Einstein metrics on a class of five-dimensional homogeneous spaces

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Abstract. We prove that there is exactly one homothety class of invariant Einstein metrics in each space $SU(2) \times SU(2)/SO(2)_r$ ($r \in Q$, $|r| \neq 1$) defined below.

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It is well-known [J] that any homogeneous Einstein manifold M^n for $n \leq 4$ is a Riemannian symmetric space. On the other hand, little is known about five-dimensional homogeneous Einstein manifolds which are not locally symmetric (cf. [B, p. 186]).

In this paper, we study a special family of homogeneous spaces $M_r^5 = SU(2) \times SU(2)/SO(2)_r$, where $SO(2)_r, r \in Q$, denotes the subgroup of all product matrices of the form:

$$\begin{pmatrix} e^{2\pi it} & 0 \\ 0 & e^{-2\pi it} \end{pmatrix} \times \begin{pmatrix} e^{2\pi irt} & 0 \\ 0 & e^{-2\pi irt} \end{pmatrix} \qquad (t \in R).$$

We prove the existence, up to a homothety, of a unique invariant Einstein metric on each $M_r^5 (|r| \neq 1)$. These metrics are never naturally reductive.

1. Preliminaries.

Let $su(2)$ denote the Lie algebra of $SU(2)$ provided by the scalar product $B(x,y) = -\frac{1}{2}\operatorname{Retr} xy$. We consider an orthonormal basis $\{x_1, x_2, x_3\}$ of $su(2)$ such that $[x_1, x_2] = x_3, [x_1, x_3] = -x_2, [x_2, x_3] = x_1$ and, moreover, the Lie algebra h of $H = SO(2)_r (r \in Q)$ is of the form $h = R \cdot (x_1, rx_1)$. Put $G = SU(2) \times SU(2)$, then $g = su(2) \oplus su(2)$ is the corresponding Lie algebra. Consider the scalar product on g given by $B \mid_{g \times g} = B_{su(2)} + B_{su(2)}$. Then we have a B-orthogonal decomposition $g = h \oplus p_1 \oplus p_2 \oplus p_3$, where $p_1 = R \cdot (rx_1, -x_1), p_2 = R \cdot (x_2, 0) + R \cdot (x_3, o), p_3 = R \cdot (0, x_2) + R \cdot (0, x_3)$. Moreover, p_1, p_2, p_3 are irreducible invariant subspaces w.r. to the adjoint representation $\operatorname{ad} h$ on $p = p_1 \oplus p_2 \oplus p_3$ and $p_1 \not\simeq p_2, p_1 \not\simeq p_3$ w.r. to this representation.

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Lemma 1.1. *We have $p_2 \not\simeq p_3$ for $|r| \neq 0, 1$ with respect to the adjoint representation of h on p.*

PROOF: Suppose that there exists an isomorphism $\varrho : p_2 \to p_3$ such that $\operatorname{ad} W \circ \varrho = \varrho \circ \operatorname{ad} W$ for every $W \in h$. Put $A = (x_2, 0), B = (x_3, 0) \in p_2$. Then we can write $\varrho(A) = (0, x), \varrho(B) = (0, y)$, where $x, y \in \operatorname{span}(x_2, x_3)$. For $W = (x_1, rx_1)$, we have $[W, \varrho(A)] = \varrho([W, A]) = \varrho(B)$, and also $[W, \varrho(B)] = -\varrho(A)$. Hence we get $r[x_1, x] = y, r[x_1, y] = -x$. Further, since the Lie bracket $[x, y]$ on $su(2)$ coincides with the usual vector cross-product, we obtain immediately $r\|[x_1, x]\| = r\|x\| = \|y\|$ and $r\|[x_1, y]\| = r\|y\| = \|x\|$. Hence the equality $r^2 = 1$ holds, which is a contradiction. □

Corollary 1.1. *For $|r| \neq 0, 1$, every $\operatorname{Ad}(H)$-invariant scalar product $\langle\cdot,\cdot\rangle$ on p has, up to a constant factor, the following form:*

$$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_\circ |_{p_2 \times p_1} + t\langle\cdot,\cdot\rangle_\circ |_{p_2 \times p_2} + s\langle\cdot,\cdot\rangle_\circ |_{p_2 \times p_2}, \tag{1.1}$$

where $t, s \in R^+$ and $\langle\cdot,\cdot\rangle_\circ = B |_{g \times g}$.

