On absolute summability factors for $|\overline{N}, p_n|_k$ summability

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Abstract. In this paper a theorem on $|\overline{N}, p_n|_k$ summability factors, which generalizes a theorem of Mishra and Srivastava [MS] on $[C, 1]_k$ summability factors, has been proved.

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1. Introduction.

Let $\sum_0^\infty a_n$ be a given infinite series with partial sums (s_n). By u_n^δ we denote the n-th Cesàro mean of order δ ($\delta > -1$ and δ is real) of the sequence (s_n). The series $\sum a_n$ is said to be summable $|C, \delta|_k, k \geq 1$, if (see [F])

$$\sum_{n=1}^{\infty} n^{k-1} |u_n^\delta - u_{n-1}^\delta|^k < \infty. \tag{1.1}$$

Let (p_n) be a sequence of positive real constants such that

$$P_n = \sum_{u=0}^{n} p_u \to \infty \quad \text{as} \quad n \to \infty. \tag{1.2}$$

The sequence-to-sequence transformation

$$t_n = \frac{1}{P_n} \sum_{u=0}^{n} p_u s_u \tag{1.3}$$

defines the sequence (t_n) of the $(\overline{N}, p_n)$ means of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [H, p. 57]). The series $\sum a_n$ is said to be summable $|\overline{N}, p_n|_k, k \geq 1$, if (see [B])

$$\sum_{n=1}^{\infty} (P_n/p_n)^{k-1} |t_n - t_{n-1}|^k < \infty. \tag{1.4}$$

In the special case when $p_n = 1$ for all values of n, $|\overline{N}, p_n|_k$ summability is the same as $|C, 1|_k$ summability.

Let K be a positive constant. If $g > 0$, then $f = O(g)$ means $|f| < K \cdot g$ and $f = o(g)$ means $f/g \to 0$ (see [H, p. XVI]).

2. Mishra and Srivastava [MS] proved the following theorem for $|C, 1|_k$ summability.

Theorem A. *Let* (X_n) *be a positive non-decreasing sequence and be there sequences* (β_n) *and* (λ_n) *such that*

$$|\Delta\lambda_n| \leq \beta_n, \tag{2.1}$$

$$\beta_n \to 0 \quad as \quad n \to \infty, \tag{2.2}$$

$$|\lambda_n| X_n = O(1) \quad as \quad n \to \infty, \tag{2.3}$$

$$\sum_{n=1}^{\infty} n X_n |\Delta\beta_n| < \infty. \tag{2.4}$$

If

$$\sum_{n=1}^{m} \frac{1}{n} |s_n|^k = O(X_n) \quad as \quad m \to \infty, \tag{2.5}$$

then the series $\sum a_n \lambda_n$ *is summable* $|C, 1|_k, k \geq 1$.

3. The aim of this paper is to generalize Theorem A for $|\overline{N}, p_n|_k$ summability. Now, we shall prove the following theorem.

Theorem. *Let* (X_n) *be a positive non-decreasing sequence and the sequences* (λ_n) *and* (β_n) *are such that conditions* (2.1)–(2.3) *of Theorem A are satisfied. Furthermore, if*

$$\sum_{n=1}^{\infty} P_n X_n |\Delta\beta_n| < \infty, \tag{3.1}$$

$$\sum_{n=1}^{m} \frac{p_n}{P_n} |s_n|^k = O(X_m) \quad as \quad m \to \infty, \tag{3.2}$$

and

$$1 = O(p_n) \quad as \quad n \to \infty \tag{3.3}$$

then the series $\sum a_n \lambda_n$ *is summable* $|\overline{N}, p_n|_k, k \geq 1$.

Remark. It should be noted that if we take $p_n = 1$ for all values of n, then the conditions (3.1) and (3.2) will be reduced to the conditions (2.4) and (2.5), respectively. Also notice that in this case condition (3.3) is obvious.

