A remark on Nehari-type oscillation criteria for self-adjoint linear differential equations

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Abstract. Oscillation criteria of Nehari-type for the equation $(-1)^n(x^\alpha y^{(n)})^{(n)}+q(x)y=0$, $\alpha\in\mathbf{R}$, are established. These criteria impose no sign restriction on the function $q(x)$ and generalize some recent results of the second author.

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1. Introduction.

Consider a self-adjoint linear differential equation of the even order

$$(-1)^n(x^\alpha y^{(n)})^{(n)}+q(x)y=0, \tag{1.1}$$

where α is a real constant and $q\in C[a,\infty)$, $a>0$. Equation (1.1) is said to be oscillatory at ∞ if for every $b>a$ there exist $x_1,x_2\in(b,\infty)$, $x_1<x_2$, and a nontrivial solution y of (1.1) such that $y^{(i)}(x_1)=0=y^{(i)}(x_2)$, $i=0,\dots,n-1$. The points x_1,x_2 are said to be (mutually) conjugate relative to (1.1).

Nehari [11] investigated the special case $\alpha=0$, $n=1$, $q(x)\le 0$ and proved that (1.1) is oscillatory at ∞ provided

$$\lim_{x\to\infty} x^{1-\sigma}\int_x^\infty q(t)t^\sigma\,dt<-1-\frac{\sigma^2}{4(1-\sigma)}.$$

This result was generalized and extended by several authors [4], [5], [8],[9], [10] and in [6], [7] the following results were proved.

Theorem A. *Let $\alpha\notin\{1,2,\dots,2n-1\}$ and*

$$q(x)\le 0 \quad \textit{for large} \quad x. \tag{1.2}$$

If $\alpha+\sigma<2n-1$, suppose

$$\liminf_{x\to\infty} x^{2n-1-\alpha-\sigma}\int_x^\infty q(t)t^\sigma\,dt<-B_{n,\alpha,\sigma}-\frac{\binom{\sigma/2}{n}^2(n!)^2}{2n-1-\alpha-\sigma}, \tag{1.3$_1$}$$

if $\alpha+\sigma>2n-1$, *suppose*

$$(1.3_2)\qquad \liminf_{x\to\infty} x^{2n-1-\alpha-\sigma}\int_1^x q(t)t^\sigma\,dt < -\bar{B}_{u,\alpha,\sigma}-\frac{\binom{\sigma/2}{n}^2(n!)^2}{2n-1-\alpha-\sigma},$$

$B_{n,\alpha,\sigma}$, $\bar{B}_{n,\alpha,\sigma}$ *being nonnegative real constants depending on* n,α,σ. *Then* (1.1) *is oscillatory at* ∞. *Moreover, the assumption* (1.2) *can be omitted for* $\sigma=0$ *if we replace in* ($1.3_{1,2}$) lim inf *by* lim sup.

The precise values of the constants $B_{n,\alpha,\sigma}$, $\bar{B}_{n,\alpha,\sigma}$ were computed in [6] and will be given later.

Theorem B. *Let* (1.2) *hold and* $\alpha=2n-1-2k$, $k=0,1,\dots,n-1$. *If*

$$(1.4)\qquad \liminf_{x\to\infty} \ln x\int_x^\infty q(t)t^{2k}dt < -(k!(n-1-k)!)^2,$$

then (1.1) *is oscillatory at* ∞. *Moreover, for* $k=0$, *the assumption* (1.2) *can be omitted if we replace in* (1.4) lim inf *by* lim sup.

The aim of this paper is to find further values of the constants σ and k for which (1.2) is not necessary if we replace in ($1.3_{1,2}$), (1.4) lim inf by lim sup. In this case we formulate Theorem B in a more general form. We also give the outline of an alternative method of computation of the constants $B_{n,\alpha,\sigma}$, $\bar{B}_{n,\alpha,\sigma}$, which may be a little simpler than that given in [6], [7].

The main idea of the proofs of our statemennts is the same as in [6], [7] and it is based on the following theorem.

Theorem C. *Equation* (1.1) *is oscillatory at* ∞ *if and only if for every* $b>a$ *there exist* $x_1,x_2\in(b,\infty)$, $x_1<x_2$, *and a nontrivial function* $v\in \overset{\circ}{W}{}_2^n(x_1,x_2)$ *such that*

$$I(v;x_1,x_2)=\int_{x_1}^{x_2}[x^\alpha(v^{(n)}(x))^2+q(x)v^2(x)]dx\le 0.$$

Recall that the Sobolev space $\overset{\circ}{W}{}_2^n(x_1,x_2)$ consists of the functions $v(x)$ whose $(n-1)$-th derivative is absolutely continuous, $v^{(n)}\in L_2(x_1,x_2)$ and $v^{(i)}(x_1)=0=v^{(i)}(x_2)$, $i=0,\dots,n-1$.

2. Auxiliary statements.

Seft-adjoint linear differential equations of the even order are closely related to the linear Hamiltonian systems (LHS). If y is a solution of (1.1), then $u=(y,...,y^{(n-1)})$, $v=((-1)^{n-1}(x^\alpha y^{(n)})^{(n-1)},\dots,x^\alpha y^{(n)})$ is a solution of the system

$$(2.1)\qquad u'=Au+B(x)v,\quad v'=C(x)u-A^Tv,$$

where

$$(2.2)\qquad \begin{aligned} B(x)&=\operatorname{diag}\{0,\dots,0,x^{-\alpha}\},\\ C(x)&=\operatorname{diag}\{q(x),0,\dots,0\},\end{aligned}$$

$$A = A_{i,j} = \begin{cases} 1, & \text{for } j = i+1, \quad i = 1, \dots, n-1, \\ 0, & \text{elsewhere.} \end{cases}$$

We say that the solution (u, v) of (2.1) is generated by the solution y of (1.1). Two points $x_1, x_2 \in [a, \infty)$ are said to be conjugate relative to (2.1) if there exists a nontrivial solution (u, v) of (2.1) such that $u(x_1) = 0 = u(x_2)$. System (2.1) is said to be disconjugate on an interval I whenever there exists no pair of points of I which are conjugate relative to (2.1) and this system is said to be nonoscillatory at ∞ if there exists $b \in (a, \infty)$ such that (2.1) is disconjugate on (b, ∞). It is obvious that $x_1, x_2 \in (a, \infty)$ are conjugate relative to (1.1) if and only if they are conjugate relative to (2.1) with A, B, C given by (2.2).

