Stable points of unit ball in Orlicz spaces

Marek Wisła

Abstract. The aim of this paper is to investigate stability of unit ball in Orlicz spaces, endowed with the Luxemburg norm, from the "local" point of view. Firstly, those points of the unit ball are characterized which are stable, i.e., at which the map $z \to \{(x,y) : \frac{1}{2}(x+y) = z\}$ is lower-semicontinuous. Then the main theorem is established: An Orlicz space $L^\varphi(\mu)$ has stable unit ball if and only if either $L^\varphi(\mu)$ is finite dimensional or it is isometric to $L^\infty(\mu)$ or φ satisfies the condition Δ_r or Δ_r^0 (appropriate to the measure μ and the function φ) or $c(\varphi) < \infty, \varphi(c(\varphi)) < \infty$ and $\mu(T) < \infty$. Finally, it is proved that the set of all stable points of norm one is dense in the unit sphere $S(L^\varphi(\mu))$.

Keywords: stable point, stable unit ball, extreme point, Orlicz space

Classification: 46E30

1. Introduction.

A convex set C of a real Hausdorff topological space X is called **stable** if the midpoint map $\Phi : C \times C \to C, \Phi(x,y) = \frac{1}{2}(x+y)$ is open with respect to the inherited topology in C [1], [4]. Stable compact sets have been investigated in [5], [9], [13]. Stability is a useful tool in studying extremal operators between Banach spaces [1]. Further, the set of extreme points of a stable set is closed. Thus "stability" arguments can be applied to the description of extreme points of the unit ball of $C(K,X)$, K being a compact Hausdorff space and X a Banach space, namely, applying the Michael selection theorem [7],

$$f \in \operatorname{Ext} B(C(K,X)) \Longleftrightarrow f(k) \in \operatorname{Ext} B(X) \quad \text{for every} \quad k \in K$$

provided the unit ball $B(X)$ of X is stable.

Finite dimensional Banach spaces (with $\dim X > 2$) can have non-stable unit balls, for let $X = \mathbb{R}^3$ and

$$B = \operatorname{conv}(\{(x,y,0) : x^2 + y^2 \le 1\} \cup \{(\pm 1, 0 \pm 1)\}).$$

A full description of stable convex subsets of finite dimensional topological spaces can be found in [11]. The above defined set B cannot be a unit ball of any generalized finite dimensional Orlicz space – this is due to the fact that every such space has stable unit ball [3], [15]. This property is no longer true in the infinite dimensional case; even in the case of classical Orlicz sequence spaces ℓ_φ : ℓ_φ has stable unit ball if and only if either ℓ_φ is isometric to ℓ^∞ or φ satisfies the condition Δ_2 [14].

The aim of this paper is to extend the latter result to the case of Orlicz spaces $L^\varphi(\mu)$ of functions defined on an arbitrary measure space via the description of stable points of the unit ball $B(L^\varphi(\mu))$.

2. Basic definition and auxiliary results.

Let (T, Σ, μ) be a measure space with a nonnegative, σ-finite and complete measure μ ($\mu \not\equiv 0$) and let $\varphi : \mathbb{R} \to [0, +\infty]$ be a convex, even, non-identically equal to 0, vanishing at 0 and left-continuous for $a > 0$ function such that $c(\varphi) = \sup\{a : \varphi(a) < \infty\} > 0$. By an **Orlicz space $L^\varphi(\mu)$** ([8], [10]), we mean the set of all measurable functions $x : T \to \mathbb{R}$ such that $I_\varphi(\lambda x) < \infty$ for some $\lambda > 0$, where **the modular** I_φ is defined by

$$I_\varphi(x) = \int_T \varphi(x(t))\, d\mu.$$

$L^\varphi(\mu)$ is equipped with the equality "almost everywhere" (a.e.) and **the Luxemburg norm** [6]

$$\|x\|_\varphi = \inf\{\lambda > 0 : I_\varphi(\lambda^{-1} x) \le 1\}.$$

(Note that $\|x\|_\varphi \le 1$ iff $I_\varphi(x) \le 1$; $I_\varphi(x) = 1$ implies $\|x\|_\varphi = 1$; $I_\varphi(x) < 1 \Rightarrow$ ($\|x\|_\varphi = 1$ iff $I_\varphi(\lambda x) = +\infty$ for every $\lambda > 1$); $\|x_n - x\|_\varphi \to 0$ iff $I_\varphi(\lambda(x_n - x)) \to 0$ for every $\lambda > 0$.) The subspace

$$E^\varphi(\mu) = \{x \in L^\varphi(\mu) : I_\varphi(\lambda x) < \infty \ \text{ for every } \ \lambda > 0\}$$

is called **the space of finite elements**.

Let r be any number greater than 1. The function φ is said to satisfy **the condition** Δ_r ($\varphi \in \Delta_r$ in short) if:

(a) there exists a constant $c > 1$ such that $\varphi(ra) \le c\varphi(a)$ for every a (respectively, every $a \ge a_0, \varphi(a_0) < \infty$) provided μ is atomless and infinite (respectively, finite);

(b) there exist $b > 0, c > 1$ and a nonnegative sequence (d_n) such that $\Sigma_n d_n < \infty$, and $\varphi(ra)\mu(e_n) \le c\varphi(a)\mu(e_n) + d_n$ for every a with $\varphi(a)\mu(e_n) \le b$ and every $n \in N$ provided μ is purely atomic and $\{e_n : n \in N \subseteq \mathbb{N}\}$ is the set of all atoms of T.

(c) a combination of (a) and (b) if T has both an atomless and purely atomic part.

If $c(\varphi) = \infty$, then

$$\varphi \in \Delta_r \ \text{ for some } \ r > 1 \Longleftrightarrow \varphi \in \Delta_r \ \text{ for every } \ r > 1 \Longleftrightarrow \varphi \in \Delta_2\,.$$

The above equivalences remain true if μ is atomless (then $\varphi \in \Delta_r$ for some $r > 1$ implies that $c(\varphi) = \infty$). If μ is purely atomic with $\Sigma_n \mu(e_n) = \infty$ and $\varphi \in \Delta_r$ for some $r > 1$, then φ vanishes only at 0 (indeed, $d_n \ge \varphi(ra(\varphi))\mu(e_n)$ for every $n \in \mathbb{N}$, where $a(\varphi) = \sup\{a : \varphi(a) = 0\}$). Thus the above equivalences hold true also in the case of purely atomic measure μ with an infinite number of atoms provided $0 < \inf_n \mu(e_n) \le \sup_n \mu(e_n) < \infty$ – no matter whether φ takes only finite values or not (if $\varphi \in \Delta_{r_0}$, then evidently $\varphi \in \Delta_r$ for every $1 < r \le r_0$; for $r > r_0$, consider $b_r = \varphi(a' r_0/r) \cdot \inf_n \mu(e_n) > 0$, where $a' = \sup\{a > 0 : \varphi(a) \le$

$b_{r_0}/\sup_n \mu(e_n)\} > 0$). If $\dim L^{\varphi}(\mu) < \infty$ (i.e., T consists of a finite number of atoms), then $\varphi \in \Delta_r$ for some $r > 1$ if and only if $L^{\varphi}(\mu)$ is not isometric to $L^{\infty}(\mu)$ (take any $a_0 \in (a(\varphi), c(\varphi))$, $1 < r < c(\varphi)/a_0$ and put $b = \varphi(a_0) \cdot \inf_n \mu(e_n) > 0, d_n = \varphi(ra_0) \cdot \sup_n \mu(e_n) < \infty$). However, if $0 < a(\varphi) \leq c(\varphi) < \infty$ then φ does not satisfy the condition Δ_r for any $r > c(\varphi)/a(\varphi)$.

