Comment.Math.Univ.Carolin. 32,2 (1991)265–272

Generating real maps on a biordered set

Antonio Martinon

Abstract. Several authors have defined operational quantities derived from the norm of an operator between Banach spaces. This situation is generalized in this paper and we present a general framework in which we derivate several maps $X \to \mathbb{R}$ from an initial one $X \to \mathbb{R}$, where X is a set endowed with two orders, $\leq$ and $\leq^*$, related by certain conditions. We obtain only three different derivated maps, if the initial map is bounded and monotone.

Keywords: derivated map, biordered set, admissible order

Classification: 06A10, 47A53

1. Introduction.

We consider an infinite dimensional Banach space (over the real or the complex numbers), say X. The set of all the closed infinite dimensional subspaces of $X, S(X)$, is ordered by

$$M \leq N \text{ if and only if } M \subset N.$$

Also, we can define another order in $S(X)$:

$$M \leq^* N \text{ if and only if } M \subset N \text{ and } \dim(N/M) < \infty .$$

Both orders are related by the two following properties:

(1) If $M \leq^* N$, then $M \leq N$.

(2) If $M \leq N$ and $P \leq^* N$, then $M \cap P \leq^* M$.

If T is a linear and continuous operator from an infinite dimensional Banach space X into a Banach space Y, we consider the map

$$n : S(X) \to \mathbb{R}; \quad n(M) := n(TJ_M) := \|TJ_M\|,$$

where J_M is the injection of M into X and $\|\cdot\|$ denotes the norm. B. Gramsch (1969) (see [SC]) defined the operational quantity

$$in(T) := \inf_{M \leq X} n(TJ_M),$$

which can be used to characterize when an operator T is an upper semi-Fredholm operator (closed range and finite dimensional kernel): $in(T) > 0$. Independently,

Supported in part by DGICYT grant PB88–0417.

A.A. Sedaev (1970) [SE] and A. Lebow and M. Schechter (1971) [LS] consider the operational quantity

$$i^*n(T) := \inf_{M \leq^* X} n(TJ_M).$$

This quantity verifies that $i^*n(T) = 0$, if and only if T is a compact operator (the image of the closed unit ball of X is relatively compact). With a different definition, i^*n has been considered by H.-O. Tylli [TY]. The equality of both definitions has been showed in [GM2], [MA2]. Finally, M. Schechter (1972) [SC] defined the following operational quantity:

$$sin(T) := \sup_{M \leq X} in(TJ_M) = \sup_{M \leq X} \inf_{N \leq M} n(TJ_N).$$

This quantity verifies: $sin(T) = 0$, if and only if T is a strictly singular operator (if TJ_M is an injection, then M is finite dimensional).

If we consider the set of all the closed infinite codimensional subspaces of Y, $S'(Y)$, where Y is an infinite dimensional Banach space, then we define two orders in $S'(Y)$:

$$U \leq V \text{ if and only if } U \supset V;$$
$$U \leq^* V \text{ if and only if } U \supset V \text{ and } \dim(U/V) < \infty .$$

Now we obtain the following properties which relate $\leq$ with $\leq^*$,

(1) If $U \leq^* V$, then $U \leq V$.

(2) If $U \leq V$ and $W \leq^* V$, then $U + W \leq^* U$.

Let T be a linear and continuous operator from a Banach space X into an infinite dimensional Banach space Y. From the map

$$n' : S'(Y) \to \mathbb{R}; \quad n'(U) := n(Q_U T) := \|Q_U T\|,$$

where Q_U denotes the quotient map of Y onto Y/U, L. Weis (1976) [WE] derived the operational quantity

$$in'(T) := \inf_{U \leq 0} n'(Q_U T)$$

which can be used to characterize a class of operators: $in'(T) > 0$ if and only if T is a lower semi-Fredholm operator (closed and finite codimensional range). Independently, A.S. Fajnshtejn and V.S. Shulman (1982) (see [FA]) and J. Zemanek (1983) [ZE] consider the operational quantity

$$i^*n'(T) := \inf_{U \leq^* 0} n'(Q_U T).$$

This quantity verifies that $i^*n'(T) = 0$, if and only if T is a compact operator. A.S. Fajnshtejn [FA] has showed that the quantity i^*n' agrees with the Hausdorff measure of noncompactness, which was introduced by Goldenstein, Gohberg and

Markus (1957) (see [BG]). Finally, L. Weis (1976) [WE] defined the following operational quantity:

$$sin'(T) := \sup_{U\leq 0} in'(Q_U T) = \sup_{U\leq 0} \inf_{V\leq U} n'(Q_V T).$$

This quantity verifies: $sin'(T) = 0$, if and only if T is a strictly cosingular operator (if $Q_U T$ is a surjection, then U is finite codimensional).