Simultaneously with (2.1) consider the matrix system

$$U' = AU + B(x)V, \quad V' = C(x)U - A^T V, \tag{2.3}$$

where U, V are $n \times n$ matrices. A solution (U, V) of (2.3) is said to be self-conjugate if $U^T(x)V(x) - V^T(x)U(x) \equiv 0$. A self-conjugate solution (U_0, V_0) of (2.3) is said to be principal at ∞ if $U_0(x)$ is nonsingular for large x and

$$\lim_{x \to \infty} \Big(\int_{x_0}^{x} U_0^{-1}(s)B(s)U_0^{T-1}(s)ds \Big)^{-1} = 0.$$

Let (U_1, V_1) be a solution of (2.3) which is linearly independent of (U_0, V_0) (i.e. (U_0, V_0), (U_1, V_1) form the base of the solution space of (2.3)), then

$$\lim_{x \to \infty} U_1^{-1}(x)U_0(x) = 0.$$

The principal solution of (2.3) at ∞ is determined uniquely up to a right multiple by a constant nonsingular $n \times n$ matrix and exists if and only if (2.1) is nonoscillatory at ∞. A solution (U_1, V_1) is said to be nonprincipal at ∞ if

$$\lim_{x \to \infty} \Big(\int_{x_0}^{x} U_1^{-1}(s)B(s)U_1^{T-1}(s)ds \Big)^{-1} = M,$$

where M is a nonsingular $n \times n$ matrix. For a more detailed information concerning LHS (2.1) and their principal solutions see, e.g., [1].

Lemma 1 [1, Chap. II]**.** *Let* (U, V) *be a self-conjugate solution of* (2.1) *such that the matrix* $U(x)$ *is nonsingular on* $I \subseteq [a, \infty)$, $x_0 \in I$. *Then*

$$(\tilde{U}(x), \tilde{V}(x)) = \Big(U(x) \int_{x_0}^{x} U^{-1}(s)B(s)U^{T-1}(s)ds,$$
$$V(x) \int_{x_0}^{x} U^{-1}(s)B(s)U^{T-1}(s)ds + U^{T-1}(x) \Big)$$

is also a self-conjugate solution of (2.3).

Lemma 2 [1, Chap. I]. *Let* (2.1) *be disconjugate on an interval* $I \subseteq [a,\infty)$ *and let* $x_1, x_2 \in I$, $x_1 \neq x_2$, $u_1, u_2 \in R^n$ *be arbitrary. There exists a unique solution* (u,v) *of* (2.1) *for which* $u(x_1)=u_1$, $u(x_2)=u_2$.

Now consider the linear differential equation

$$y^{(n)} + q_{n-1}(x)y^{(n-1)} + \cdots + q_0(x)y = 0, \tag{2.4}$$

where $q_i \in C[a,\infty)$, $i=0,\dots,n-1$. Concerning this equation, we shall need another definition of disconjugacy for linear differential equations, introduced by Nehari. Equation (2.4) is said to be disconjugate in the sense of Nehari, shortly N-disconjugate, on an interval $I \subseteq [a,\infty)$ whenever every nontrivial solution of this equation has at most $(n-1)$ zeros on I, every zero counted according to its multiplicity, (2.4) is said to be eventually N-disconjugate if there exists $b \in [a,\infty)$ such that this equation is N-disconjugate on (b,∞).

Recall briefly oscillation properties of solutions of (2.4). A system of solutions $y_1,\dots,y_n$ of (2.4) is said to form a Markov system of solutions on $I \subseteq [a,\infty)$ if n Wronskians

$$W(y_1,\dots,y_k) = \begin{vmatrix} y_1 & \cdots & y_k \\ \vdots & & \vdots \\ y_1^{(k-1)} & \cdots & y_k^{(k-1)} \end{vmatrix},$$

$k=1,\dots,n$, are positive throughout I. The system $y_1,\dots,y_n$ is said to form a Descartes system of solutions on I if all Wronskians $W(y_{i_1},\dots,y_{i_k})$, $1 \le i_1 < \cdots < i_k \le n$, $k=1,\dots,n$, are positive throughout I.

Lemma 3 [1, Chap. III]. *Equation* (2.4) *is eventually N-disconjugate if and only if there exists* $b \in [a,\infty)$ *such that* (2.4) *possesses a Markov system of solutions on* (b,∞) *satisfying the additional condition*

$$\begin{gathered} y_i > 0 \quad \text{for large} \quad x, i=1,\dots,n \\ y_{k-1} = o(y_k) \quad \text{for} \quad x \to \infty, \quad k=2,\dots,u. \end{gathered} \tag{2.5}$$

Moreover, a Markov system of solutions of (2.4) *satisfying* (2.5) *form the Descartes system of solutions for large* x.