Note that if $c(\varphi) = \infty$ and $L^{\varphi}(\mu)$ is finite dimensional, then $L^{\varphi}(\mu) = E^{\varphi}(\mu)$. If $c(\varphi) = \infty$ and $\dim L^{\varphi}(\mu) = \infty$, the equality $L^{\varphi}(\mu) = E^{\varphi}(\mu)$ holds if and only if $\varphi \in \Delta_2$ (cf. [8, Theorem 8.13, p. 52], see also the proof of Lemma 5 below), thus, applying the Lebesgue dominated convergence theorem, we have then

$$\big(I_{\varphi}(x) = 1 \Longleftrightarrow \|x\|_{\varphi} = 1\big) \quad \text{if and only if} \quad \varphi \in \Delta_2\,.$$

In fact, we can replace the condition Δ_2 by Δ_r for some $r > 1$ in the last equivalence. Then the assumption $c(\varphi) = \infty$ is used in the "if" part of the proof only, so, in any case, we have that if $\varphi \notin \Delta_r$ for any $r > 1$, then there exists $x \in L^{\varphi}(\mu)$ such that $\|x\|_{\varphi} = 1$ but $I_{\varphi}(x) < 1$ and that is what we need in the sequel.

If Σ contains only a finite number, say m, of atoms, then the Orlicz space $L^{\varphi}(\mu)$ can be identified with the finite dimensional **generalized Orlicz space** $\ell^{(\varphi_1,\ldots,\varphi_m)}$ which consists of all (finite) sequences $x = (x_n)_{n=1}^m$ with $I_{\varphi}(x) = \Sigma_{n=1}^m \varphi_n(\lambda x_n) < \infty$ for some $\lambda > 0$, where $\varphi_n(a) = \varphi(a)\mu(\{e_n\})$, $n = 1, 2, \ldots, m$, yielded with the Luxemburg norm. The unit ball of that space is stable [3], [15] *independently* of the shape of the function φ.

The infinite dimensional case of Orlicz spaces was investigated by A. Suarez–Granero [12]: $B(L^{\varphi}(\mu))$ is stable provided φ takes only finite values and $\varphi \in \Delta_2$. In the sequel, we shall use that result, but in a somewhat weakened form (cf. Proposition 1 below).

Let C be a convex set of a topological vector space and let $\Phi : C \times C \to C$ be defined by $\Phi(x, y) = \frac{1}{2}(x + y)$. A point $z \in C$ is called **stable** (or C is said to be **stable at** z, cf. [11, p. 197]) if for every $(x, y) \in C \times C$ with $\frac{1}{2}(x+y) = z$ and every open neighborhood W of (x, y), $\Phi(W)$ is an open neighborhood of z. Equivalently, z is stable if and only if the mapping

$$C \ni \zeta \mapsto \Phi^{-1}(\zeta) \in C \times C$$

is lower-semicontinuous at z, that is, if, for any open set $W \subset C \times C$ with $W \cap \Phi^{-1}(z) \neq \emptyset$, there exists an open neighborhood $U \subset C$ of z such that $W \cap \Phi^{-1}(\zeta) \neq \emptyset$ for every $\zeta \in U$. Therefore C is stable if and only if every point $z \in C$ is stable.

Let us note that every extreme point is stable and that the stability of a point z with $\|z\| < 1$ can be deduced from the fact that every open convex set is stable. Further, $L^{\infty}(\mu)$ has the 3.2 intersection property, so its unit ball is stable (cf. [1]).

Let us turn back to the Suarez–Granero result. Omitting the assumption that φ takes only finite values (it is superfluous there) we have

Proposition 1 [12]**.** *Every point $z \in B(L^{\varphi}(\mu))$ with $I_{\varphi}(z) = 1$ is stable.*

Note. In order to establish stability of a point $z \in B(L^\varphi(\mu))$ it is sufficient to restrict the investigation to the case of $L^\varphi(\mu)$ being neither finite dimensional nor isometric to $L^\infty(\mu)$ and to the case of z with norm one, modular less than one and not being an extremal point of $B(L^\varphi(\mu))$.

Then z will be stable if for every $\varepsilon > 0$ and every distinct x, y of norm one with $\frac{1}{2}(x+y) = z$ (if e.g. $\|x\|_\varphi < 1$ then $\|z\|_\varphi < 1$) there exists $\delta > 0$ such that for every w with $\|w - z\|_\varphi < \delta$ we can find $u, v \in B(L^\varphi(\mu))$ satisfying the following conditions:

$$\|u - x\|_\varphi < \varepsilon, \quad \|v - y\|_\varphi < \varepsilon \quad \text{and} \quad w = \frac{1}{2}(u+v).$$

If $x, y \in B(L^\varphi(\mu))$, $z = \frac{1}{2}(x+y)$ and $I_\varphi(z) = 1$, then an easy calculation shows that $\varphi(z(t)) = \frac{1}{2}[\varphi(x(t)) + \varphi(y(t))]$ for almost every $t \in T$ (i.e., φ is affine on each non one-point interval $[\min\{|x(t)|, |y(t)|\}, \max\{|x(t)|, |y(t)|\}]$ for a.e. $t \in T$). The next proposition provides conditions under which the converse implication holds true as well.

Proposition 2. *Assume that $L^\varphi(\mu)$ is neither finite dimensional nor isometric to $L^\infty(\mu)$. Let $z \in B(L^\varphi(\mu))$ and define, for $n = 2, 3, \dots$,*

$$A_n = \{t \in T : |z(t)| < (1 - \frac{1}{n})c(\varphi)\}, \quad \text{if } c(\varphi) < \infty \text{ and } \varphi(c(\varphi)) < \infty$$

and $A_n = T$ otherwise. If $\|z\chi_{A_n}\|_\varphi = 1$ for some $n \geq 2$, then the following conditions are equivalent:

(i) $I_\varphi(z) < 1$,
(ii) *there exist a set $E \subset A_n$ of positive measure and functions $x, y \in B(L^\varphi(\mu))$ such that $\frac{1}{2}(x+y) = z$, $\|z\chi_E\|_\varphi < 1$ and*

$$2\varphi(z(t)) < \varphi(x(t)) + \varphi(y(t)) \quad \text{for every } t \in E.$$

Proof: We should only prove the implication (i) $\Rightarrow$ (ii). Let $T = M \cup S$, where M, S denote, respectively, the purely atomic and atomless part of the measure space (T, Σ, μ). Then either $\|z\chi_{M\cap A_n}\|_\varphi = 1$ or $\|z\chi_{S\cap A_n}\|_\varphi = 1$. Indeed, otherwise $I_\varphi(\lambda z\chi_{M\cap A_n}) \leq 1$ and $I_\varphi(\lambda z\chi_{S\cap A_n}) \leq 1$ for some $\lambda > 1$, so $I_\varphi(\lambda z\chi_{A_n}) < \infty$ and, in virtue of the Lebesgue dominated convergence theorem, $I_\varphi(\lambda' z\chi_{A_n}) < 1$ for some $1 < \lambda' \leq \lambda$, i.e., $\|z\chi_{A_n}\|_\varphi \leq (\lambda')^{-1} < 1$ – a contradiction.

