An integral estimate for weak solutions to some quasilinear elliptic systems

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Abstract. We prove an integral estimate for weak solutions to some quasilinear elliptic systems; such an estimate provides us with the following regularity result: weak solutions are bounded.

Keywords: quasilinear elliptic systems, weak solutions, integral estimates, regularity

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Let Ω be a bounded open subset of $\mathbb{R}^n$ and $u \in \mathbb{R}^N$; let us fix a real number $q \geq 2$; we set

$$V(u) = (1 + |u|^2)^{1/2}, \quad W(u) = V^{(q-2)/2}(u)\, u. \tag{1}$$

We are concerned with weak solutions $u :\longrightarrow \mathbb{R}^N$ to the quasilinear system

$$-\sum_{i=1}^{n} D_i\left(V^{q-2}(u(x)) \sum_{j=1}^{n} \sum_{\beta=1}^{N} A_{i\,j}^{\alpha\beta}(x, u(x))\, D_j u^\beta(x)\right) = 0 \tag{2}$$

$\forall\, x \in \Omega, \forall\, \alpha = 1, \dots, N$, where the coefficients $A_{i\,j}^{\alpha\beta}$ are elliptic, that is, there exist positive constants m, M such that

$$m|\xi| \leq \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} A_{i\,j}^{\alpha\beta}(x, u)\xi_j^\beta\, \xi_i^\alpha \leq M|\xi|^2 \tag{3}$$

$\forall\ \xi \in \mathbb{R}^{nN},\ \forall\ u \in \mathbb{R}^N,\ \forall\ x \in \Omega.$ Quasilinear elliptic systems, considered just before, arise, when we deal with the integral functional

$$\int_\Omega \left(1 + |Dv(x)|^2\right)^{q/2}\, dx \tag{4}$$

and we write the Euler equation: after an integration by parts, we get a system of type (2), (3), in which u is the gradient of the minimizer v of (4): [G], [M].

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In order to develop the regularity theory in Campanato's spaces $\mathfrak{L}^{P,\lambda}$, we need good estimates for solutions to some particular systems, namely those in which the coefficients $A_{i\,j}^{\alpha\beta}(x,u)$ are constant:

$$A_{i\,j}^{\alpha\beta}(x,u) \equiv A_{i\,j}^{\alpha\beta}. \tag{5}$$

This is the way, followed in the past, for dealing with the case $q = 2$ [G] and the case of nonlinear systems of a different type [C1]. Throughout this paper, we are concerned with systems (2), (3), in which the coefficients $A_{i\,j}^{\alpha\beta}$ are constant, that is, (5) holds. Before stating the estimate, we must say what we mean when we talk about "weak solutions" to the elliptic systems (2), (3), (5): we agree that $u : \Omega \longrightarrow \mathbb{R}^N$ is a weak solution to (2), (3), (5), if

$$u \in H^{1,2}(\Omega), \quad V^{q-2}(u)|u|^2 \in L^1(\Omega), \quad V^{q-2}(u)|Du|^2 \in L^1(\Omega) \tag{6}$$

and

$$\int_\Omega V^{q-2}(u(x)) \sum_{i,j=1}^{n} \sum_{\alpha,\beta=1}^{N} A_{i\,j}^{\alpha\beta}\, D_j u^\beta(x)\, D_i \phi^\alpha(x)\, dx = 0 \tag{7}$$

for each test function $\phi : \Omega \longrightarrow \mathbb{R}^N$ such that

$$\phi \in H^{1,2}(\Omega), \quad V^{q-2}(u)|\phi|^2 \in L^1(\Omega), \quad V^{q-2}(u)|D\phi|^2 \in L^1(\Omega). \tag{8}$$

Let us call ${}^*H_0^{1,2}(\Omega; u)$ the set of all ϕ verifying (8). Campanato proved the following estimate:

Theorem 1 (Campanato [C2]). *Let u be a weak solution to* (2), (3), (5); *if the coefficients $A_{i\,j}^{\alpha\beta}$ satisfy*

$$A_{i\,j}^{\alpha\beta} = \delta_{i\,j}\, \delta^{\alpha\beta}, \tag{9}$$

then

$$\int_{B(x^0,r)} |W(u)|^2\, dx \le \left(\frac{r}{s}\right)^n \int_{B(x^0,s)} |W(u)|^2\, dx \tag{10}$$

$\forall\, x^0 \in \Omega$, $\forall\, r, s : 0 < r \le s < \operatorname{dist}(x^0, \partial\Omega)$; *where $\delta_{i\,j}, \delta^{\alpha\beta}$ are Kronecker's symbols* ($\delta_{i\,j} = 1$, *if* $i = 1$ *and* $\delta_{i\,j} = 0$, *if* $i \neq j$), $B(x^0, \sigma) = \{x \in \mathbb{R}^N : |x - x^0| < \sigma\}$ *and* $W(u)$ *is defined in* (1).