Lemma 4 [1, Chap. III]. *Let* $y_1,\dots,y_n$ *be a Descartes system of solutions of* (2.4) *for large* x *satisfying* (2.5). *If* $1 \le i_1 < i_2 < \cdots < i_k \le n$, $1 \le j_1 < j_2 < \cdots < j_k \le n$, $1 \le k \le n$, *are distinct* k*-tuples such that* $i_l \le j_l$, $l=1,\dots,k$, *then*

$$\lim_{x\to\infty} W(y_{i_1},\dots y_{i_k})/W(y_{j_1},\dots y_{j_k}) = 0.$$

Lemma 5. *Let* $u_1,\dots,u_n,v_1,\dots,v_n$ *be real-valued functions of the class* C^{n-1} *and let*

$$U(x) = (u_j^{(i-1)}(x))_{i,j=1}^n, \quad V(x) = (v_j^{(i-1)}(x))_{i,j=1}^n$$

be their Wronski matrices. If $U(x)$ is nonsingular, then

$$[U^{-1}(x)V(x)]_{i,j} = \frac{W(u_1, \dots, u_{i-1}, v_j, u_{i+1}, \dots, u_n)}{W(u_1, \dots, u_n)}.$$

Proof: Taking into account the rule for computation of the entries of the inverse matrix and the rule for the product of two matrices, the conclusion can be verified by a direct computation. □

Lemma 6 [1, Chap. III]. *Let $y_1, \dots y_n \in C^{n-1}$ be real-valued functions, $y_j \neq 0$. Then*

$$W(y_1, \dots, y_n) = \\ (-1)^{j+1} W((y_1/y_j)', \dots, (y_{j-1}/y_j)', (y_{j+1}/y_j)', \dots, (y_n/y_j)').$$

3. Main results.

Theorem 1. *Let $\alpha \notin \{1, 2, \dots, 2n-1\}$, $\sigma/2 \in \{0, 1, \dots, n-1\} \cup \{n-\alpha, n-a+1, \dots, 2n-1-\alpha\}$. If $\alpha + \sigma < 2n-1$, suppose*

$$\limsup_{x\to\infty} x^{2n-1-\alpha-\sigma} \int_x^\infty q(t) t^\sigma \, dt < -B_{n,\alpha,\sigma} - \frac{\binom{\sigma/2}{n}^2 (n!)^2}{2n-1-\alpha-\sigma}, \tag{3.1$_1$}$$

if $\alpha + \sigma > 2n-1$, suppose

$$\limsup_{x\to\infty} x^{2n-1-\alpha-\sigma} \int_1^x q(t) t^\sigma \, dt < -\bar{B}_{n,\alpha,\sigma} - \frac{\binom{\sigma/2}{n}^2 (n!)^2}{2n-1-\alpha-\sigma}. \tag{3.1$_2$}$$

Then (1.1) is oscillatory at ∞.

Proof: Let $h(x) = x^{\sigma/2}$, $f(x) = (-1)^n \frac{(2n-1)!}{[(n-1)!]^2} \int_0^x t^{n-1}(t-1)^{n-1} \, dt$, $0 \le x \le 1$.
Case I. $\alpha + \sigma < 2n-1$.
Define

$$y(x) = \begin{cases} 0, & \text{for } a \le x \le Q \\ g(x), & \text{for } Q \le x \le R \\ R^{\tau/2} h(x), & \text{for } R \le x \le S \\ R^{\tau/2} h(x) f(\frac{T-x}{T-S}), & \text{for } S \le x \le T \\ 0, & \text{for } T \le x, \end{cases} \tag{3.2}$$

where $\tau = 2n-1-\alpha-\sigma$ and $g(x)$ is the solution of the equation

$$(x^\alpha y^{(n)})^{(n)} = 0 \tag{3.3}$$

satisfying the boundary conditions

$$g^{(i)}(Q)=0,\quad g^{(i)}(R)=R^{\tau/2}h^{(i)}(R),\quad i=0,\dots,n-1. \tag{3.4}$$

Note that such a solution always exists since the linear Hamiltonian system corresponding to (3.3) is disconjugate on $[a,\infty)$. Using the results of [6], we get

$$\lim_{R\to\infty}\int_Q^R x^\alpha (y^{(n)}(x))^2 dx = B_{n,\alpha,\sigma},$$

where

$$\begin{aligned} B_{n,\alpha,\sigma} &= 0 \quad \text{for} \quad \alpha > 2n-1 \\ &= \frac{(n!)^2\binom{\sigma/2}{n-k}^2}{\binom{n}{k}} \int_0^1 x^{-\alpha}\Big[\sum_{i=n-k}^{n-1}(-x)^i\binom{n+i-\alpha}{k}\cdot \\ &\quad \cdot\binom{\sigma/2+\alpha+k-2n}{i+k+n}\binom{\sigma/2+\alpha-n-i-1}{n-i-1}\Big]^2 dx, \\ &\quad \text{for} \quad 2n-1-2k<\alpha<2n+1-2k, \quad k=1,\dots,n-1 \\ &\quad \text{and} \quad \alpha<1, \quad k=n, \\ \lim_{S\to\infty}&\int_R^S x^\alpha (y^{(n)})^2 dx = \frac{\binom{\sigma/2}{n}(n!)^2}{2n-1-\alpha-\sigma} =: \varrho \\ \lim_{S\to\infty}&\int_S^{2S} x^\alpha (y^{(n)})^2 dx = 0. \end{aligned} \tag{3.5}$$

To show that (1.2) can be omitted if we replace (1.3_1) by (3.1_1), we proceed as follows. Since $\alpha\notin\{1,2,\dots,2n-1\}$, the function $g(x)$ is of the form

$$g(x)=\sum_{i=0}^{n-1}(a_i x^i + b_i x^{n-\alpha+i}),$$

where a_i, b_i are real constants, and hence

$$g(x)x^{-\sigma/2}=\sum_{i=0}^{n-1} a_i x^{i-\sigma/2}+b_i x^{n-\alpha+i-\sigma/2}.$$

Since σ is such that $x^{\sigma/2}$ is a solution of (3.3), one of the exponents of x on the right-hand side of the last expression equals 0. It follows that $[g(x)x^{-\sigma/2}]'$ is a linear combination of $(2n-1)$ functions, each of them is a power of x. Order these functions according to their exponents and denote them $y_1,\dots,y_{2n-1}$ (y_{2n-1} corresponds to the greatest power). It is not difficult to verify that these functions form the Markov system of solutions of a certain $(2n-1)$-order linear differential equation which is, by

Lemma 1, N-disconjugate on $[a,\infty)$. Since $[g(x)x^{-\sigma/2}]'$ is a solution of this equation and $[g(x)x^{-\sigma/2}]^{(j)}_{x=Q} = 0 = [g(x)x^{-\sigma/2}]^{(i)}_{x=R}$, $i=1,\dots,n-1$, $[g(x)x^{-\sigma/2}]'$ does not vanish on (R,Q), i.e. $g(x)x^{-\sigma/2}$ is monotonic on this interval. The function $f'(x)$ is a polynomial of order $(2n-2)$ for which $f^{(i)}(0)=0=f^{(i)}(1)$, $i=1,\dots,n-1$, hence the existence of $x_0\in(0,1)$ such that $f'(x_0)=0$ would imply $f'(x)\equiv 0$ on $(0,1)$ – a contradiction.