4. We need the following lemma for the proof of our theorem.

Lemma. *Under the conditions of the theorem, we have*

$$P_n X_n \beta_n = o(1) \quad as \quad n \to \infty, \tag{4.1}$$

$$\sum_{n=1}^{\infty} p_n X_n \beta_n < \infty, \tag{4.2}$$

$$\sum_{n=1}^{\infty} \beta_n X_n < \infty. \tag{4.3}$$

PROOF: Since $\beta_n \to 0$ as $n \to \infty$, by (2.2), we have that

$$\beta_n = \sum_{u=n}^{\infty} \Delta\beta_u \,. \tag{4.4}$$

Since $(X_n P_n)$ is increasing, we have

$$P_n X_n \beta_n \leq \sum_{u=n}^{\infty} P_u |\Delta\beta_u| X_u < \infty,$$

by (3.1). Hence

$$P_n X_n \beta_n = o(1) \quad \text{as} \quad n \to \infty.$$

Since (X_n) is increasing, using (4.4), we have that

$$\begin{aligned}\sum_{n=1}^{\infty} p_n X_n \beta_n &\leq \sum_{n=1}^{\infty} p_n X_n \sum_{u=n}^{\infty} |\Delta\beta_u| = \sum_{u=1}^{\infty} |\Delta\beta_u| \sum_{n=1}^{u} p_n X_n \\ &\leq \sum_{u=1}^{\infty} X_u |\Delta\beta_u| \sum_{n=1}^{u} p_n = \sum_{u=1}^{\infty} P_u X_u |\Delta\beta_u| < \infty,\end{aligned}$$

by (3.1).

Finally, we have that

$$\sum_{n=1}^{\infty} X_n \beta_n = O(1) \sum_{n=1}^{\infty} p_n X_n \beta_n < \infty,$$

by (3.3) and (4.2). This completes the proof of the lemma. □

5. Proof of the theorem.

Let (T_n) be the $(\overline{N}, p_n)$ means of the series $\sum a_n \lambda_n$. Then, by definition, we have

$$T_n = \frac{1}{P_n} \sum_{u=0}^{n} p_u \sum_{r=0}^{u} a_r \lambda_r = \frac{1}{P_n} \sum_{u=0}^{n} (P_n - P_{u-1}) a_u \lambda_u \,.$$

Further, for $n \geq 1$, we have

$$T_n - T_{n-1} = \frac{p_n}{P_n P_{n-1}} \sum_{u=1}^{n} P_{n-1} a_u \lambda_u \,.$$

Using Abel's transformation, we get that

$$\begin{aligned}T_n - T_{n-1} &= \frac{p_n}{P_n P_{n-1}} \sum_{u=1}^{n-1} \Delta(P_{u-1}\lambda_u) s_u + \frac{p_n s_n \lambda_n}{P_n} = -\frac{p_n}{P_n P_{n-1}} \sum_{u=1}^{n-1} p_u s_u \lambda_u \\ &\quad + \frac{p_n}{P_n P_{n-1}} \sum_{u=1}^{n-1} P_u s_u \Delta\lambda_u + \frac{p_n s_n \lambda_n}{P_n} = T_{n,1} + T_{n,2} + T_{n,3},\end{aligned}$$

say. To complete the proof of the theorem, by Minkowski's inequality for $k \geq 1$, it is sufficient to show that

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}|T_{n,r}|^k < \infty, \quad \text{for} \quad r = 1, 2, 3.$$

Now, applying Hölder's inequality with the indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, we have that

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}|T_{n,1}|^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{u=1}^{n-1} p_u|s_u| \; |\lambda_u| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{u=1}^{n-1} p_u|s_u|^k|\lambda_u|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{u=1}^{n-1} p_u \right\}^{k-1}$$

$$= O(1) \sum_{u=1}^{m} p_u|s_u|^k|\lambda_u|^k \sum_{n=u+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O(1) \sum_{u=1}^{m} \frac{p_u}{P_u}|s_u|^k|\lambda_u|^k .$$