The proof follows from Lemma 1.1 and the Schur's lemma.

We construct a scalar product $(\cdot,\cdot)$ on $g = su(2) \oplus su(2)$ by setting $(\cdot,\cdot) = \langle\cdot,\cdot\rangle |_{p \times p} + \langle\cdot,\cdot\rangle_\circ |_{h \times h}$. Then we consider the following $(\cdot,\cdot)$-orthonormal basis of g:

$$E_1 = (r\alpha x_1, -\alpha x_1),\ E_2 = 1/\sqrt{t}(x_2, 0),\ E_3 = 1/\sqrt{t}(x_3, 0),$$
$$E_4 = 1/\sqrt{s}(0, x_2),\ E_5 = 1/\sqrt{s}(0, x_3),\ E_6 = (\alpha x_1, r\alpha x_1),$$

where $\alpha = (r^2 + 1)^{-1/2}$.

It is obvious that $h = R{\cdot}E_6, p_1 = R{\cdot}E_1, p_2 = R{\cdot}E_2 + R{\cdot}E_3$ and $p_3 = R{\cdot}E_4 + R{\cdot}E_5$.

Lemma 1.2. *We have the following multiplication table:*

$$[E_1, E_2] = r\alpha E_3,\ [E_1, E_3] = -r\alpha E_2,\ [E_1, E_4] = -\alpha E_5,\ [E_1, E_5] = \alpha E_4,$$
$$[E_2, E_3] = r\alpha t^{-1} \cdot E_1 + \alpha t^{-1} \cdot E_6,\ [E_2, E_4] = [E_2, E_5] = [E_3, E_4] = [E_3, E_5] = 0,$$
$$[E_4, E_5] = \alpha s^{-1} \cdot E_1 + r\alpha s^{-1} \cdot E_6,\ [E_3, E_6] = \alpha E_2,\ [E_5, E_6] = r\alpha E_4,$$

where $\alpha = (r^2 + 1)^{-1/2}$.

The proof is straightforward and can be omitted.

Corollary 1.2. *The multiplication table of Lemma 1.2 implies that:*

$$[p_1, p_1] = 0,\ [p_1, p_2] = p_2,\ [p_1, p_3] = p_3,\ [p_2, p_3] = 0,$$
$$[p_2, p_2] \subset p_1 \oplus h,\ [p_3, p_3] \subset p_1 \oplus h.$$

2. The computation of the sectional curvatures.

In this part, we shall use the notation of the previous part. First, we see that every G-invariant Riemannian metric on $M_r^5(|r| \neq 0, 1)$ is determined, up to a homothety, by an $\mathrm{Ad}(H)$-invariant scalar product of the form (1.1). Further, the sectional curvatures of such metric can be calculated by means of the standard formula (see [B]):

$$
\begin{aligned}
\langle R(X,Y)Y,X\rangle &= (-3/4)\langle [X,Y]_p, [X,Y]_p\rangle - \langle [[X,Y]_h, Y], X\rangle - \\
&- (1/2)\langle Y, [X,[X,Y]_p]_p\rangle - (1/2)\langle X, [Y,[Y,X]_p]_p\rangle + \\
&+ \langle u(X,Y), u(X,Y)\rangle - \langle u(X,X), u(Y,Y)\rangle,
\end{aligned}
\tag{2.1}
$$

where $X, Y \in p, \langle\cdot,\cdot\rangle$ is the corresponding scalar product on p and the mapping $u : p \times p \to p$ is defined by the formula:

$$
2\langle u(X,Y), Z\rangle = \langle [Z,X]_p, Y\rangle + \langle [Z,Y]_p, X\rangle \tag{2.2}
$$

for all $Z \in p$.