Let $1 < \varrho < 2$ be a number such that $(1 - \frac{1}{n})\varrho \leq 1$. The rest of the proof will be split into two parts.

1°) $\|z\chi_{S\cap A_n}\|_\varphi = 1$. Since $I_\varphi(\varrho z\chi_{S\cap A_n}) = +\infty$, the set

$$D = \{t \in A_n \cap S : 2\varphi(z(t)) < \varphi(\varrho z(t))\}$$

is of positive measure. We claim that there exists $1 < \varrho' \leq \varrho$ such that $\varphi(\varrho' z(t)) < \infty$ on some subset $F \subseteq D$ of positive measure. That is clear ($\varrho = \varrho'$) when either

$c(\varphi) = \infty$ or $c(\varphi) < \infty$ with $\varphi(c(\varphi)) < \infty$. Suppose that $c(\varphi) < \infty, \varphi(c(\varphi)) = \infty$ and that $\varphi(\varrho' z(t)) = \infty$ for every $1 < \varrho' \leq \varrho$ and a.e. $t \in D$. Then $\varrho'|z(t)| \geq c(\varphi)$ for every $1 < \varrho' \leq \varrho$; so $|z(t)| \geq c(\varphi)$ for a.e. $t \in D$. Hence $I_\varphi(z) = \infty$ – a contradiction. To simplify the notation we shall assume that $\varrho' = \varrho$.

Applying the fact that $\varphi(\varrho z(t)) < \infty$ for $t \in F$, we can find a set $E \subset F$ such that $I_\varphi(\varrho z \chi_E) \leq 1 - I_\varphi(z) < 1$. Thus $\|z\chi_E\|_\varphi \leq \varrho^{-1} < 1$. Define

$$x = z\chi_{T\setminus E} + \varrho z \chi_E, \quad y = z\chi_{T\setminus E} + (2-\varrho) z\chi_E \,.$$

Plainly, $x, y \in B(L^\varphi(\mu))$. Further, for every $t \in E$,

$$\varphi(x(t)) + \varphi(y(t)) \geq \varphi(\varrho z(t)) > 2\varphi(z(t)).$$

2°) $\|z\chi_{M\cap A_n}\|_\varphi = 1$. By assumptions, $c(\varphi) > 0$ and the set $M \cap A_n$ is infinite. Without loss of generality, we can identify $M \cap A_n$ with the set $\mathbb{N}$ of all natural numbers.

Since $I_\varphi(z\chi_\mathbb{N}) < 1$, there exists $p \in \mathbb{N}$ such that

$$I_\varphi(z\chi_{\{p,p+1,\dots\}}) < 2\eta,$$

where $\eta = 1 - I_\varphi(z) > 0$.

Define $\langle p, m\rangle = \{p, p+1, \dots, m\}$ if $m \geq p$, $\langle p, m\rangle = \emptyset$ otherwise. Further, let

$$h(m) = I_\varphi(z\chi_{T\setminus\langle p,m\rangle}) + I_\varphi(\varrho z\chi_{\langle p,m\rangle}), \quad m \in \mathbb{N}.$$

Since $I_\varphi(\varrho z\chi_\mathbb{N}) = \infty, h(m) \to \infty$ as $m \to \infty$. Let $q := \min\{m \geq p-1 : h(m) < 1\}$. Then $0 \leq h(q) < 1$ and $1 \leq h(q+1) \leq \infty$. Since, in any case, the interval $[\varphi(z_{q+1}), \varphi(\varrho z_{q+1})]$ is contained in the range of φ (which is equal to $[0, \varphi(c(\varphi))] \cup \{\infty\}$ if $c(\varphi) < \infty$ and to $[0,\infty)$ if $c(\varphi) = \infty$) we can find $1 < \sigma \leq \varrho < 2$ such that $I_\varphi(\overline{x}) = 1$, where

$$\overline{x} = z\chi_{T\setminus\langle p,q+1\rangle} + \varrho z\chi_{\langle p,q\rangle} + \sigma z\chi_{\{q+1\}} \,.$$

Note that $I_\varphi(\varrho z\chi_{T\setminus\langle p,q+1\rangle}) = \infty$ (otherwise $I_\varphi(\sigma z) < \infty$, so $\|z\|_\varphi < 1$). Using similar arguments, we infer the existence of the numbers $r \in \mathbb{N}, r \geq q+1$ and $1 < \tau \leq \varrho < 2$ such that $I_\varphi(y) = 1$, where

$$y = z\chi_{T\setminus\langle p,r+1\rangle} + (2-\varrho)z\chi_{\langle p,q\rangle} + (2-\sigma)z\chi_{\{q+1\}} + \varrho z\chi_{\langle q+2,r\rangle} + \tau z\chi_{\{r+1\}} \,.$$

Put

$$x = z\chi_{T\setminus\langle p,r+1\rangle} + \varrho z\chi_{\langle p,q\rangle} + \sigma z\chi_{\{q+1\}} + (2-\varrho)z\chi_{\langle q+2,r\rangle} + (2-\tau)z\chi_{\{r+1\}} \,.$$

Obviously $\frac{1}{2}(x+y) = z$ and $I_\varphi(x) \leq I_\varphi(\overline{x}) = 1$. Further

$$I_\varphi(x) \geq I_\varphi(\overline{x}) - I_\varphi(z\chi_{\langle q+2,r+1\rangle}) \geq 1 - I_\varphi(z\chi_{\{p,p+1,\dots\}}) > 1 - 2\eta.$$

Finally, observe that φ is not affine on at least one interval from the following ones:

$$[(2-\varrho)|z(m)|,\ \varrho|z(m)|],\quad m\in\langle p,q\rangle\cup\langle q+2,r\rangle;$$
$$[(2-\sigma)|z(q+1)|,\ \sigma|z(q+1)|];\quad [(2-\tau)|z(r+1)|,\ \tau|z(r+1)|].$$

Indeed, otherwise,

$$I_\varphi(x)+I_\varphi(y)=2I_\varphi(z)=2(1-\eta);$$

so $I_\varphi(x)=1-2\eta$ – a contradiction.

Taking $E=\{i\}$, where $i\in\langle p,r+1\rangle$ is that index for which φ is not affine on the corresponding interval, all the requirements of (ii) are satisfied and the proof of Proposition 2 is concluded. □

3. Main results.

Theorem 3. *A point $z\in B(L^\varphi(\mu))$ is stable if and only if at least one of the following conditions is satisfied:*

(i) *$L^\varphi(\mu)$ is finite dimensional,*
(ii) *$L^\varphi(\mu)$ is isometric to $L^\infty(\mu)$,*
(iii) $\|z\|_\varphi<1$,
(iv) $I_\varphi(z)=1$,
(v) *$c(\varphi)<\infty, \varphi(c(\varphi))<\infty$ and $\|z\chi_{A_n}\|_\varphi<1$ for every $n=2,3,\dots$, where*

$$A_n=\{t\in T:|z(t)|<(1-\frac{1}{n})\,c(\varphi)\}.$$

PROOF: ($\Leftarrow$) The sufficiency of each of the conditions (i) $\div$ (iv) was discussed in Section 2.