Now, using the second mean value theorem of integral calculus, we have

$$\int_Q^T qy^2dx = \int_Q^R qg^2\,dx + R^\tau\int_R^S qx^\sigma\,dx + R^\tau\int_S^T qx^\sigma f^2(\frac{T-x}{s-x})\,dx =$$

$$= \int_Q^R qx^\sigma(gx^{-\sigma/2})^2\,dx + R^\tau\int_R^S qx^\sigma\,dx + R^\tau\int_S^T qx^\sigma f^2(\frac{T-x}{T-s})\,dx =$$

$$= R^\tau\int_{\xi_1}^{\xi_2} qx^\sigma\,dx,$$

where $\xi_1\in(Q,R)$, $\xi_2\in(S,T)$. According to (3.1_1), the integral $\int_{\xi_1}^{\xi_2} qx^\sigma dx$ is negative if ξ_1,ξ_2 are sufficiently large (i.e. Q and S are sufficiently large), hence $R^\tau\int_{\xi_1}^{\xi_2} qx^\sigma\,dx \le \xi_1^\tau\int_{\xi_1}^{\xi_2} qx^\sigma\,dx$. To finish the proof, we proceed in the same way as in [6]. Let $\delta>0$ be sufficiently small. By (3.1), there exists $Q\in[a,\infty)$ such that

$$\xi_1^\tau\int_{\xi_1}^{\infty} qx^\sigma\,dx < -B_{n,\alpha,\sigma}-\varrho-4\delta \tag{3.6}$$

whenever $\xi_1>Q$. By (3.5), (3.6), R,S can be chosen such that

$$I_{R,S} = \int_R^S x^\alpha(y^{(n)})^2\,dx < \varrho+\delta$$

$$I_{S,2S} = \int_S^{2S} x^\alpha(y^{(n)})^2\,dx < \delta$$

$$I_{Q,R} = \int_Q^R x^\alpha(y^{(n)})^2\,dx < B_{n,\alpha,\sigma}+\delta$$

and

$$\xi_1^\tau\int_{\xi_1}^{\xi_2} qx^\sigma\,dx < -B_{n,\alpha,\sigma}-\varrho-3\delta$$

whenever $\xi_2>S$. Consequently, $I(y;Q,T) = I_{Q,R}+I_{R,S}+I_{S,T}+\int_Q^T qy^2\,dx < B_{n,\alpha,\sigma}+\delta+\varrho+\delta+\delta-B_{n,\alpha,\sigma}-\varrho-3\delta = 0$ if $T=2S$.

II. Case $\alpha+\sigma>2n-1$.

Define

$$y(x) = \begin{cases} 0, & \text{for } a\le x\le Q,\\ S^{\tau/2}x^{\sigma/2}f(\frac{x-Q}{R-Q}), & \text{for } Q\le x\le R,\\ S^{\tau/2}x^{\sigma/2}, & \text{for } R\le x\le S,\\ g(x), & \text{for } S\le x\le T,\\ 0, & \text{for } x\ge T,\end{cases}$$

where $g(x)$ is the solution of (3.3) satisfying the boundary conditions $g^{(i)}(S) = S^{\tau/2}(x^{\sigma/2})^{(i)}_{x=S}$, $g^{(i)}(T) = 0$, $i = 0, \dots, n-1$, τ and $f(x)$ are the same as above. To prove that the function $g(x)x^{-\sigma/2}$ is monotonic on (S,T), we can proceed in the same way as in the first part of the proof and the exact value of $\bar{B}_{n,\alpha,\sigma}$,

$$\begin{aligned}
\bar{B}_{n,\alpha,\sigma} &= 0 \quad \text{for} \quad \alpha < 1 \\
&= \frac{n!\binom{\sigma/2}{k}^2}{\binom{n}{k}} \int_1^\infty x^{-\alpha} \Big[\sum_{i=0}^{n-k-1} (-x)^i \binom{n+i-k-\alpha}{n-k} \\
&\quad \binom{\sigma/2-n+\alpha}{i} \binom{\sigma/2-n-i-1+\alpha}{n-k-i-1} \Big]^2 dx \\
&\quad \text{for} \quad 2n-1-2k < \alpha < 2n+1-2k, \quad k = 1, \dots, n-1 \\
&\quad \text{and} \quad 2n-1-2k < \alpha, \quad k = 0
\end{aligned}$$

was calculated in [6]. The proof is complete. □

Remark 1. Recall that the numbers $\binom{\beta}{n}$, β-real number, n-natural number, are defined as follows: $\binom{\beta}{n} = (1/n!)\beta(\beta-1)\dots(\beta-n+1)$. Consequently, if $\sigma/2 \in \{0, \dots, n-1\}$, the second constant on the right-hand side of $(3.1_{1,2})$ equals zero.

In the proof we have just finished, we did not pay attention to the exact computation of the constants $B_{n,\alpha,\sigma}$, $\bar{B}_{n,\alpha,\sigma}$ since this had been done in [6]. Now make some remarks concerning the calculation of these constants. Our observations are based on a relation between equation (1.1) and LHS (2.1).