Since $|\lambda_n| = O(1/X_n) = O(1)$, by (2.3), we have that

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}|T_{n,1}|^k = O(1) \sum_{u=1}^{m} \frac{p_u}{P_u}|s_u|^k|\lambda_u| \; |\lambda_u|^{k-1}$$

$$= O(1) \sum_{u=1}^{m} \frac{p_u}{P_u}|s_u|^k|\lambda_u| = O(1) \sum_{u=1}^{m-1} \Delta|\lambda_u| \sum_{r=1}^{u} \frac{p_r}{P_r}|s_r|^k + O(1)|\lambda_m| \sum_{u=1}^{m} \frac{p_u}{P_u}|s_u|^k$$

$$= O(1) \sum_{u=1}^{m-1} |\Delta\lambda_u| X_u + O(1)|\lambda_m| X_m = O(1) \sum_{u=1}^{m-1} \beta_u X_u + O(1)|\lambda_m| X_m = O(1)$$

as $m \to \infty$, by virtue of (2.1), (2.3), (3.2) and (4.3).

Using the conditions (2.1), (3.3) and applying Hölder's inequality as in $T_{n,1}$, we have that

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}|T_{n,2}|^k \leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{u=1}^{n-1} P_u|\Delta\lambda_u| \; |s_u| \right\}^k$$

$$\leq \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{u=1}^{n-1} P_u\beta_u|s_u| \right\}^k = O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}^k} \left\{ \sum_{u=1}^{n-1} p_u P_u\beta_u|s_u| \right\}^k$$

$$= O(1) \sum_{n=2}^{m+1} \frac{p_n}{P_n P_{n-1}} \left\{ \sum_{u=1}^{n-1} (P_u\beta_u)^k p_u|s_u|^k \right\} \times \left\{ \frac{1}{P_{n-1}} \sum_{u=1}^{n-1} p_u \right\}^{k-1}$$

$$= O(1) \sum_{u=1}^{m} (P_u\beta_u)^k p_u|s_u|^k \sum_{n=u+1}^{m+1} \frac{p_n}{P_n P_{n-1}} = O(1) \sum_{u=1}^{m} (P_u\beta_u)^k \frac{p_u}{P_u}|s_u|^k .$$

Since $P_n\beta_n = O(1/X_n) = O(1)$, by (4.1), we have

$$\sum_{n=2}^{m+1}(P_n/p_n)^{k-1}|T_{n,2}|^k = O(1)\sum_{u=1}^{m}(P_u\beta_u)^{k-1}P_u\beta_u\frac{p_u}{P_u}|s_u|^k$$
$$= O(1)\sum_{u=1}^{m}P_u\beta_u\frac{p_u}{P_u}|s_u|^k = O(1)\sum_{u=1}^{m-1}\Delta(P_u\beta_u)\sum_{r=1}^{u}\frac{p_r}{P_r}|s_r|^k$$
$$+ O(1)P_m\beta_m\sum_{u=1}^{m}\frac{p_u}{P_u}|s_u|^k$$
$$= O(1)\sum_{u=1}^{m-1}|\Delta(P_u\beta_u)|X_u + O(1)P_m\beta_m X_m = O(1)\sum_{u=1}^{m-1}P_u|\Delta\beta_u|X_u$$
$$+ O(1)\sum_{u=1}^{m-1}p_{u+1}\beta_{u+1}X_u + O(1)P_m\beta_m X_m = O(1) \quad \text{as} \quad m\to\infty,$$

by virtue of (3.1), (3.2), (4.1) and (4.2). Finally, as in $T_{n,1}$, we get that

$$\sum_{n=1}^{m}(P_n/p_n)^{k-1}|T_{n,3}|^k = \sum_{n=1}^{m}\frac{p_n}{P_n}|\lambda_n|^k|s_n|^k = O(1) \quad \text{as} \quad m\to\infty.$$

Therefore, we get that

$$\sum_{n=1}^{m}(P_n/p_n)^{k-1}|T_{n,r}|^k = O(1) \quad \text{as} \quad m\to\infty, \quad \text{for} \quad r = 1, 2, 3.$$

This completes the proof of the theorem. □

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