Let us assume that none of the conditions (i) $\div$ (iv) is satisfied, but (v) holds. It is easy to check that $\varphi(c(\varphi))>0$ (otherwise $L^\varphi(\mu)$ is isometric to $L^\infty(\mu)$). Let x,y be arbitrary elements of $B(L^\varphi(\mu))$ such that $x\neq y$, $\frac{1}{2}(x+y)=z$ and $\|x\|_\varphi=\|y\|_\varphi=1$. Fix $0<\varepsilon<1$ and take $\alpha\in(\frac{1}{2},1)$ such that $\|x-\overline{x}\|_\varphi<\frac{1}{3}\cdot\varepsilon$ and $\|y-\overline{y}\|_\varphi<\frac{1}{3}\cdot\varepsilon$, where

$$\overline{x}=\alpha x+(1-\alpha)y,\quad \overline{y}=(1-\alpha)x+\alpha y.$$

Evidently, $\frac{1}{2}(\overline{x}+\overline{y})=z$ and $\overline{x}\neq\overline{y}$. Further, $x(t)=\overline{x}(t)$ iff $y(t)=\overline{y}(t)$ iff $x(t)=y(t)=z(t)$. Thus $\|\overline{x}\chi_{A_n}\|_\varphi<1$ and $\|\overline{y}\chi_{A_n}\|_\varphi<1$ for every $n\geq 2$.

Let $B_n=\{t\in T:(1-\frac{1}{n})c(\varphi)\leq|z(t)|<c(\varphi)\}$, $n\geq 2$, and $C=\{t\in T:|z(t)|=c(\varphi)\}$. We have $\overline{x}(t)=z(t)=\overline{y}(t)$ for a.e. $t\in C$. Further, for every w with $I_\varphi(w)<\infty, |w(t)|\leq c(\varphi)$ for a.e. $t\in T$. Since $\varphi(c(\varphi))>0$, $\mu(B_n)\to 0$ as $n\to\infty$; so

$$\int_{B_n}\varphi(w(t))\,d\mu\leq\varphi(c(\varphi))\mu(B_n)\to 0\quad\text{ as }\ n\to\infty$$

uniformly on $\{w\in L^\varphi(\mu):I_\varphi(w)<\infty\}$.

By the convexity of I_φ and the inequality $I_\varphi(z) < 1$, we infer that $I_\varphi(\overline{x}) < 1$ and $I_\varphi(\overline{y}) < 1$. Thus, there exists $0 < \beta < 1$ such that

$$\max\{I_\varphi(\overline{x}), I_\varphi(\overline{y})\} + 2\beta < 1.$$

Next, fix $n \geq 6/\varepsilon$ with $\varphi(c(\varphi))\mu(B_n) < \beta$. Since $\|\overline{x}\chi_{A_n}\|_\varphi < 1, I_\varphi(\lambda\overline{x}\chi_{A_n}) \leq 1$ for some $\lambda > 1$. In virtue of the Lebesgue dominated convergence theorem, we can find $\gamma > 1$ such that

$$I_\varphi(\gamma\overline{x}\chi_{A_n}) \leq I_\varphi(\overline{x}\chi_{A_n}) + \beta$$

and, analogously,

$$I_\varphi(\gamma\overline{y}\chi_{A_n}) \leq I_\varphi(\overline{y}\chi_{A_n}) + \beta.$$

Let $\delta = \min\{(1-\frac{1}{\gamma})\beta, \frac{\varepsilon}{6}\}$ and take an arbitrary $w \in B(L^\varphi(\mu))$ with $\|w-z\|_\varphi < \delta$. Then $\|\frac{\gamma}{\gamma-1}(w-z)\|_\varphi < \beta < 1$, so $I_\varphi(\frac{\gamma}{\gamma-1}(w-z)) \leq \beta$, for $I_\varphi(x) \leq \|x\|_\varphi$ for every $x \in B(L^\varphi(\mu))$. Finally, let

$$u(t) = \begin{cases} w(t) & \text{if } t \in C \cup B_n, \\ \overline{x}(t) + w(t) - z(t) & \text{if } t \in A_n; \end{cases}$$

$$v(t) = \begin{cases} w(t) & \text{if } t \in C \cup B_n, \\ \overline{y}(t) + w(t) - z(t) & \text{if } t \in A_n. \end{cases}$$

We claim that $\|u - x\|_\varphi < \varepsilon$ and $\|v - y\|_\varphi < \varepsilon$. Since $\overline{x}(t) = z(t)$ for a.e. $t \in C$,

$$I_\varphi(\delta^{-1}(u-\overline{x})\chi_C) = I_\varphi(\delta^{-1}(w-z)\chi_C) \leq I_\varphi(\delta^{-1}(w-z)) \leq 1;$$

so $\|(u-\overline{x})\chi_C\|_\varphi \leq \delta < \varepsilon/6$. Further, since $\frac{1}{2}(\overline{x}(t) + \overline{y}(t)) = z(t)$ for a.e. $t \in T$, $|\overline{x}(t) - z(t)| \leq \frac{1}{n} \cdot c(\varphi)$ for a.e. $t \in B_n$. Thus

$$\begin{aligned} I_\varphi(3\varepsilon^{-1}(u-\overline{x})\chi_{B_n}) = I_\varphi\left(\frac{1}{2}(6\varepsilon^{-1}(w-z)\chi_{B_n}) + \frac{1}{2}(6\varepsilon^{-1}(z-\overline{x})\chi_{B_n})\right) \leq \\ \leq \frac{1}{2}I_\varphi(6\varepsilon^{-1}(w-z)) + \frac{1}{2}\varphi(\frac{6}{\varepsilon n} \cdot c(\varphi))\mu(B_n) \leq \\ \leq \frac{1}{2}I_\varphi(\delta^{-1}(w-z)) + \frac{1}{2}\varphi(c(\varphi))\mu(B_n) \leq \frac{1}{2}(1+\beta) \leq 1; \end{aligned}$$

so $\|(u-\overline{x})\chi_{B_n}\|_\varphi \leq \varepsilon/3$. Therefore

$$\begin{aligned} \|u-x\|_\varphi \leq \|u-\overline{x}\|_\varphi + \|x-\overline{x}\|_\varphi \leq \\ \leq \|(u-\overline{x})\chi_C\|_\varphi + \|(u-\overline{x})\chi_{B_n}\|_\varphi + \|(w-z)\chi_{A_n}\|_\varphi + \|x-\overline{x}\|_\varphi < \varepsilon. \end{aligned}$$

Analogously, $\|v - \overline{y}\| < \varepsilon$.

