alpha\in\bigcap_{F\in\mathcal{F}}\Big[\bigcup_{i\in F} U_{i\alpha}\Big].$$

PROOF: For $U=\bigcup_{i<\omega} U_{i\alpha}$ assume $D\subset\omega^*\setminus[U]$ to be σ-compact. Then $X=\omega\cup U\cup D$ is σ-compact and, so, normal. Since $\omega\subset X$, then $[X]_{\beta\omega}=\beta X$ is a Čech–Stone compactification of X. Since U and D are closed in X, then $[U]\cap[D]=[U]_{\beta X}\cap[D]_{\beta X}=\emptyset$. Hence $[U]$ is a P-set.

In every $U_{i\alpha}$ we can find a cellular family of C-many nonempty clopen sets $\{V_{i\beta}\colon \beta<C\}$ and put $V_\beta=\bigcup_{i<\omega} V_{i\beta}$ for any $\beta<C$. Since $[U]=\beta U$ by the standard arguments, then $[V_\beta]$ are disjoint clopen subsets of $[U]$.

Thus $[V_{\beta_0}]\cap P=\emptyset$ for some $\beta_0<C$, and by the first paragraph of this proof, this implies $[V_{\beta_0}]\cap[P]=\emptyset$. So we can choose a_α to be any point of $\bigcap_{F\in\mathcal{F}}\big[\bigcup_{i\in F} V_{i\beta_0}\big]$. □

From now on every point a_α satisfies the conditions of Lemma 5.

Lemma 6. *Let* O *be any clopen neighbourhood of* p. *If* $\alpha>\mathcal{K}(O)$ *for some* $\alpha<\omega_1$, *then* $a_\alpha\in O$.

PROOF: For $F=\{i\in\omega\colon p_i\in O\}$ we get $F\in\mathcal{F}$. For any $i\in F$ there is $\alpha_i\le\mathcal{K}(O)$ such that $O_{i\alpha_i}\subset O$. Then $\alpha>\alpha_i$ implies $U_{i\alpha}\subset O_{i\alpha_i}\subset O$ by our construction and

$$a_\alpha \in \Big[\bigcup_{i\in F} U_{i\alpha}\Big] \subset \Big[\bigcup_{i\in F} O_{i\alpha_i}\Big] \subset O.$$

□

Lemma 7. *The set* $\{a_\alpha\colon \alpha<\omega_1\}$ *is discrete. Hence* p *is a Kunen point.*

Proof: For any $\alpha<\omega_1$ let O be any clopen neighbourhood of p such that $a_\alpha\notin O$. Then $a_\beta\in O$ for every $\beta>\mathcal{K}(O)$ by the previous lemma. Since the sets

$$C=\bigcup\{U_{i\alpha}\colon i<\omega\} \qquad \text{and} \qquad D=\bigcup\{U_{i\beta}\colon\ i<\omega,\ \beta\neq\alpha \text{ and } \beta\le\mathcal{K}(O)\}$$

are σ-compact, open and disjoint, $[C]\cap[D]=\emptyset$. Since $a_\alpha\in[C]$ and $a_\beta\in[D]$ if $\beta\neq\alpha$ and $\beta\le\mathcal{K}(O)$, then the open set $\omega^*\setminus\big(O\cup[D]\big)$ contains a_α and non of $a'_\beta s$ for $\beta\neq\alpha$. □

It implies that $\omega^*\setminus\{p\}$ is not normal by E. van Douwen. We shall give now another proof. Denote $A=\{a_\alpha\colon \alpha<\omega_1 \text{ even}\}$ and $B=\{a_\alpha\colon \alpha<\omega_1 \text{ odd}\}$.

Lemma 8. *Since* $p=[A]\cap[B]$, p *is a butterfly-point.*

Proof: By Lemma 6 we get $p\in[A]\cap[B]$. On the other hand, let O be any clopen neighbourhood of p. Then

$$\begin{aligned}[A]\cap[B]\setminus O &\subset [\{a_\alpha\colon \alpha\le\mathcal{K}(O) \text{ even}\}]\cap[\{a_\alpha\colon \alpha\le\mathcal{K}(O) \text{ odd}\}]\\ &\subset \Big[\bigcup\{U_{i\alpha}\colon i<\omega \text{ and } \alpha\le\mathcal{K}(O) \text{ even}\}\Big]\\ &\quad\cap\Big[\bigcup\{U_{i\alpha}\colon i<\omega \text{ and } \alpha\le\mathcal{K}(O) \text{ odd}\}\Big]=\emptyset,\end{aligned}$$

because ω^* is an F-space. □

Lemma 9. *The space* $\omega^*\setminus\{p\}$ *is not normal.*

Proof: For any continuous map $f\colon \omega^*\setminus\{p\}\to[0,1]$ it is enough to show that $f(A)\cap f(B)\neq\emptyset$.