Let $g(x)$ be the solution of (3.3) satisfying (3.4), denote by (u,v) the solution of the corresponding LHS

$$u' = Au + B(x)v, \quad v' = -A^T v \tag{3.7}$$

generated by g and let $H(x)$ be the solution of

$$H' = AH, \quad H(0) = I \quad \text{(the identity matrix)}. \tag{3.8}$$

Then $(H,0)$ is the solution of the matrix system associated with (3.7) and using Lemma 1 one can directly verify that

$$\begin{aligned}
u(x) &= H(x) \int_Q^x H^{-1}BH^{T-1}\,ds\Big(\int_Q^R H^{-1}BH^{T-1}\,dx\Big)^{-1} H^{-1}(R)C(R) \\
v(x) &= H^{T-1}(x)\Big(\int_Q^R H^{-1}BH^{T-1}\,dx\Big)^{-1} H^{-1}(R)C(R),
\end{aligned}$$

where $C(R) = R^{\tau/2}\ (R^{\sigma/2}, \sigma/2\, R^{\sigma/2-1}, \dots, \binom{\sigma/2}{k} k! R^{\sigma/2-k+1}, \dots, \binom{\sigma/2}{n} n! R^{\sigma/2-n+1})$.

It holds

$$\int_Q^R x^\alpha (g^{(n)})^2\,dx = \int_Q^R v^T(x)B(x)v(x)\,dx =$$
$$= C^T(R)H^{T-1}(R)(\int_Q^R H^{-1}BH^{T-1}\,dx)^{-1}\int_Q^R H^{-1}BH^{T-1}\,dx$$
$$(\int_Q^R H^{-1}BH^{T-1}\,dx)^{-1}H^{-1}(R)C(R) =$$
$$= C^T(R)H^{T-1}(R)(\int_Q^R H^{-1}BH^{T-1}\,dx)^{-1}H^{-1}(R)C(R).$$

Since

$$H(x) = \begin{bmatrix} 1 & x & x^2/2! & \dots & x^{n-1}/(n-1)! \\ 0 & 1 & x & \dots & x^{n-2}/(n-2)! \\ \vdots & & & & \\ 0 \dots\dots\dots & & & 0 & 1 \end{bmatrix}$$

hence

$$H^{-1}(x) = \begin{bmatrix} 1 & -x & x^2/2! & \dots & (-1)^{n-1}x^{n-1}/(n-1)! \\ 0 & 1 & -x & \dots & (-1)^{n-2}x^{n-2}/(n-2)! \\ \vdots & & & & \\ 0 \dots\dots\dots & & & 0 & 1 \end{bmatrix},$$

and concerning the vector $d(R) = H^{-1}(R)C(R)$, we have

$$d_i(R) = R^{\tau/2}\sum_{k=0}^{n-i}(-1)^k\frac{R^k}{k!}\binom{\sigma/2}{k+i}(k+i)!R^{\sigma/2-k-i} =$$
$$= R^{\frac{2n-1-\alpha}{2}-i}\sum_{k=0}^{n-i}(-1)^k\frac{1}{k!}\binom{\sigma/2}{k+i}(k+i)!.$$

Now it suffices to compute the matrix $(\int_Q^R H^{-1}BH^{T-1}dx)^{-1}$.

Consider the case $\alpha+\sigma < 2n-1$, the case $\alpha+\sigma > 2n-1$ can be treated analogously. By a routine computation, we get

$$\int_Q^x H^{-1}BH^{T-1}\,dt = \left[\frac{(-1)^{i+j}t^{2n+1-\alpha-i-j}}{(n-i)!(n-j)!(2n-\alpha-i-j+1)}\,\Big|_Q^x\right]_{i,j=1}^n.$$

If $\alpha > 2n-1$, $(H,0)$ is a nonprincipal solution of (3.7), see [3]. Hence

$$\lim_{x\to\infty}(\int_Q^x H^{-1}BH^{T-1}\,dt)^{-1} = M,$$

M being a nonsingular $n \times n$ matrix, and $\lim_{R\to\infty} d_i(R) = 0$, $i = 1, \dots, n$. Consequently,

$$B_{n,\alpha,\sigma} = \lim_{R\to\infty} d^T(R)(\int_Q^R H^{-1}BH^{T-1}dx)^{-1}d(R) = 0.$$

Denote

$$y_j = \frac{x^{2n-\alpha-j}}{(n-j)!}\sum_{i=1}^n \frac{(-1)^{i+j}}{(n-i)!(2n-\alpha-i-j+1)i!}, \quad j = 1, \dots, n$$
$$D = d_{i,j} = \left[\frac{(-1)^{i+j}Q^{2n+1-i-j-\alpha}}{(n-i)!(n-j)!(2n-\alpha-i-j+1)}\right]_{i,j=1}^n$$

and let (U_1, V_1) be the matrix solution of (3.7) generated by $y_1, \dots, y_n$. If $\alpha < 1$, then $(H, 0)$ is the principal solution of this system and

$$\lim_{x\to\infty}(H^{-1}(x)U_1(x) + D)^{-1}H^{-1}(x)U_1(x) =$$
$$= \lim_{x\to\infty}(I + U_1^{-1}(x)H(x)D)^{-1} = I,$$

i.e.

$$\lim_{x\to\infty}(H^{-1}(x)U_1(x) + D)^{-1} =$$
$$= \lim_{x\to\infty}(\int_Q^x H^{-1}BH^{T-1}\,ds)^{-1} = \lim_{x\to\infty} U_1^{-1}(x)H(x).$$

Denote $M_j = \frac{(-1)^j}{(n-j)!}\sum_{i=1}^n \frac{(-1)^i}{(n-i)!i!(2n+1-\alpha-i-j)}$.