To prove the stability of z, we should show that $u, v \in B(L^\varphi(\mu))$. We have

$$I_\varphi(u) = I_\varphi(w\chi_C) + I_\varphi(w\chi_{B_n}) + I_\varphi\big(\frac{1}{\gamma}(\gamma\overline{x}\chi_{A_n}) + (1-\frac{1}{\gamma})\frac{\gamma}{\gamma-1}(w-z)\chi_{A_n}\big)$$
$$\le \varphi(c(\varphi))\mu(C) + \varphi(c(\varphi))\mu(B_n) + \frac{1}{\gamma}\cdot I_\varphi(\gamma\overline{x}\chi_{A_n}) + (1-\frac{1}{\gamma})\cdot I_\varphi(\frac{\gamma}{\gamma-1}(w-z)\chi_{A_n})$$
$$\le I_\varphi(\overline{x}\chi_C) + \beta + \frac{1}{\gamma}\cdot\big(I_\varphi(\overline{x}\chi_{A_n}) + \beta\big) + (1-\frac{1}{\gamma})\cdot I_\varphi(\frac{\gamma}{\gamma-1}(w-z))$$
$$\le I_\varphi(\overline{x}) + 2\beta < 1.$$

($\Rightarrow$) Let us suppose that none of the conditions (i) $\div$ (v) is satisfied. Let $n \in \mathbb{N}$ be any number such that $\|z\chi_{A_n}\| = 1$ if $c(\varphi) < \infty$ and $\varphi(c(\varphi)) < \infty$, and put $n = 1, A_n = T$ in the other case.

By the lower semicontinuity of I_φ and in virtue of Proposition 2, we can find $\varepsilon > 0, x, y \in B(L^\varphi(\mu))$ with $\frac{1}{2}(x+y) = z$ and a set $E \subset A_n$ of positive measure such that $\|z\chi_E\|_\varphi < 1$ and

$$2I_\varphi(z\chi_E) < I_\varphi(u\chi_E) + I_\varphi(v\chi_E)$$

for every $u, v \in B(L^\varphi(\mu))$ with $\|u-x\|_\varphi < \varepsilon$ and $\|v-y\|_\varphi < \varepsilon$.

Let $0 < \delta < 2/n$ and fix $k \in \mathbb{N}$ with $k > 2\delta^{-1} > n$. Since $I_\varphi(z) < 1, \|z\chi_E\|_\varphi < 1$ and $\|z\chi_{A_n}\|_\varphi = 1$, we have $I_\varphi(\lambda z\chi_{A_n\setminus E}) = \infty$ for every $\lambda > 1$. Let us take, if $c(\varphi) < \infty$ and $\varphi(c(\varphi)) < \infty$, any countable covering $(E_i)_{i=1}^\infty$ of the set $A_n \setminus E$ consisting of pairwise disjoint sets $E_i \subset A_n \setminus E$ of positive and finite measure and put $a_i = \varphi^{-1}(i)$,

$$E_i = \{t \in T\setminus E : a_{i-1} \le |z(t)| < a_i\}, \quad i = 1, 2, \dots ,$$

in the other cases (since $I_\varphi(z) < 1, \mu(E_i) < \infty$ for each $i \in \mathbb{N}$). Define

$$h(m) = \sum_{i=1}^{m} I_\varphi((1+\frac{1}{k})z\chi_{E_i}) + \sum_{i=m+1}^{\infty} I_\varphi(z\chi_{E_i}), \quad m = 0, 1, 2, \dots$$

(with the usual convention $\sum_\emptyset = 0$). We have

$$\varphi((1+\frac{1}{k})z(t)) \le \begin{cases} \varphi(c(\varphi)) < \infty & \text{for } t \in A_n, \text{ if } c(\varphi) < \infty, \varphi(c(\varphi)) < \infty, \\ \varphi((1+\frac{1}{k})a_i) < \infty & \text{for } t \in E_i, \text{ if } c(\varphi) = \infty \text{ or} \\ & c(\varphi) < \infty \text{ and } \varphi(c(\varphi)) = \infty. \end{cases}$$

Thus $h(m) < \infty$ for every $m \in \mathbb{N}$. Further, $h(m) \to \infty$ as $m \to \infty$, since $I_\varphi((1+\frac{1}{k})z\chi_{A_n\setminus E}) = \infty$.

Let $p = \max\{m \ge 0 : h(m) < 1\}$ and let $0 < s \le k^{-1}$ be such a number that $I_\varphi(w) = 1$, where

$$w(t) = \begin{cases} (1+\frac{1}{k})z(t) & \text{for } t \in \bigcup_{i=1}^p E_i, \\ (1+s)z(t) & \text{for } t \in E_{p+1}, \\ z(t) & \text{otherwise.} \end{cases}$$

Suppose that there are $u, v \in B(L^{\varphi}(\mu))$ such that $\|u - x\|_{\varphi} < \varepsilon, \|v - y\|_{\varphi} < \varepsilon$ and $\frac{1}{2}(u + v) = w$. Then, by the convexity of φ, we have

$$\varphi(a + \eta) \geq \varphi'_{+}(a)\eta + \varphi(a)$$

for every $\eta \in \mathbb{R}$ and $|a| < c(\varphi)$, where φ'_{+} denotes the right hand side derivative of φ. Therefore

$$\begin{aligned}
2 &\geq I_{\varphi}(u) + I_{\varphi}(v) = \\
&= I_{\varphi}(u\chi_E) + I_{\varphi}(v\chi_E) + I_{\varphi}([w + u - v]\chi_{T\setminus E}) + I_{\varphi}([w + v - w]\chi_{T\setminus E}) > \\
&> 2I_{\varphi}(z\chi_E) + 2I_{\varphi}(w\chi_{T\setminus E}) + \int_{T\setminus E} \varphi'_{+}(w(t))[u(t) + v(t) - 2w(t)]\, d\mu = \\
&= 2I_{\varphi}(w) = 2.
\end{aligned}$$

This contradiction ends the proof of the theorem. □

Let $\{e_n : n \in N\}, N \subseteq \mathbb{N}$, be the set of all atoms of T and let $r > 1$. We shall say that a function φ satisfies the **condition** Δ_r^0 (on T) – $\varphi \in \Delta_r^0$ in short – if

- there exist $a_0 > 0$ and $c > 1$ such that $0 < \varphi(a_0) < \infty$ and

  $$\varphi(ra) \leq c\varphi(a) \quad \text{for every } |a| \leq a_0,$$

  provided the atomless part of T is of positive measure;
- there exist $a_0 > 0, b > 0, c > 1$ and a nonnegative sequence (d_n) such that $\sum_n d_n < \infty, 0 < \varphi(a_0) < \infty$ and

  $$\varphi(ra)\mu(e_n) \leq c\varphi(a)\mu(e_n) + d_n$$

  for every $|a| \leq a_0$ with $\varphi(a)\mu(e_n) \leq b$ and every $n \in N$ provided μ is purely atomic.

If $\varphi \in \Delta_r^0$ for some $r > 1$ on the atomless part of T which is of positive measure, then, evidently, $\varphi \in \Delta_r^0$ on the whole set T. Further, if the measure of the atomless part of T is either infinite or equal to zero and $\varphi \in \Delta_r$ for some $r > 1$, then $\varphi \in \Delta_r^0$. Thus $\varphi \in \Delta_r^0$ for some $r > 1$ provided $\dim L^{\varphi}(\mu) < \infty$ and $L^{\varphi}(\mu)$ is not isometric to $L^{\infty}(\mu)$.

If $\varphi \in \Delta_r^0$ for some $r > 1$, then, for any number $a' \in (a(\varphi), c(\varphi))$, we can find $1 < r' \leq c(\varphi)/a'$ such that $\varphi \in \Delta_{r'}^0$ with a' instead of a (consider $r' = \min\{r, c(\varphi)/a'\} > 1$, $c' = \max\{c, \varphi(r'a')/\varphi(a_0)\}$ and $b' = b$). Therefore, if $\varphi \in \Delta_r^0$ for some $r > 1$ and $\|x\|_{\infty} < c(\varphi)$, then

$$\|x\|_{\varphi} = 1 \Longleftrightarrow I_{\varphi}(x) = 1.$$

Note that $\varphi \in \Delta_r^0$ for some $r > 1$ iff $\varphi \in \Delta_2^0$ provided φ takes only finite values.