If we consider the injection modulus and the surjection modulus, instead of the norm, there can be obtained new operational quantities. If T is a linear and continuous operator, then the injection modulus of T is defined by

$$j(T) := \inf\{\|Tx\| : x \in B_X\},$$

and the surjection modulus of T by

$$q(T) := \sup\{\varepsilon > 0 : \varepsilon B_Y \subset T B_X\},$$

where B_X is the closed unit ball of X. M. Schechter (1972) [SC] considers the following operational quantities:

$$sj(T) := \sup_{M\leq X} j(TJ_M),$$
$$s^*j(T) := \sup_{M\leq^* X} j(TJ_M).$$

He verifies that $sj(T) = 0$, if and only if T is a strictly singular operator and $s^*j(T) > 0$, if and only if T is an upper semi-Fredholm operator. The author (1989) [MA1], [MA2] has defined the operational quantity

$$isj(T) := \inf_{M\leq X} sj(TJ_M) = \inf_{M\leq X} \sup_{N\leq M} j(TJ_N)$$

and showed that $isj(T) > 0$, if and only if T is an upper semi-Fredholm operator. The quantities iq, siq and i^*q, similarly defined, verify $iq = siq = i^*q = 0$. J. Zemanek (1983) [ZE] defines the following operational quantities:

$$sq'(T) := \sup_{U\leq 0} q(Q_U T),$$
$$s^*q'(T) := \sup_{U\leq^* 0} q(Q_U T),$$

where 0 is the null subspace of Y. They verify that $sq'(T) = 0$, if and only if T is a strictly cosingular operator and $s^*q'(T) > 0$, if and only if T is a lower semi-Fredholm operator. The author (1989) [MA1], [MA2] has defined the operational quantity

$$isq'(T) := \inf_{U\leq 0} sq'(Q_U T) = \inf_{U\leq 0} \sup_{V\leq U} q(Q_V T)$$

and showed that $isq'(T) > 0$, if and only if T is a lower semi-Fredholm operator. The quantities ij', sij' and i^*j', similarly defined, verify $ij = sij = i^*j' = 0$.

It is possible to consider other operational quantities by using inf and sup: $isin, i^*s^*si^*n, \ldots$, but there are only three different quantities: in, i^*n, sin. Analogously it occurs with n', j and q' [MA2].

If we consider a space ideal $\mathbb{A}$ (in the sense of A. Pietsch [PI]) and the set $S_{\mathbb{A}}(X)$ (respectively $S'_{\mathbb{A}}(Y)$), defined as the set of all the subspaces M of X (U of Y) such that $M(Y/U)$ does not belong to $\mathbb{A}$, then we can define operational quantities of a similar way as above. This procedure is used in [GM1], [GM3], [MA2] to define classes of operators which generalize the classes of the semi-Fredholm operators, strictly singular operators and strictly cosingular operators.

In this paper, we consider a general situation. Let X be a set endowed with two orders, $\leq$ and $\leq^*$, related by similar conditions of (1) and (2). We show that if $a : X \to \mathbb{R}$ is bounded and monotone, then we obtain only three new maps: ia, sia, i^*a (if a is increasing) or sa, isa, s^*a (if a is decreasing).

2. Generating real maps on an ordered set.

In this paper, $(X, \leq)$ is a (partially) ordered set. We denote $B(X, \mathbb{R})$ the set of bounded maps of X in R. We define the maps i and s on $B(X, \mathbb{R})$ in the following way: for $a \in B(X, \mathbb{R})$ and $x \in X$,

$$ia(x) := \inf_{z \leq x} a(z),$$
$$sa(x) := \sup_{z \leq x} a(z).$$

Note that sa is the infimum of all increasing maps $b \in B(X, \mathbb{R})$ such that $a \leq b$ and ia is the supremum of all decreasing maps $c \in B(X, \mathbb{R})$ such that $c \leq a$. That is, sa is the lower hull of the family $\{b \in B(X, \mathbb{R}) : a \leq b, b \text{ increasing }\}$ and ia is the upper hull of the family $\{c \in B(X, \mathbb{R}) : c \leq a, c \text{ decreasing }\}$ [BO, IV, S5, No. 5].