In the next lines we will prove the following

Theorem 2. *Let u be a weak solution to* (2), (3), (5); *if the coefficients $A^{\alpha\beta}_{i\,j}$ satisfy*

$$A^{\alpha\beta}_{i\,j} = a_{i\,j}\, b^{\alpha\beta} \tag{11}$$

for every $i,j = 1,\dots,n$ and for every $\alpha,\beta = 1,\dots,N$, where $a_{i\,j}, b^{\alpha\beta}$ are real numbers such that there exist positive constants ν, L for which

$$\nu|\eta|^2 \le \sum_{i,j=1}^{n} a_{i\,j}\eta_j\eta_i \le L|\eta|^2 \quad \forall\, \eta \in \mathbb{R}^n, \tag{12}$$

$$a_{i\,j} = a_{j\,i} \qquad \forall\, i,j = 1,\dots,n, \tag{13}$$

$$\det\,(b^{\alpha\beta}) \neq 0, \tag{14}$$

then, for $c = (L/\nu)^n$, we have

$$\int_{B(x^0,r)} |W(u)|^2\, dx \le c\left(\frac{r}{s}\right)^n \int_{B(x^0,s)} |W(u)|^2\, dx \tag{15}$$

$\forall\, x^0 \in \Omega,\ \forall\, r,s : 0 < r \le s < \operatorname{dist}(x^0,\partial\Omega)$.

Remark. The inequality (15) tells us that $|W(u)|^2$ is locally bounded; since $|u| \le |W(u)|$ (because of (1) and $q \ge 2$), we get that u is locally bounded, too.

Proof of Theorem 2: We will prove Theorem 2 by reducing to the case treated by Campanato in this way:
Step 1. We get rid of the matrix $(b^{\alpha\beta})$ by using the new test function $\psi = {}^t b\phi$, where ${}^t b$ is the transpose of the matrix $b = (b^{\alpha\beta})$.
Step 2. We find a linear transformation $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that its Jacobian matrix diagonalizes the matrix $a = (a_{i\,j}) : JGa\,{}^tJG = Id$.
Step 3. We consider the new function $v = u \circ G^{-1}$; we prove that v satisfies the hypotheses of Campanato's Theorem 1.
Step 4. We write the estimate (10) for v.
Step 5. We come back to u by changing variables and we get the estimate (15).

The previous technique, consisting in diagonalizing the matrix and changing variables, has been employed in [FH], [L]. Now we will exploit all the details. Since $b^{\alpha\beta}$ is constant, we have

$$\sum_{\alpha,\beta} b^{\alpha\beta}\, D_j u^\beta D_i \phi^\alpha = \sum_\beta D_j u^\beta D_i \Big(\sum_\alpha b^{\alpha\beta}\,\phi^\alpha\Big); \tag{16}$$

we set $\psi^\beta = \sum_{\alpha=1}^N b^{\alpha\beta}\,\phi^\alpha$; since we assumed $\det\,(b^{\alpha\beta}) \neq 0$, we have

$$\psi \in {}^*H^{1,2}_0(\Omega;u) \Longleftrightarrow \phi \in {}^*H^{1,2}_0(\Omega;u). \tag{17}$$

We recall that u satisfies (7) with $A_{i\,j}^{\alpha\beta} = a_{i\,j} b^{\alpha\beta}$: by means of (16) and (17), we get

$$\int_\Omega V^{q-2}(u(x)) \sum_{i,j=1}^{n} a_{i\,j} \sum_{\alpha,\beta=1}^{N} D_j u^\beta(x)\, D_i \psi^\beta(x)\, dx = 0 \tag{18}$$

for every $\psi \in {}^*H_0^{1,2}(\Omega; u)$. Now we are looking at the matrix $a = (a_{i\,j})$: it is real, symmetric and positive, so we can find an orthonormal basis for $\mathbb{R}^n$ consisting of eigenvectors of the matrix a: let $w^1, w^2, \dots, w^n$ be such a basis where each w^s has the scalar components w_j^s, $j = 1, \dots, n$. Let λ^s be the real positive (because of the ellipticity (12)) eigenvalue corresponding to the eigenvector w^s; let us consider the following linear transformation $G : \mathbb{R}^n \longrightarrow \mathbb{R}^n$, where every component G_s is defined in this way:

$$G_s(x) = \sum_{j=1}^{n} (\lambda^s)^{-1/2} w_j^s x_j.$$

Let $JG = (JG_{rs})$ $r, s = 1, \dots, n$ be the Jacobian matrix of the linear transformation G; such a matrix diagonalizes the matrix $a = (a_{i\,j})$, that is,

$$\sum_{i,j=1}^{n} JG_{ri} a_{i\,j}\, JG_{sj} = \delta_{rs} \qquad \forall\, r, s = 1, \dots, n; \tag{19}$$

moreover, we have

$$L^{-n/2} \le |\det JG| \le \nu^{-n/2}, \tag{20}$$

$$\frac{1}{L}|x-y|^2 \le |G(x) - G(y)|^2 \le \frac{1}{\nu}|x-y|^2 \quad \forall\, x, y \in \mathbb{R}^n. \tag{21}$$