Before we present a theorem on stability of the unit ball in $L^{\varphi}(\mu)$, we shall prove an auxiliary lemma.

Lemma 4. *Let M, S be the purely atomic and atomless part of T, respectively. If φ does not satisfy the condition Δ_r^0 for every $r>1$ on M (respectively on S) and $\mu(M)=\infty$ (respectively $\mu(S)=\infty$), then there exists a sequence (x_k) of simple functions with disjoint supports such that*

$$\|x_k\|_\infty < \frac{1}{2k}\cdot\min\{1,c(\varphi)\},\ I_\varphi(x_k)<2^{-k}\ \text{ and }\ I_\varphi((1+\frac{1}{k})x_k)\geq 1$$

for every $k\geq 1$. Thus, the point $x=\sum_k x_k\in B(L^\varphi(\mu))$ is not stable.

Proof: (a) Assume that φ does not satisfy the condition Δ_r^0 for any $r>1$ on S and $\mu(S)=\infty$. Take, for every $k\in\mathbb{N}$,

$$a_k=\frac{1}{2k}\cdot\min\{1,c(\varphi)\},\ c_k=2^{k+1},\ r_k=1+\frac{1}{k}\,.$$

Then we can find a sequence (β_k) of positive numbers such that $0<\beta_k<a_k$ and $\varphi((1+\frac{1}{k})\beta_k)>2^{k+1}\varphi(\beta_k)$. Further, we can choose a sequence (T_k) of pairwise disjoint measurable sets with $\varphi(\beta_k)\mu(T_k)=1/2^{k+1}$ for every $k\in\mathbb{N}$. Then the sequence $x_k=\beta_k\xi_{T_k}, k\in\mathbb{N}$, possesses all the required properties.

(b) Now, let $\mu(M)=\infty$ and assume that φ does not satisfy the condition Δ_r^0 for any $r>1$ on $M=\{e_n:n\in\mathbb{N}\}$. Take c_k,r_k as above and put $a_k=\frac{1}{2kr_k}\cdot\min\{1,c(\varphi)\}$ and $b_k=1/2^{k+1}$ for $k\in\mathbb{N}$. Further, let

$$\alpha_n(k):=\sup\{\varphi(r_ka)\mu(e_n):0\leq a\leq a_k,\varphi(a)\mu(e_n)\leq b_k,\varphi(r_ka)>c_k\varphi(a)\}$$

for $n,k\in\mathbb{N}$. Then, for every $k,n\in\mathbb{N}$, we have $0\leq\alpha_n(k)<\infty$ and

$$\varphi(r_ka)\mu(e_n)\leq c_k\varphi(a)\mu(e_n)+\alpha_n(k)$$

for every $|a|\leq a_k$ with $\varphi(a)\mu(e_n)\leq b_k$. Since $\varphi\notin\Delta_{r_k}^0$, $\sum_n\alpha_n(k)=\infty$ for every $k\in\mathbb{N}$. Thus we can find a sequence (N_k) of pairwise disjoint subsets of $\mathbb{N}$ such that $\alpha_n(k)>0$ for every $n\in N_k$ and $\sum_{n\in N_k}\alpha_n(k)>2$ for every $k\in\mathbb{N}$. By the definition of $\alpha_n(k)$, for every $k\in\mathbb{N}$ and $n\in N_k$, we can find a number $\beta_n(k)$ such that

$$0\leq\beta_n(k)\leq a_k,\ \varphi(\beta_n(k))\mu(e_n)\leq b_k,\ \varphi(r_k\beta_n(k))>c_k\varphi(\beta_n(k))$$

and

$$\varphi(r_k\beta_n(k))\mu(e_n)>\alpha_n(k)-2^{-n}.$$

Thus $\sum_{n\in N_k}\varphi(r_k\beta_n(k))\mu(e_n)>1$ for every $k\in\mathbb{N}$. Define, for $k\in\mathbb{N}$,

$$n_k=\max\{p\in N_k:\sum_{n\in N_k,n\leq p}\varphi(r_k\beta_n(k))\mu(e_n)<1\},$$

$m_k=\min\{p\in N_k:p>n_k\}$ and

$$x_k=(\beta_n(k)\chi_{N_k\cap\{1,\dots,m_k\}}(n))_{n=1}^\infty\,.$$

Then $\|x_k\|_\infty \le a_k$,

$$I_\varphi(x_k) = \sum_{n\in N_k, n\le n_k} \varphi(\beta_n(k))\mu(e_n) + \varphi(\beta_{m_k}(k))\mu(e_{m_k}) \le$$
$$\le \sum_{n\in N_k, n\le n_k} \frac{1}{c_k}\cdot\varphi(r_k\beta_n(k))\mu(e_n) + b_k < \frac{1}{c_k} + b_k = \frac{1}{2^k}$$

and

$$I_\varphi((1+\frac{1}{k})x_k) = \sum_{n\in N_k, n\le m_k} \varphi(r_k\beta_n(k))\mu(e_n) \ge 1.$$

(c) Let $x = \sum_k x_k$. Obviously, $\|x\|_\varphi = 1$ and $I_\varphi(x) < 1$. Further, since $\|x\|_\infty \le \frac{1}{2}\cdot\min\{1, c(\varphi)\}$, $A_n = \{t\in T : |x(t)| \le (1-\frac{1}{n})c(\varphi)\} = T$ for every $n\ge 2$. Thus $\|x\chi_{A_n}\|_\varphi = \|x\|_\varphi = 1$ for every $n \ge 2$, so, by virtue of Theorem 3, x is not stable. □

Theorem 5. *The unit ball of the Orlicz space $L^\varphi(\mu)$ is stable if and only if at least one of the following conditions is satisfied:*

(i) *$L^\varphi(\mu)$ is finite dimensional;*
(ii) *$L^\varphi(\mu)$ is isometric to $L^\infty(\mu)$;*
(iii) *φ satisfies the condition Δ_r for some $r > 1$;*
(iv) *φ satisfies the condition Δ_r^0 for some $r > 1$ provided $c(\varphi) < \infty$ and $\varphi(c(\varphi)) < \infty$;*
(v) *φ satisfies the condition Δ_r^0 for some $r > 1$ on the purely atomic part of T provided $c(\varphi) < \infty, \varphi(c(\varphi)) < \infty$ and the measure of the atomless part of T is finite;*
(vi) *$c(\varphi) < \infty, \varphi(c(\varphi)) < \infty$ and $\mu(T) < \infty$.*

Proof: ($\Leftarrow$) The sufficiency of (i) and (ii) is obvious. If $\varphi\in\Delta_r$ for some $r > 1$ and $c(\varphi) = \infty$, then $B(L^\varphi(\mu))$ is stable by virtue of the Suarez–Granero theorem [12].