$$[U_1^{-1}H]_{i,j} =$$
$$\frac{W(M_1x^{2n-1-\alpha}, .., M_{i-1}x^{2n-\alpha-i+1}, x^{j-1}/(j-1)!, M_{i+1}x^{2n-\alpha-i-1}, .., M_nx^{n-\alpha})}{W(M_1x^{2n-1-\alpha}, \dots, M_nx^{n-\alpha})} =$$
$$\frac{1}{M_i(j-1)!}\frac{W(x^{2n-\alpha-1}, \dots, x^{2n-\alpha-i+1}, x^{j-1}, x^{2n-i-\alpha-1}, \dots, x^{n-\alpha})}{W(x^{2n-1-\alpha}, \dots, x^{n-\alpha})} =$$
$$\frac{x^{n(j-1)}}{M_i(j-1)!}\frac{W(x^{2n-\alpha-j}, \dots, x^{2n-\alpha-i-j+2}, 1, x^{2n-\alpha-i-j}, \dots, x^{n-\alpha-j+1})}{1!2!\dots(n-1)!(-1)^{n(n+3)/2}x^{n(n-\alpha)}}.$$

Using Lemma 6 for computing the Wronskian in the nominator, we get

$$W(x^{2n-\alpha-j}, \dots, x^{2n-\alpha-i-j+2}, 1, x^{2n-\alpha-i-j}, \dots, x^{n-\alpha-j+1}) =$$
$$= (-1)^{j+1}W((2n-\alpha-j)x^{2n-\alpha-j-1}, \dots, (2n-\alpha-i-j+2)x^{2n-\alpha-i-j+1},$$

$$(2n-\alpha-i-j)x^{2n-\alpha-i-j-1},\dots,(n-\alpha-j+1)x^{n-\alpha-j}) =$$
$$= x^{(n-1)(n-\alpha-j)}\binom{2n-\alpha-j}{n}n!\cdot\frac{W(x^{n-1},\dots,x^{n-i+1},x^{n-i-1},\dots,1)}{(2n-\alpha-i-j+1)} =$$
$$= x^{(n-1)(n-\alpha-j)+i-1}\frac{\binom{2n-\alpha-j}{n}n!}{2n-\alpha-i-j+1}\cdot\frac{1!2!\dots(n-1)!}{(n-i)!}(-1)^{(n-1)(n+2)/2}.$$

Combining these computations with the previous ones, we have

$$[U_1^{-1}H]_{i,j} = x^{\alpha-2n+i+j-1}\cdot\frac{\binom{2n-\alpha-j}{n}(-1)^{n+1}n!}{(n-i)!(2n-\alpha-i-j+1)}\cdot\frac{1}{(j-1)!M_i}$$

and

$$I_{Q,R} = \sum_{i,j=1}^{n} d_i(R)d_j(R)[U_1^{-1}(R)H(R)]_{i,j} =$$
$$= \sum_{i,j=1}^{n}\left(\sum_{k=0}^{n-i}\frac{(-1)^k}{k!}\binom{\sigma/2}{k+i}(k+i)!\right)\left(\sum_{k=0}^{n-j}\frac{(-1)^k}{k!}\binom{\sigma/2}{k+j}(k+j)!\right)\cdot$$
$$\cdot\frac{\binom{2n-\alpha-j}{n}(-1)^{n+1}n!}{(n-i)!(2n-\alpha-i-j+1)(j-1)!M_i}.$$

If $\alpha\in(2n-1-2k,2n+1-2k)\setminus\{1,\dots,2n-1\}$, then

$$d^T(R)(\int_Q^R H^{-1}BH^{T-1}\,dx)^{-1}d(R) =$$
$$= d^T(R)(U_1(R)-H(R)D)^{-1}H(R)d(R).$$

Denote $\tilde{y}_j = y_j - \sum_{i=1}^{n}\frac{x^{i-1}}{(i-1)!}d_{ij}$, then

$$[(U_1(R)+H(R)D)^{-1}H(R)]_{i,j} = \frac{W(\tilde{y}_1,\dots,\tilde{y}_{i-1},x^{j-1}/(j-1)!,\tilde{y}_{i+1},\dots,\tilde{y}_n)}{W(\tilde{y}_1,\dots,\tilde{y}_n)}|_{x=R}.$$

By means of Lemma 4, one can verify that if $\max\{i,j\}>k$, then

$$[(U_1+HD)^{-1}H]_{i,j} = o(R^{\alpha+i+j+1-2n}) \quad\text{as}\quad R\to\infty$$

and thus

$$d^T(R)(\int_Q^R H^{-1}BH^{T-1}\,dx)^{-1}d(R) =$$
$$= \sum_{i,j=1}^{k} d_i(R)d_j(R)[(U_1(R)+H(R)D)^{-1}H(R)]_{i,j},$$

whereby the entries of $[(U_1+HD)^{-1}H]$ can be computed in the same way as in the case $\alpha<1$.

Now turn our attention to the case when $\alpha\in\{1,3,\dots,2n-1\}$ which is treated by Theorem B. This theorem can be modified in the following way.

Theorem 2. *Let $\alpha \in \{1, 3, \dots, 2n-1\}$, $k \in \{0, \dots, n-1\}$, $c_0, \dots, c_{k-1} \in R$. If*

$$\limsup_{x\to\infty} \ln x \int_x^\infty q(t)(c_0 + \cdots + c_{k-1}t^{k-1} + t^k)^2 dt < -(k!(n-1-k)!)^2, \tag{3.9}$$

then (1.1) *is oscillatory at* ∞.