Lemma 2.1. *For an* $\mathrm{Ad}(H)$*-invariant scalar product* $\langle\cdot,\cdot\rangle$ *of the form* (1.1), *the following formulas are true:*

$$
\begin{aligned}
&u(X,Y) = (t-1)/2t[X,Y], \qquad \textit{where } \ X \in p_1, Y \in p_2, \\
&u(X,Z) = (s-1)/2s[X,Z], \qquad \textit{where } \ X \in p_1, Z \in p_3, \\
&u(p_1,p_1) = u(p_2,p_2) = u(p_3,p_3) = u(p_2,p_3) = 0.
\end{aligned}
\tag{2.3}
$$

Proof: We shall use Corollary 1.2, the formula (2.2) and the notations of Corollary 1.1. Let $X \in p_1, Y \in p_2$. If $Z \in p_1$, then $[Z,X]_p = 0, [Z,Y]_p \in p_2$ and hence $\langle u(X,Y), Z\rangle = 0$. Further, if $Z \in p_3$, then $[Z,X]_p \in p_3, [Z,Y]_p = 0$ and also $\langle u(X,Y), Z\rangle = 0$. Therefore $u(X,Y) \in p_2$. Let $Z \in p_2$, then $[Z,X]_p \in p_2, [Z,Y]_p \in p_1$ and we have:

$$
2t\langle u(X,Y), Z\rangle_\circ = t\langle [Z,X]_p, Y\rangle_\circ + \langle [Z,Y]_p, X\rangle_\circ\,.
$$

But

$$
\begin{aligned}
&t\langle [Z,X]_p, Y\rangle_\circ + \langle [Z,Y]_p, X\rangle_\circ = \\
&= t\langle [Z,X], Y\rangle_\circ + \langle [Z,Y], X\rangle_\circ = \\
&= t\langle [X,Y], Z\rangle_\circ - \langle [X,Y], Z\rangle_\circ\,,
\end{aligned}
$$

since $\langle\cdot,\cdot\rangle_\circ$ is $\mathrm{Ad}(G)$-invariant. Hence $u(X,Y) = (t-1)/2t[X,Y]$ for $X \in p_1, Y \in p_2$. The other cases are treated analogously. □

Lemma 2.2. *For the sectional curvatures of $M_r^5(r \neq 0)$ we have:*

$$\begin{aligned}
&K_\sigma(E_1,E_2) = K_\sigma(E_1,E_3) = (r\alpha/2t)^2,\\
&K_\sigma(E_1,E_4) = K_\sigma(E_1,E_5) = (\alpha/2s)^2,\\
&K_\sigma(E_2,E_3) = 1/t - 3\alpha^2/4t^2,\ K_\sigma(E_2,E_4) = K_\sigma(E_2,E_5) = 0,\\
&K_\sigma(E_4,E_5) = 1/s - 3\alpha^2/4t^2,\ K_\sigma(E_4,E_3) = 0,
\end{aligned}$$

where $\alpha = (r^2+1)^{-1/2}$.

PROOF: Let us calculate $K_\sigma(E_1,E_2)$. From the formula (2.1) and Lemmas 1.2, 2.1 we have:

$$\begin{aligned}
K_\sigma(E_1,E_2) &= (-3/4)\langle r\alpha E_3, r\alpha E_3\rangle - (1/2)\langle E_2,[E_1,r\alpha E_3]_p\rangle -\\
&- (1/2)\langle E_1,[E_2,(-r\alpha)E_3]_p\rangle + ((t-1)/2t)^2\langle [E_1,E_2],\,[E_1,E_2]\rangle =\\
&= (-3/4)r^2\alpha^2 + r^2\alpha^2/2 + r^2\alpha^2/2t + ((t-1)/2t)^2 r^2\alpha^2 = (r\alpha/2t)^2\,.
\end{aligned}$$

The other sectional curvatures are calculated analogously. □

Corollary 2.1. *For the Ricci curvatures of $M_r^5(r \neq 0)$ we have:*

$$\begin{aligned}
&\mathrm{ricc}\,(E_1) = (r^2s^2+t^2)\alpha^2/2t^2s^2,\ \mathrm{ricc}\,(E_2) = (2t(r^2+1)-r^2)\alpha^2/2t^2,\\
&\mathrm{ricc}\,(E_4) = (2s(r^2+1)-1)\alpha^2/2s^2, \quad \textit{where}\ \ \alpha = (r^2+1)^{-1/2}\,.
\end{aligned}$$

The proof follows from the Lemma 2.2 by a straightforward computation.