Let $c(\varphi) < \infty, \varphi(c(\varphi)) = \infty$ and $\varphi\in\Delta_r$ for some $r > 1$. Then μ must be purely atomic and $\inf_{n\in N}\mu(e_n) \ge b/\varphi(c(\varphi)r^{-1})$, where $\{e_n : n\in N\subseteq\mathbb{N}\}$ is the set of atoms of T. Then $L^\varphi(\mu)\subseteq\ell^1\subseteq c_0$, so $\|x\|_\infty < c(\varphi)$ for every $I_\varphi(x) < \infty$. Further, $\varphi\in\Delta_r^0$, thus $I_\varphi(x) = 1$ iff $\|x\|_\varphi = 1$, so $B(L^\varphi(\mu))$ is stable.

Let $c(\varphi) < \infty, \varphi(c(\varphi)) < \infty$. Fix any x with $\|x\|_\varphi = 1$. Then $\|x\|_\infty \le c(\varphi)$. If $\|x\|_\infty < c(\varphi)$ and one of the conditions (iv), (v), (vi) is satisfied, then, by the Lebesgue dominated convergence theorem, $I_\varphi(x) = 1$, so x is stable (if $T_0\subseteq T$ is such a set that $\mu(T_0) < \infty$, then $I_\varphi(rx\chi_{T_0}) \le \varphi(r\|x\|_\infty)\mu(T_0) < \infty$, where $1 < r < \frac{c(\varphi)}{\|x\|_\infty}$).

If $\|x\|_\infty = c(\varphi)$, then $\|x_n\|_\infty < \|x\|_\infty = c(\varphi)$ for every $n\ge 2$, where $x_n = x\chi_{\{t:|x(t)|<(1-\frac{1}{n})c(\varphi)\}}$. Thus $I_\varphi(x_n) < I_\varphi(x) = 1$, and, by assumptions, $I_\varphi(rx_n) < \infty$ for some $r > 1$, so $\|x_n\|_\varphi < 1$ for every $n\ge 2$, hence, by Theorem 3, $B(L^\varphi(\mu))$ is stable.

($\Rightarrow$) Assume that none of the conditions (i) $\div$ (vi) is satisfied.

Let $c(\varphi) = \infty$ or $\varphi(c(\varphi)) = \infty$. Then $\varphi \notin \Delta_r$ for any $r > 1$, so there exists $x \in L^\varphi(\mu)$ such that $\|x\|_\varphi = 1$, but $I_\varphi(x) < 1$ (cf. [8], [16, Lemma 3.2]). By Theorem 3, $B(L^\varphi(\mu))$ is not stable.

Let $c(\varphi) < \infty, \varphi(c(\varphi)) < \infty$. Then $\mu(T) = \infty$. Let M, S be the purely atomic and atomless parts of T, respectively. If $\mu(S) < \infty$, then, by Lemma 4, $B(L^\varphi(\mu))$ is not stable for $\varphi \notin \Delta_r^0$ on M for every $r > 1$. Finally, if $\mu(M) < \infty$, then $\mu(S) = \infty$ and $\varphi \notin \Delta_r^0$ on S for every $r > 1$, so, once again, by Lemma 4, $B(L^\varphi(\mu))$ is not stable and the proof is finished. □

It is easy to observe that the theorem on stability of the unit ball in Orlicz sequence spaces ($\mu(e_n) = 1$ for every $n \in \mathbb{N}$) presented in [14] can be deduced from Theorem 5 (note that $\varphi \in \Delta_r$ for some (every) $r > 1$ iff $\varphi \in \Delta_r^0$ for some (every) $r > 1$ iff $\varphi \in \Delta_2$ in that case). In the case of an atomless measure we have the following

Corollary 6. *Let μ be an atomless measure. Then the unit ball of the Orlicz space $L^\varphi(\mu)$ is stable if and only if one of the following conditions is satisfied:*

(i) *$L^\varphi(\mu)$ is isometric to $L^\infty(\mu)$;*
(ii) *$c(\varphi) = \infty$ and φ satisfies the condition Δ_2;*
(iii) *$c(\varphi) < \infty, \varphi(c(\varphi)) < \infty$ and either $\mu(T) < \infty$ or φ satisfies the condition Δ_r^0 for some $r > 1$.*

Corollary 7. *If the function φ takes only finite values, then the unit ball of the Orlicz space $L^\varphi(\mu)$ is stable if and only if either $L^\varphi(\mu)$ is finite dimensional or φ satisfies the condition Δ_2.*

4. Topological structure of stable points in the unit sphere.

It is evident that the set of stable points of $B(L^\varphi(\mu))$ is dense in $B(L^\varphi(\mu))$. We shall show that the similar result is valid if we replace "the unit ball" by "the unit sphere" $S(L^\varphi(\mu))$; more precisely, the set of all stable points of $B(L^\varphi(\mu))$ with norm one (it will be denoted by Stab) is dense in $S(L^\varphi(\mu))$.

Proposition 8. *Let $G_n = \{z \in L^\varphi(\mu) : \|z\|_\infty > (1 - \frac{1}{n}) \cdot c(\varphi)\}$ if $c(\varphi) < \infty$ and $\varphi(c(\varphi)) < \infty$; $G_n = \emptyset$ otherwise; $n = 1, 2, \dots$. Then, for every n sufficiently large, the set $S(L^\varphi(\mu)) \setminus (\mathrm{Stab} \cup G_n)$ is of the first Baire category in $S(L^\varphi(\mu)) \setminus G_n$ with the topology induced from $L^\varphi(\mu)$.*

Note. If $c(\varphi) < \infty$, then $L^\varphi(\mu) \subset L^\infty(\mu)$ and, by the closed graph theorem, the corresponding identity map is continuous. Thus the sets G_n are open in $L^\varphi(\mu)$.

PROOF: In virtue of Theorem 5, we can assume that $L^\varphi(\mu)$ is neither finite dimensional nor isometric to $L^\infty(\mu)$. Thus $\varphi(c(\varphi)) > 0$. Let $q \geq 2$ be any number such that $\varphi(\frac{q-1}{q+1} \cdot c(\varphi)) > 0$. Let us fix $n \geq q$. Then

$$S(L^\varphi(\mu)) \setminus (\mathrm{Stab} \cup G_n) = \bigcup_{m=1}^{\infty} H_m,$$

where $H_m = \{x \notin G_n : I_\varphi(x) \le 1 - \frac{1}{m}, \|x\|_\varphi = 1\}$, $m = 1, 2, \ldots$. Since I_φ is lower semicontinuous, each H_m is closed in $S(L^\varphi(\mu)) \setminus G_n$. Suppose that $\bigcup_m H_m$ is not of the first Baire category, i.e. $\operatorname{int} H_m \neq \emptyset$ for some (fixed from now on) $m \in \mathbb{N}$. Then there exists $z \in H_m$ and an open neighborhood $U \subset S(L^\varphi(\mu)) \setminus G_n$ of z such that

$$\sup_{x\in U} I_\varphi(x) \le 1 - \frac{1}{m}\,.$$

We claim that $I_\varphi(\lambda z\chi_H) = \infty$ for every $\lambda > 1$, where

$$H = \{t \in T : |z(t)| \le \frac{n-1}{n+1} \cdot c(\varphi)\} \ \text{ if } \ c(\varphi) < \infty \ \text{ and } \ \varphi(c(\varphi)) < \infty,$$

and $H = T$ otherwise. If $H = T$, then there is nothing to prove. Assume that $c(\varphi) < \infty, \varphi(c(\varphi)) < \infty$ and suppose that $I_\varphi(\lambda z\chi_H) < \infty$ for some $\lambda > 1$. Let $1 < \lambda' \le \min\{\lambda, (1 + \frac{1}{n})\}$. Since $|z(t)| \le (1 - \frac{1}{n})c(\varphi)$ for a.e. $t \in T$, we have

$$\begin{aligned} I_\varphi(\lambda' z) &\le I_\varphi(\lambda z\chi_H) + I_\varphi((1 + \frac{1}{n})z\chi_{T\setminus H}) \le \\ &\le I_\varphi(\lambda z\chi_H) + KI_\varphi((z\chi_{T\setminus H}) < \infty, \end{aligned}$$

where

$$K = \varphi(c(\varphi))/\varphi(\frac{n-1}{n+1} \cdot c(\varphi)) < \infty.$$

Hence $\|z\|_\varphi < 1$ – a contradiction. Consequently, the set H can neither be empty nor consist of a finite number of atoms.