For every $i<\omega$ we choose $\alpha_i<\omega_1$ so that $p\notin O_{i\alpha_i}$ and put $W=\bigcup_{i\in\omega}O_{i\alpha_i}$. Since $Y=\omega\cup W$ is regular and σ-compact, it is normal. Since W is closed in Y, the restriction f/W has a continuous extension $g\colon Y\to[0,1]$. Since $\omega\subset Y\subset\beta\omega$, then g has a continuous extension $\tilde g\colon \beta\omega\to[0,1]$. For its restriction $h=\tilde g/\omega^*$ onto ω^* we have $h^{-1}h(p)=\bigcap_{i\in\omega}O_i$ for some clopen $O_i\subset\omega^*$. If $\alpha>\sup_{i<\omega}\alpha_i$ for some $\alpha<\omega_1$, then $a_\alpha\in\big[\bigcup_{i<\omega}U_{i\alpha}\big]\subset\big[\bigcup_{i<\omega}O_{i\alpha_i}\big]$, i.e. $a_\alpha\in[W]\setminus\{p\}$. Since $f/W=h/W$, then $f(a_\alpha)=h(a_\alpha)$. If $\alpha>\sup_{i\in\omega}\mathcal{K}(O_i)$, then $a_\alpha\in\bigcap_{i\in\omega}O_i$ by Lemma 6 and, so, $h(a_\alpha)=h(p)$. But then $h(p)\in f(A)\cap f(B)$. □

Lemma 10. *There is a strongly discrete subset of* $\{a_\alpha\colon \alpha<\omega_1\}$ *of cardinality* ω_1.

Proof: We shall construct by induction on $\lambda < \omega_1$ both a countable set $A_\lambda \subset \{a_\alpha\colon \alpha < \omega_1\}$ and a cellular family of clopen neighbourhoods of its points $\mathcal{B}_\lambda = \{Oa\colon a \in A_\lambda\}$ so that $A_\lambda \subsetneq A_\gamma$ if $\lambda < \gamma < \omega_1$ and $Oa \cap P = \emptyset$ for any $Oa \in \mathcal{B}_\lambda$.

First we put $A_0 = \{a_0\}$ and choose $\mathcal{B}_0 = \{Oa_0\}$ so that $Oa_0 \cap P = \emptyset$.

If A_α and $\mathcal{B}_\alpha$ have been constructed for every $\alpha < \lambda$ for some limit ordinal $\lambda < \omega_1$, then we put $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ and $\mathcal{B}_\lambda = \bigcup_{\alpha<\lambda} \mathcal{B}_\alpha$.

Assume A_λ and B_λ have been constructed for some ordinal $\lambda < \omega_1$. Then $V_a = \omega^* \setminus Oa$ is a clopen neighbourhood of P for each $Oa \in \mathcal{B}_\lambda$. Let $\alpha > \sup_{a\in A_\lambda} \mathcal{K}(V_a)$ for some $\alpha < \omega_1$. Then $a_\alpha \in \big[\bigcup_{i\in\omega} U_{i\alpha}\big]$. For any $i \in \omega$ and $a \in A_\lambda$ we have $U_{i\alpha} \subset O_{i\beta} \subset V_a$ for some $\beta \le \mathcal{K}(V_a)$, i.e. $U_{i\alpha} \cap O_a = \emptyset$. Hence $\big[\bigcup_{i\in\omega} U_{i\alpha}\big] \cap \big[\bigcup_{a\in A_\lambda} Oa\big] = \emptyset$, because ω^* is an F-space, and $a_\alpha \notin \big[\bigcup_{a\in A_\lambda} Oa\big]$. There is a clopen neighbourhood Oa_α, which does not intersect neither P nor $\bigcup_{a\in A_\lambda} Oa$. We put $A_{\lambda+1} = A_\lambda \cup \{a_\alpha\}$ and $\mathcal{B}_{\lambda+1} = \mathcal{B}_\lambda \cup \{Oa_\alpha\}$.

Finally, the set $\bigcup_{\alpha<\omega_1} A_\alpha$ is as required. □

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