Proof: Let $h(x) = c_0 + \cdots + c_{k-1}x^{k-1} + x^k$, $\tilde{h}(x) = x^k$. Define

$$y(x) = \begin{cases} 0, & \text{for } a \le x \le Q, \\ g(x), & \text{for } Q \le x \le R, \\ h(x), & \text{for } R \le x \le S, \\ f(x), & \text{for } S \le x \le T, \\ 0, & \text{for } x \ge T, \end{cases}$$

where g, f are the solutions of (3.3) satisfying (3.4) and

$$f^{(i)}(S) = h^{(i)}(S), \quad f^{(i)}(T) = 0, \quad i = 0, \dots, n-1, \tag{3.10}$$

respectively. Further, let $\tilde{g}, \tilde{f}$ be the solutions of (2.3) satisfying (3.4) and (3.10) with h replaced by $\tilde{h}$. It was proved in [7] that

$$\lim_{R\to\infty} \ln R \int_Q^R x^\alpha (\tilde{f}^{(n)})^2 dx = \gamma$$

and

$$\lim_{T\to\infty} \int_S^T x^\alpha (\tilde{f}^{(n)})^2 dx = 0,$$

where $\gamma = (k!(n-k-1)!)^2$. We shall show that also $\lim_{R\to\infty} \ln R \int_Q^R x^\alpha (g^{(n)})^2\, dx = \gamma$, $\lim_{T\to\infty} \int_S^T x^\alpha (f^{(n)})^2 dx = 0$. Let $(\tilde{u}, \tilde{v})$ be the solution of (3.7) generated by $\tilde{g}$. According to (3.4), Lemma 1 and Lemma 2, one can directly verify that

$$\tilde{u}(x) = H(x) \int_Q^x H^{-1} B H^{T-1} ds (\int_Q^R H^{-1} B H^{T-1} ds) e_{k+1}$$
$$\tilde{v}(x) = H^{T-1}(x) (\int_Q^R H^{-1} B H^{T-1} dx)^{-1} e_{k+1},$$

where $e_{k+1} = (0, \dots, 0, 1, 0, \dots, 0) \in R^n$ with the number 1 on the $(k+1)$-th place. $\int_Q^R x^\alpha (\tilde{g}^{(n)})^2 dx = \int_Q^R v^T B v dx = e_{k+1}^T (\int_Q^R H^{-1} B H^{T-1} dx) e_{k+1}$. By Lemma 1, $(U_1(x), V_1(x)) = (H(x) \int_Q^x H^{-1} B H^{T-1} ds, H^{T-1}(x))$ is a solution of the matrix system corresponding to (3.7) and this solution is generated by the solutions $y_j(x) =$

$(-1)^j \int_Q^x \frac{(x-t)^{n-1}}{(n-1)!} \frac{t^{n-\alpha-j}}{(n-j)!}\, dt$, $j = 1,\dots,n$, see [2]. $\int_Q^x H^{-1}BH^{T-1}ds = H^{-1}(x)U_1(x)$, hence

$$(\int_Q^R H^{-1}BH^{T-1}\, ds)^{-1} = \frac{W(y_1,\dots,y_{i-1},x^{j-1}/(j-1)!,y_{i+1},\dots,y_n)}{W(y_1,\dots,y_n)}|_{x=R}$$

and thus

$$\lim_{R\to\infty} \ln R \int_Q^R x^\alpha (g^{(n)})^2\, dx = \lim_{R\to\infty} \ln R e_{k+1}^T (\int_Q^R H^{-1}BH^{T-1}\, dx)^{-1} e_{k+1} =$$

$$= \lim_{R\to\infty} \ln R \frac{W(y_1,\dots,y_k,x^k/k!,y_{k+2},\dots,y_n)}{W(y_1,\dots,y_n)}\; |_{x=R} = \gamma.$$

Let (u,v) be the solution of (3.7) generated by g. We have

$$u(x) = H(x)\int_Q^x H^{-1}BH^{T-1}\, ds (\int_Q^R H^{-1}BH^{T-1}\, dx)^{-1} c$$

$$v(x) = H^{T-1}(\int_Q^R H^{-1}BH^{T-1}\, dx)^{-1} c,$$

where $c = (c_0,\dots,c_{k-1},1,0,\dots,0) \in R^n$.

$$\int_Q^R x^\alpha (g^{(n)})^2\, dx = c^T(\int_Q^R H^{-1}BH^{T-1}\, dx)^{-1} c =$$

$$= \sum_{i,j=1}^{k+1} c_{i-1}c_{j-1}(\int_Q^R H^{-1}BH^{T-1}\, dx)^{-1},$$

where $c_{k+1} = 1$.

$$(\int_Q^R H^{-1}BH^{T-1}\, dx)^{-1}_{i,j=1} = \frac{W(y_1,\dots,y_{i-1},x^{j-1}/(j-1)!,y_{i+1},\dots,y_n)}{W(y_1,\dots,y_{i-1},x^k/k!,y_{i+1},\dots,y_n)} \cdot$$

$$\frac{W(y_1,\dots,y_{i-1},x^k/k!,y_{i+1},\dots,y_n)}{W(y_1,\dots,y_k,x^k/k!,y_{k+2},\dots,y_n)} \cdot \frac{W(y_1,\dots,y_k,x^k/k!,y_{k+2},\dots,y_n)}{W(y_1,\dots,y_n)}.$$

The third term in the last product multiplied by $\ln x$ tends to γ as $x\to\infty$. The first two terms are bounded if $i \le k+1$ (Lemma 4) and if $i < k+1$ or $j < k+1$, by a direct computation, one can verify that the least of them is $O(\frac{1}{x})$ as $x\to\infty$ (i.e., multilplied by x remains bounded). Consequently,

$$\lim_{R\to\infty} \ln R \int_Q^R x^\alpha (g^{(n)})^2\, dx =$$

$$= \lim_{R\to\infty} \ln R \sum_{i,j=1}^{k+1} c_{i-1}\, c_{j-1}(\int_Q^R H^{-1}BH^{T-1}\, dx)^{-1}_{i,j} =$$

$$= \lim_{R\to\infty} \ln R(\int_Q^R H^{-1}BH^{T-1})^{-1}_{k+1,k+1} = \gamma.$$

Similarly $\lim_{T\to\infty}\int_S^T x^\alpha(\tilde f^{(n)})^2\,dx=0$ implies $\lim_{T\to\infty}\int_S^T x^\alpha(f^{(n)})^2=0$.