3. Invariant Einstein metrics on M_r^5.

We start with

Lemma 3.1. *Let $\langle\cdot,\cdot\rangle$ be an* $\mathrm{Ad}(H)$*-invariant scalar product on p of the form* (1.1). *Then the invariant Einstein metrics on $M_r^5(|r| \neq 0,1)$ are defined by the formulas*

$$t = \frac{|r|(2y^2+1)}{2|y|(r^2+1)}, \quad s = \frac{2y^2+1}{r^2+1}\,, \tag{3.1}$$

where y is any real root of the equation $8|y|^3 - 8|r|y^2 + 4|y| - |r| = 0$.

PROOF: Since $SO(2)_r$ acts transitively on p_1, p_2, p_3 and preserves the Ricci curvature, then $\langle\cdot,\cdot\rangle$ is Einsteinian iff $\mathrm{ricc}(E_1) = \mathrm{ricc}(E_2) = \mathrm{ricc}(E_4)$. But from Corollary 2.1 we see that this is equivalent to the formulas:

$$\begin{cases} \frac{r^2s^2+t^2}{s^2} = 2t(r^2+1) - r^2 \\ \frac{r^2s^2+t^2}{t^2} = 2s(r^2+1) - 1 \end{cases} \qquad (t,s \in R^+)$$

or

$$\begin{cases} 2\{t(r^2+1) - r^2\} = t^2/s^2 \\ 2\{s(r^2+1) - 1\}/r^2 = s^2/t^2 \end{cases} \qquad (t,s \in R^+).$$

From here we can express the parameter t in two different ways:

$$\begin{cases} t = \frac{r^2}{4(r^2+1)\{s(r^2+1)-1\}} + \frac{r^2}{r^2+1} \\ t = \frac{s|r|}{\sqrt{2\{s(r^2+1)-1\}}} \end{cases} \qquad (t, s \in R^+, s > \frac{1}{r^2+1}) .$$

Hence we get the equation

$$\frac{r^2}{4(r^2+1)\{s(r^2+1)-1\}} + \frac{r^2}{r^2+1} = \frac{s|r|}{\sqrt{2\{s(r^2+1)-1\}}} .$$

After the substitution $s(r^2+1)-1 = 2y^2$, we obtain $8|y|^3 - 8|r|y^2 + 4|y| - |r| = 0$. Further, for the discriminant $\mathcal{D}$, we get $\mathcal{D} = \frac{1}{6^3}(r^4 - \frac{61}{2^5} \cdot r^2 + 1) > 0$, and hence we have only one real root. The formulas (3.1) now follow easily, and this completes the proof. □

Hence we obtain the first part of the following

Theorem 3.1. *For each $r \in Q(|r| \neq 0, 1)$, there exists, up to a homothety, a unique invariant Einstein metric on M_r^5. This metric is never naturally reductive.*

Proof of the second part: In fact, we only have to compare our metrics with the family of naturally reductive metrics of the type I from [K–V]. This can be done by a direct computation. □

Remark 1. Let us note that the property "never naturally reductive" means "not naturally reductive whatever is the group representation $M_r^5 = G/H(G \subset I(M_r^5))$ and whatever is the $\mathrm{Ad}(H)$-invariant decomposition $g = h \oplus p$", cf. [K–V].

Remark 2. For $r = 0$, we obtain the decomposable homogeneous space $S^2 \times S^3$, where the invariant Einstein metrics are well-known. All of them are naturally reductive (cf. [B]).

The cases M_{-1}^5, M_1^5 are to be studied separately.

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