We can iterate the procedure obtaining many derivated maps from a : $isa, ssa, sissia, \ldots$. If a is monotone, we only obtain two different new maps.

We will denote a increasing by $a_\uparrow$ and a decreasing by $a^\downarrow$.

Proposition 1. *Suppose $(X, \leq)$ is an ordered set and $a \in B(X, \mathbb{R})$ is monotone.*

(1) *If $a_\uparrow$, then $ia^\downarrow, sia_\uparrow$, and they are the only different derivated maps which are obtained from a using i and s. Moreover,*

$$ia^\downarrow \leq sia_\uparrow \leq a_\uparrow\,.$$

(2) *If $a^\downarrow$, then $sa_\uparrow, isa^\downarrow$, and they are the only different derivated maps which are obtained from a using i and s. Moreover,*

$$a^\downarrow \leq isa^\downarrow \leq sa_\uparrow\,.$$

Proof: We give a proof in several steps. For every a (monotone or not), we obtain that

$$ia^{\downarrow} \leq a \leq sa_{\uparrow}\,. \tag{1}$$

Moreover,

$$(-a)_{\uparrow} \Leftrightarrow a^{\downarrow}; \quad i(-a) = -sa. \tag{2}$$

In the "first generation", we obtain ia and sa. If $a_{\uparrow}$, then $a = sa$, hence

$$a_{\uparrow} \Rightarrow ia^{\downarrow} \leq a = sa_{\uparrow}\,. \tag{3}$$

Analogously

$$a^{\downarrow} \Rightarrow ia = a^{\downarrow} \leq sa_{\uparrow}\,. \tag{4}$$

In the "second generation": If $a_{\uparrow}$, then we obtain iia and sia. Because $ia^{\downarrow}$, by (4), it is $iia = ia$. On the other hand, by (1), it is $ia \leq sia$ and $sia \leq sa = a$. Hence

$$a_{\uparrow} \Rightarrow ia^{\downarrow} \leq sia_{\uparrow} \leq a_{\uparrow}\,. \tag{5}$$

Analogously, by (2),

$$a^{\downarrow} \Rightarrow a^{\downarrow} \leq isa^{\downarrow} \leq sa_{\uparrow}\,. \tag{6}$$

In the "third generation": If $a_{\uparrow}$, then we obtain $isia$ and $ssia$. Because $sia_{\uparrow}$, using (3), it is $ssia = sia$. On the other hand, using (5), it is

$$iis = ia \leq isia \leq ia,$$

hence $ia = isia$. Analogously, by (2), if $a^{\downarrow}$, then $iisa = sa$ and $sisa = sa$. □

3. Generating real maps on a biordered set.

Let $\leq^*$ be another order on X (that is, $(X, \leq^*)$ is an ordered set). If $a \in B(X, \mathbb{R})$ is $*$-monotone ($a_{\uparrow *}$ or $a^{\downarrow *}$), then using i^* and s^* (defined using $\leq^*$ instead of $\leq$), by Proposition 1, we can write

$$\begin{aligned} a_{\uparrow *} \Rightarrow & i^* a^{\downarrow *} \leq s^* i^* a_{\uparrow *} \leq a_{\uparrow *}\,, \\ a^{\downarrow *} \Rightarrow & a^{\downarrow *} \leq i^* s^* a^{\downarrow *} \leq s^* a_{\uparrow *}\,. \end{aligned}$$

In the following results, we consider the case a monotone (for $\leq$), when $\leq^*$ verifies a certain condition related to $\leq$.

If $(X, \leq)$ and $(X, \leq^*)$ are ordered sets, we say that $\leq^*$ is admissible with regard to $\leq$, if

(1) $x \leq^* y \Rightarrow x \leq y$, and moreover,

(2) $y \leq x$ and $z \leq^* x \Rightarrow \exists\, y \cap z$ and $y \cap z \leq^* y$,

$y \cap z$ being the infimum of $\{y, z\}$ for $\leq$. If $\leq^*$ is admissible with regard to $\leq$, then $(X, \leq, \leq^*)$ will be called a biordered set.

