

Two remarks on the maximal-ideal space of H^∞

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Abstract. The topology of the maximal-ideal space of H^∞ is discussed.

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In this note I discuss two properties of the topology of \mathbf{M} , the maximal-ideal space of \mathbf{H}^∞ , which is the Banach algebra of analytic functions which are bounded in the unit disc $\mathbf{D} = \{z: |z| < 1\}$. The collection of homomorphisms from \mathbf{H}^∞ to \mathbf{C} is referred to as \mathbf{M} (which technically is the collection of the kernels of these homomorphisms). Each $f \in \mathbf{H}^\infty$ is identified with the function on \mathbf{M} which takes every homomorphism ϕ to $\phi(f)$. (I shall use “ z ” for the identity function or for a complex number, according to whatever may be convenient.) If $\lambda \in \mathbf{C}$ and $|\lambda| = 1$, a compact subset \mathbf{M}_λ (called the fiber over λ) of \mathbf{M} is defined to be the collection of homomorphisms ϕ for which $\phi(z) = \lambda$. A great deal is known about \mathbf{M} and about the fibers \mathbf{M}_λ . See [1] and [2]. The disc \mathbf{D} is identified with a subset of \mathbf{M} by having each point z correspond to the homomorphism evaluation at z .

Two comments I wish to make are in regard to the fibers. In particular, \mathbf{M}_1 is known to be connected; the proof relies on a difficult theorem [1, page 88]. I shall present an easy proof by use of a different difficult theorem, the Corona theorem [2, pages 185, 315], which is that \mathbf{D} is dense in \mathbf{M} . The second comment regards subsets of \mathbf{D} which may or may not have \mathbf{M}_1 in their closures.

Proposition 1. *The fiber \mathbf{M}_1 is connected.*

PROOF: The portion of the disc $\{z = re^{i\theta}: R < r < 1 \text{ and } |\theta| < \delta\}$, where δ is a small positive number, has \mathbf{M}_1 in its closure $C(R, \delta)$. This is obvious from the fact that \mathbf{D} is dense in \mathbf{M} . The set $C(R, \delta)$ is compact and connected, being the closure of a connected set in a compact space. Form $\bigcap_{R \rightarrow 1} C(R, \delta)$, the intersection of nested sets, to obtain $C(\delta)$, again compact and containing \mathbf{M}_1 . Now take

$\bigcap_{\delta \rightarrow 0} C(\delta)$, once again obtaining a compact and connected space. Obviously, this last contains \mathbf{M}_1 and nothing else. \square

Proposition 2. *Let U be a subset of \mathbf{D} . If the closure \bar{U} of U in \mathbf{C} does not contain an arc of the circle $\mathbf{T} = \{z : |z| = 1\}$ which has 1 in its (relative) interior, then the closure of U in \mathbf{M} does not contain all of \mathbf{M}_1 .*

PROOF: Recall the pseudo-hyperbolic metric $\varrho(z, u) = |z - u|/|1 - \bar{u}z|$ between two points of \mathbf{D} . Convergence of $\prod \varrho(z, u_n)$ at any point z of the disc (and therefore at all points of the disc) is the necessary and sufficient condition that there be a (convergent) Blaschke product with $\{u_n : n = 1, 2, \dots\}$ as its zeroes. If no u_n is 0 and they are all distinct, the standard Blaschke product $B(z)$ with $\{u_n : n = 1, 2, \dots\}$ as its zeroes is this:

$$B(z) = B(z, \{u_1, u_2, \dots\}) = \prod \left[\frac{-\bar{u}_n}{u_n} \frac{z - u_n}{1 - \bar{u}_n z} \right].$$

(Note that $B(0) = \prod |u_n| > 0$.)

Observe that if $\delta > 0$, then for every θ , $\varrho(z, re^{i\theta}) \rightarrow 1$ uniformly on $\{z : |z| \leq 1, |z - e^{i\theta}| \geq \delta\}$ as $r \rightarrow 1^-$.

If the closure \bar{U} of U in \mathbf{C} omits small arcs $A_n = \{e^{i\theta} : \theta \in J_n\}$, where $J_n = [a_n, b_n]$ are pairwise disjoint intervals in $(0, 1)$ and $b_n \rightarrow 0$ as $n \rightarrow \infty$, we shall see that the closure of U in \mathbf{M} does not contain all of \mathbf{M}_1 . (There is no loss of generality in using $(0, 1)$ instead of $(-1, 0) \cup (0, 1)$. And if $(-\varepsilon, \varepsilon)$ is disjoint from \bar{U} , the proposition is obvious.) For each J_n let $K_n = (a'_n, b'_n)$, where a'_n and b'_n are any numbers for which $a_n < a'_n < b'_n < b_n$, and let r_n be positive numbers converging to 1 so that the open sets $C_n = \{re^{i\theta} : r_n < r < 1, \theta \in K_n\}$ are nonempty, are contained in \mathbf{D} , and are disjoint from \bar{U} .

For any set V of points (not including 0) of \mathbf{D} let $B(z, V)$ be the Blaschke product having V as its set of zeroes, as indicated above, if it converges. We shall choose a sequence V by induction. First select u_1 to be any point of C_1 so that $\inf\{\varrho(z, u_1) : z \in \bar{U}\} > 1/2$. Next pick $u_2 \in C_2$ so that $\inf\{\varrho(z, u_2) : z \in \bar{U} \cup \{u_1\}\} > 3/4$. And choose $u_3 \in C_3$ so that $\inf\{\varrho(z, u_3) : z \in \bar{U} \cup \{z_1, z_2\}\} > 7/8$. Etc. We eventually pick $u_n \in C_n$ so that $\inf\{\varrho(z, u_n) : z \in \bar{U} \cup \{u_1, u_2, \dots, u_{n-1}\}\} > 1 - 2^{-n}$. Etc. We obtain a sequence u_n which converges to 1.

The choices of u_n and the above inequalities show that

$$B(z) = B(z, \{u_1, u_2, \dots\})$$

converges and $|B(z)| \geq c > 0$ for some c and all $z \in \bar{U}$. This means that if a homomorphism Φ is in the cluster set of U in \mathbf{M} , then $|\Phi(B)| \geq c > 0$. However, the sequence u_n has a cluster set in \mathbf{M}_1 , and for any homomorphism φ in that cluster set $\varphi(B) = 0$.

This means that the closure of U in \mathbf{M} does not contain all of \mathbf{M}_1 , since it omits such φ . \square

Remarks. The above proof suggests two additional facts: (1) The converse of Proposition 2 is false, even for open U . (2) If X is a nonempty proper subset of $(\mathbf{M} - \mathbf{D})$ which is a union of fibers \mathbf{M}_λ , then there is no subset of \mathbf{D} with cluster set in $(\mathbf{M} - \mathbf{D})$ equal to X .

I shall leave the proof of these as an exercise for those interested.

REFERENCES

- [1] Hoffman K., *Banach Spaces of Analytic Functions*, Dover, New York, 2007.
- [2] Garnett J. B., *Bounded Analytic Functions*, Graduate Texts in Mathematics, 236, Springer, New York, 2007.

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