We set $v = u \circ G^{-1}$ and we get $v \in H^{1,2}(G(\Omega)), V^{q-2}(v)|v|^2 \in L^1(G(\Omega))$, $V^{q-2}(v)|Dv|^2 \in L^1(G(\Omega))$. We set $z = \psi \circ G^{-1}, x = G^{-1}(y)$ and we change the variables in (18): we get

$$\int_{G(\Omega)} V^{q-2}(v(y)) \sum_{r,s=1}^{n} \Big(\sum_{i,j=1}^{n} JG_{ri} a_{i\,j} JG_{sj} \Big) \sum_{\beta=1}^{N} D_s v^\beta(y) \cdot D_r z^\beta(y)\, dy = 0 \qquad \forall\, z \in {}^*H_0^{1,2}(G(\Omega); v). \tag{22}$$

We agree that $Du, D\psi$ mean derivatives with respect to x of u and ψ, while Dv, Dz mean derivatives with respect to y of v and z. Since JG diagonalizes the matrix a, that is, (19) holds, we have proved that v satisfies

$$\int_{G(\Omega)} V^{q-2}(v) \sum_{s=1}^{n} \sum_{\beta=1}^{N} D_s v^\beta D_s z^\beta\, dy = 0 \qquad \forall\, z \in {}^*H_0^{1,2}(G(\Omega); v). \tag{23}$$

So we can apply Campanato's Theorem 1:

$$\int_{B(y^0,t)} |W(v)|^2\,dy \le \left(\frac{t}{R}\right)^n \int_{B(y^0,R)} |W(v)|^2\,dy, \tag{24}$$

$\forall\, y^0 \in G(\Omega),\ \forall\, t, R:\ 0 < t \le R < \operatorname{dist}(y^0, \partial G(\Omega))$.

Let x^0 belong to Ω and let r, R satisfy $0 < r \le \sqrt{\nu} R \le \sqrt{L} R < \operatorname{dist}(x^0, \partial\Omega)$, where ν and L are the constants in the ellipticity assumption (12); in this case $R < \operatorname{dist}(G(x^0), \partial G(\Omega))$ and, using (20), (21), (24), we get

$$\begin{aligned}
\int_{B(x^0,r)} |W(u)|^2\,dx &\le L^{n/2} \int_{B(G(x^0),r/\sqrt{\nu})} |W(u)|^2\,dx \le \\
&\le L^{n/2} \left(\frac{r/\sqrt{\nu}}{R}\right)^n \int_{B(G(x^0),R)} |W(v)|^2\,dy \le \\
&\le L^{n/2} \left(\frac{r/\sqrt{\nu}}{R}\right)^n \nu^{-n/2} \int_{B(x^0,\sqrt{L}R)} |W(u)|^2\,dx = \\
&= \left(\frac{L}{\nu}\right)^n \left(\frac{r}{\sqrt{L}R}\right)^n \int_{B(x^0,\sqrt{L}R)} |W(u)|^2\,dx.
\end{aligned}$$

We have proved the following inequality

$$\int_{B(x^0,\sqrt{L}R)} |W(u)|^2\,dx \le \left(\frac{L}{\nu}\right)^n \left(\frac{r}{\sqrt{L}R}\right)^n \int_{B(x^0,\sqrt{L}R)} |W(u)|^2\,dx \tag{25}$$

for $x^0 \in \Omega$ and $0 < r \le \sqrt{\nu} R \le \sqrt{L} R < \operatorname{dist}(x^0, \partial\Omega)$.

It is easy to check that (25) still remains true when $\sqrt{\nu} R < r \le \sqrt{L} R$, so the previous inequality (25) holds for $0 < r \le \sqrt{L} R < \operatorname{dist}(x^0, \partial\Omega)$. We set $s = \sqrt{L} R$ and we get our thesis (15):

$$\int_{B(x^0,r)} |W(u)|^2\,dx \le \left(\frac{L}{\nu}\right)^n \left(\frac{r}{s}\right)^n \int_{B(x^0,s)} |W(u)|^2\,dx.$$

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