Let E be an infinite set. The set

$$\mathcal{P}_\infty(E) := \{A \subset E : A \text{ infinite }\}$$

is a simple example of a biordered set, taking $A \leq B \Leftrightarrow A \subset B, A \leq^* B \Leftrightarrow A \subset B$ and $B \setminus A$ finite. Note that $A \leq^* B$, if and only if A belongs to the Fréchet filter on B.

Proposition 2. *Suppose $(X, \leq, \leq^*)$ is a biordered set and $a \in B(X, \mathbb{R})$ is monotone.*

(1) *If $a_\uparrow$, then $i^*a_\uparrow$ is the only derivated map which is obtained using i^* and s^*. Moreover,*
$$ia^\downarrow \leq sia_\uparrow \leq i^*a_\uparrow \leq a_\uparrow \,.$$

(2) *If $a^\downarrow$, then $s^*a^\downarrow$ is the only derivated map which is obtained using i^* and s^*. Moreover,*
$$a^\downarrow \leq s^*a^\downarrow \leq isa^\downarrow \leq sa_\uparrow \,.$$

Proof: We give only the proof of (1). (2) can be obtained analogously.

We have $i^*a_\uparrow$: let $x, y \in X$ with $x \leq y$, and let $\varepsilon > 0$. Then there exists $z \leq^* y$ such that $a(z) < i^*a(y) + \varepsilon$. As $\leq^*$ is admissible with regard to $\leq$, there exists $x \cap z \leq^* x$ and hence

$$i^*a(x) \leq a(x \cap z) \leq a(z) < i^*a(y) + \varepsilon$$

for every $\varepsilon > 0$. Consequently, $i^*a(x) \leq i^*a(y)$.

It is obvious that $ia \leq i^*a \leq s^*a = sa = a$. Moreover, using $i^*a_\uparrow$, we obtain $sia \leq si^*a = i^*a \leq a$.

In the "second generation", using i^* and s^*, we obtain i^*i^*a and s^*i^*a. Using Proposition 1, we obtain $i^*i^*a = i^*a$, because $i^*a^\downarrow*$. From $i^*a_\uparrow$ it results $s^*i^*a = i^*a$. □

Proposition 3. *Suppose $(X, \leq, \leq^*)$ is a biordered set and $a \in B(X, \mathbb{R})$ is monotone.*

(1) *If $a_\uparrow$, then i^*a, sia, ia are constant on $\{z \in X : z \leq^* x\}$ for every $x \in X$.*

(2) *If $a^\downarrow$, then s^*a, isa, sa are constant on $\{z \in X : z \leq^* x\}$ for every $x \in X$.*

Proof: We give only the proof of (2). (1) can be obtained analogously.

Let $x \in X$ and $z \leq^* x$, hence $z \leq x$. From $s^*a_\uparrow*$, we obtain $s^*a(z) \leq s^*a(x)$. From $s^*a^\downarrow$, we obtain $s^*a(z) \geq s^*a(x)$. Hence s^*a is constant on $\{z \in X : z \leq^* x\}$.

From $sa_\uparrow$, we obtain $sa(z) \leq sa(x)$. On the other hand, for every $\varepsilon > 0$ there exists $y \in X$, with $y \leq x$, such that $a(y) > sa(x) - \varepsilon$. As $\leq^*$ is admissible with regard to $\leq$, there exists $y \cap z$. Hence

$$sa(x) - \varepsilon < a(y) \leq a(y \cap z) \leq sa(z).$$

Consequently $sa(x) \leq sa(z)$ and sa is constant on $\{z \in X : z \leq^* x\}$.
It follows from (1) and $sa_\uparrow$ that isa is constant. □

Propositions 1 and 2 assure us that there is only a finite number of different derivated maps which are obtained using i and s, or i^* and s^*. The following theorem assures the same result when we use i, s, i^* and s^*.