Now we prove that the functions g/h, f/h are monotonic on (Q,R) and (S,T), respectively, in order to use the second mean value theorem of integral calculus in computing the integrals $\int_Q^R x^\alpha(g^{(n)})^2\,dx$, $\int_S^T x^\alpha(f^{(n)})^2\,dx$. Here we follow the method introduced in [3]. The fundamental system of the solutions of (3.3) consists of powers of x or powers of x multiplied by $\ln x$. Order these solutions according to their rate of growth at ∞ and denote them $u_1,\dots,u_{2n}$, i.e. $u_i=o(u_{i+1})$ as $x\to\infty$, $i=1,\dots,2n-1$. One of these solutions, say y_j, is x^k, this solution replace by h. It is not difficult to verify that this system of solutions (with h denoted again y_j) is the Descartes system of solutions for large x, i.e. $W(y_{i_1},\dots,y_{i_k})>0$ for large x, whenever $1\le i_1<i_2<\dots<i_k\le 2n$, $k=1,\dots,2n$. Denote $z_1=-(y_1/h)',\dots,z_{j-1}=-(y_{j-1}/h)'$, $z_j=(y_{j+1}/h)',\dots,z_{2n-1}=(y_{2n}/h)'$. We have $z_1=-(y_1/h)'=h^{-2}W(y_1,h)>0$, $W(z_1,z_2)=h^{-3}W(y_1,y_2,h)>0$ and similarly $W(z_1,\dots,z_i)>0$, $i=3,\dots,2n-1$, hence $z_1,\dots,z_{2n-1}$ form the Markov system of solutions of a certain $(2n-1)$ order linear differential equation which is, by Lemma 3, eventually N-disconjugate.

Since the function $g(x)$ is a solution of (3.3), we have

$$g(x)=\sum_{i=1}^{j-1}d_iy_i(x)+d_jh(x)+\sum_{i=j+1}^{2n}d_iy_i(x);$$

hence

$$[g(x)/h(x)]'=\sum_{\substack{i=1\\ i\neq j}}^{2n}(y_i/h)'d_i=-\sum_{i=1}^{j-1}d_iz_i+\sum_{i=j}^{2n-1}d_{i+1}z_i,$$

where d_i are real constants. The function $(g/h)'$ verifies boundary conditions $(g/h)^{(i)}(Q)=0=(g/h)^{(i)}(R)$, $i=1,\dots,n-1$, i.e., it has $(2n-2)$ zeros (counting multiplicity) on $[Q,R]$ and if Q is sufficiently large, the eventual N-disconjugacy of the equation with solutions $z_1,\dots,z_{2n-1}$ implies that $(g/h)'$ does not vanish on (Q,R), hence g/h is monotonic on this interval. Analogously we can prove that the function f/h is monotonic on (S,T).

The second mean value theorem of integral calculus applied to the integrals $\int_Q^R qg^2\,dx$, $\int_S^T qf^2\,dx$ gives $\int_Q^R qg^2\,dx=\int_Q^R qh^2(g/h)^2\,dx=\int_{\xi_1}^R qh^2\,dx$, $\xi_1\in(Q,R)$. Similarly $\int_S^T qf^2\,dx=\int_S^{\xi_2} qh^2\,dx$, $\xi_2\in(S,T)$. The remaining part of the proof is the same as in Theorem 1. The proof is complete. $\square$

Remark 2. Similarly to Theorem 2, Theorem 1 can be also formulated in a more general form. More precisely, if σ is such that $x^{\sigma/2}$ is a solution of (3.3), we can replace (3.1_1) by the condition

$$\limsup_{x\to\infty}x^{2n-1-\alpha-\sigma}\int_x^\infty q(t)(t^{\sigma/2}+y_1(t)+\dots+y_k(t))^2dt<-B_{n,\alpha,\sigma}-\frac{\binom{\sigma/2}{n}(n!)^2}{2n-1-\alpha-\sigma},$$

where $y_1, \dots, y_k$ are the solutions of (3.3) whose rate of growth at ∞ is slower than $x^{\sigma/2}$ (the exact value of k depends on σ and α). For example, if $\alpha < 1$ and $\sigma/2 \in \{1, \dots, n-1\}$, then $k = \sigma/2$ and $y_1 = x^{k-1}, \dots, y_k = 1$. Similarly one can modify the condition (3.1_2). To prove Theorem 1 in this modified form it suffices to show that the test function y defined by (3.2) with $h = x^{\sigma/2} + y_1 + \dots + y_k$ satisfies

$$\lim_{R\to\infty} \int_Q^R x^\alpha (y^{(n)})^2\, dx = \gamma,$$
$$\lim_{S\to\infty} \int_S^{2S} x^\alpha (y^{(n)})^2\, dx = 0,$$
$$\lim_{S\to\infty} \int_R^S x^\alpha (h^{(n)})^2\, dx = \varrho$$

and that the function g/h is monotonic on (Q, R). These statements can be proved using only a slight modification of the method used in the proof of Theorem 2. Note that in the case when the assumption (1.2) is needed (i.e. $x^{\sigma/2}$ is not a solution of (3.3)), such a modification is useless, since if $q(x) \le 0$, the integral $\int^\infty q(t)t^{\sigma/2}\, dt$ converges absolutely and an addition of a function with a slower rate of growth at ∞ than $x^{\sigma/2}$ plays no role.

Remark 3. In Theorems 1, 2, we have considered the cases $\alpha \notin \{1, 2, \dots, 2n-1\}$, $\alpha + \sigma \neq 2n-1$ and $\alpha \in \{1, 3, \dots, 2n-1\}$, $\alpha + \sigma = 2n-1$. The case α arbitrary, $\alpha + \sigma = 2n-1$ was treated in [4]. The problem of analogical conditions to (1.3), (1.4), remains open in the case $\alpha \in \{1, 2, \dots, 2n-1\}$, σ-arbitrary (a particular case $\alpha \in \{2, 4, \dots, 2n-2\}$ $n = 2, 3$, was resolved in [5]). The method of computing of constants in (1.3), (1.4) via certain Wronskians of solutions of (3.3), may turn out to be useful in filling this gap in the theory of Nehari-type oscillation criteria for (1.1).

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