Theorem 4. *Suppose $(X, \leq, \leq^*)$ is a biordered set and $a \in B(X, \mathbb{R})$ is monotone.*

(1) *If $a_\uparrow$, then $ia^\downarrow, sia_\uparrow, i^*a_\uparrow$ are the only different derivated maps obtained from a using i, s, i^* and s^*. Moreover*

$$ia^\downarrow \leq sia_\uparrow \leq i^*a_\uparrow \leq a_\uparrow\,.$$

(2) *If $a^\downarrow$, then $sa_\uparrow, isa^\downarrow, s^*a^\downarrow$ are the only different derivated maps obtained from a using i, s, i^* and s^*. Moreover*

$$a^\downarrow \leq s^*a^\downarrow \leq isa^\downarrow \leq sa_\uparrow\,.$$

Proof: Using Propositions 1, 2 and 3, and the techniques of Propositions 1 and 2, we can see that the generation process ends in a finite number of steps which are represented in the following diagrams:

$$a_\uparrow \left| \begin{array}{l} sa = a \\ s^*a = a \\ i^*a_\uparrow \left| \begin{array}{l} ii^*a = ia \\ \\ si^*a = i^*i^*a = s^*i^*a = i^*a \end{array}\right. \\ \\ ia^\downarrow \left| \begin{array}{l} iia = i^*ia = s^*ia = ia \\ sia_\uparrow \left| \begin{array}{l} isia = ia \\ \\ ssia = i^*sia = s^*sia = sia \end{array}\right. \end{array}\right. \end{array}\right.$$

$$a^\downarrow \left| \begin{array}{l} ia = a \\ i^*a = a \\ s^*a^\downarrow \left| \begin{array}{l} ss^*a = a \\ \\ is^*a = s^*s^*a = i^*s^*a = s^*a \end{array}\right. \\ \\ sa_\uparrow \left| \begin{array}{l} ssa = s^*sa = i^*sa = sa \\ isa_\uparrow \left| \begin{array}{l} sisa = sa \\ \\ iisa = s^*isa = i^*isa = isa \end{array}\right. \end{array}\right. \end{array}\right.$$

For example, using Proposition 3 we obtain $\alpha^*\beta a = \beta a$, with $\alpha, \beta \in \{i, s\}$. □

References

[BG] Banas J., Goebel K., *Measures of Noncompactness in Banach Spaces*, Marcel Dekker, Lecture Notes in Pure and Applied Math., vol.60; New York, Basel, 1980.

[BO] Bourbaki N., *Éléments de Mathématique. Topologie Générale*, Hermann, Paris, 1971.

[FA] Fajnshtejn A.S., *On measures of noncompactness of linear operators and analogs of the minimal modulus for semi-Fredholm operators* (in Russian), Spektr. Teor. Oper. **6** (1985), 182–195, Zbl 634 #47010.

[GM1] Gonzalez M., Martinon A., *Operational quantities and operator ideals*, Actas XIV Jornadas Hispano–Lusas Mat. (1989), Univ. La Laguna, vol. I, 1990, 383–387.

[GM2] Gonzalez M., Martinon A., *Operational quantities derived from the norm and measures of noncompactness*, Proc. Royal Irish Acad., to appear.

[GM3] Gonzalez M., Martinon A., *A generalization of semi-Fredholm operators*, 1990, preprint.

[LS] Lebow A., Schechter M., *Semigroups of operators and measures of noncompactness*, J. Func. Anal. **7** (1971), 1–26.

[MA1] Martinon A., *Generating operational quantities in Fredholm theory*, Actas XIV Jornadas Hispano–Lusas Mat. (1989), Univ. La Laguna, vol. I, 1990, 449–453.

[MA2] Martinon A., *Cantidades operacionales en Teoría de Fredholm (Tesis)*, Univ. La Laguna, 1989.

[PI] Pietsch A., *Operator Ideals*, North-Holland; Amsterdam, New York, Oxford, 1980.

[SC] Schechter M., *Quantities related to strictly singular operators*, Indiana Univ. Math. J. **21** (1972), 1061–1071.

[SE] Sedaev A.A., *The structure of certain linear operators* (in Russian), Mat. Issled. **5** (1970), 166–175, MR 43 #2540; Zbl. 247 #47005.

[TY] Tylli H.-O., *On the asymptotic behaviour of some quantities related to semi-Fredholm operators*, J. London Math. Soc. (2) **31** (1985), 340–348.

[WE] Weis L., *Über strikt singulare und strikt cosingulare Operatoren in Banachräumen* (Dissertation), Univ. Bonn, 1974.

[ZE] Zemanek J., *Geometric characteristics of semi-Fredholm operators and their asymptotic behaviour*, Studia Math. **80** (1984), 219–234.

Department of Mathematical Analysis, University of La Laguna, 38271 La Laguna, Tenerife, Spain

(*Received* September 18, 1990)