Let us choose a countable covering $(T_n)_{n=1}^\infty$ of H consisting of pairwise disjoint sets $T_n \subset H$ of positive and finite measure. In an analogous way as in the proof of Proposition 2, step 2°), for a given $k \ge n$, we can find the numbers $p_k \in \mathbb{N} \cup \{0\}$ and $0 < s_k \le \frac{1}{k}$ such that $I_\varphi(x_k) = 1$, where

$$x_k = (1 + \frac{1}{k})z\chi_{\overline{T}_{p_k}} + (1 + s_k)z\chi_{T_{p_k+1}} + z\chi_{T\setminus\overline{T}_{p_k+1}}$$

and $\overline{T}_k = \bigcup_{i=1}^k T_i$. Further, if $c(\varphi) < \infty$ and $\varphi(c(\varphi)) < \infty$,

$$|x_k(t)| \le (1 + \frac{1}{k})|z(t)| \le (1 + \frac{1}{n}) \cdot \frac{n-1}{n+1} \cdot c(\varphi) = (1 - \frac{1}{n}) \cdot c(\varphi)$$

for every $t \in H$. Therefore, in any case, $x_k \in S(L^\varphi(\mu)) \setminus G_n$ for every $k \ge n$. Moreover, for every $\lambda > 1$ and $k \ge \max\{n, \lambda\}$, we have

$$I_\varphi(\lambda(z - x_k)) = \sum_{i=1}^{p_k} I_\varphi(\frac{\lambda}{k} \cdot z\chi_{T_i}) + I_\varphi(\lambda s_k z\chi_{T_{p_{k+1}}}) \le \frac{\lambda}{k} I_\varphi(z) \xrightarrow[k\to\infty]{} \infty,$$

i.e., $\|z - x_k\|_\varphi \to 0$. Thus $x_k \in U$, so $I_\varphi(x_k) \le 1 - \frac{1}{m}$ for large k – a contradiction. $\square$

Theorem 9. *The set* Stab *is dense in* $S(L^{\varphi}(\mu))$.

Proof: If $c(\varphi) = \infty$ or $c(\varphi) < \infty$ but $\varphi(c(\varphi)) = \infty$, the statement is an immediate consequence of Proposition 8 and the Baire category theorem ([2]).

Assume that $c(\varphi) < \infty, \varphi(c(\varphi)) < \infty$ and let $x \in S(L^{\varphi}(\mu)) \setminus \mathrm{Stab}$. In virtue of Theorem 3, $I_{\varphi}(x) < 1$ and there exists $p \in \mathbb{N}$ such that $I_{\varphi}(\lambda x \chi_{T \setminus D_n}) = \infty$ for every $\lambda > 1$ and $n \geq p$, where $D_n = \{t \in T : |x(t)| \geq (1 - \frac{1}{n}) \cdot c(\varphi)\}$, $n = 2, 3, \ldots$. Evidently

$$\varphi((1 - \frac{1}{n})c(\varphi))\mu(D_n) \leq I_{\varphi}(x) \leq 1,$$

so $\sup_{n>m} \mu(D_n) < \infty$ for sufficiently large $m \in \mathbb{N}$. Define

$$x_n = x\chi_{T \setminus D_n} + (1 - \frac{1}{n})c(\varphi) \cdot \operatorname{sgn} x \cdot \chi_{D_n}, \quad n \geq 2.$$

Then, for every $\lambda > 0$ and $n > \max\{2, \lambda, m\}$,

$$\begin{aligned} I_{\varphi}(\lambda(x - x_n)) &= I_{\varphi}(\lambda(x - (1 - \frac{1}{n})c(\varphi) \cdot \operatorname{sgn} x) \cdot \chi_{D_n}) \leq \\ &\leq \varphi(\frac{\lambda}{n} \cdot c(\varphi))\mu(D_n) \leq \frac{\lambda}{n} \cdot \varphi(c(\varphi)) \cdot \sup_{n>m} \mu(D_n) \xrightarrow[n \to \infty]{} 0, \end{aligned}$$

i.e., $\|x - x_n\|_{\varphi} \to 0$. Further, $|x_n(t)| \leq (1 - \frac{1}{n})c(\varphi)$ for a.e. $t \in T$, i.e., $x_n \notin G_n$ for $n \geq 2$, where G_n's denote the sets defined in Proposition 8. Since $I_{\varphi}(\lambda x_n \chi_{T \setminus D_n}) = \infty$ for every $\lambda > 1$ and $n \geq p$, $\|x_n\|_{\varphi} = 1$; so $x_n \in S(L^{\varphi}(\mu)) \setminus G_n$ for $n \geq p$. But, for large n, $\mathrm{Stab} \setminus G_n$ is dense in $S(L^{\varphi}(\mu)) \setminus G_n$ by Proposition 8 and the Baire category theorem, so we can find a sequence $(y_n), y_n \in \mathrm{Stab}$ such that $\|y_n - x\|_{\varphi} \to 0$ and the proof is completed. $\square$

Corollary 10. *If* $[0, \infty) \subseteq \varphi(\mathbb{R})$, *then* Stab *is a dense* G_{δ} *subset of* $S(L^{\varphi}(\mu))$.

References

[1] Clausing A., Papadopoulou S., *Stable convex sets and extreme operators*, Math. Ann. **231** (1978), 193–200.

[2] Engelking R., *General Topology*, Polish Scientific Publishers, Warsaw, 1977.

[3] Grząślewicz R., *Finite dimensional Orlicz spaces*, Bull. Acad. Polon. Sci.: Math. **33** (1985), 277–283.

[4] Lazar A.J., *Affine functions on simplexes and extreme operators*, Israel J. Math. **5** (1967), 31–43.

[5] Lima Å., *On continuous convex functions and split faces*, Proc. London Math. Soc. **25** (1972), 27–40.

[6] Luxemburg W.A.J., *Banach function spaces*, Thesis, Delft, 1955.

[7] Michael E., *Continuous selections I*, Ann. of Math. (2) **63** (1956), 361–382.

[8] Musielak J., *Orlicz spaces and modular spaces*, Lecture Notes in Math. **1034**, Springer Verlag, 1983.

[9] O'Brien R.C., *On the openness of the barycentre map*, Math. Ann. **223** (1976), 207–212.

[10] Orlicz W., *Über eine gewisse Klasse von Räumen vom Typus B*, Bull. Intern. Acad. Pol., série A, Kraków, 1932, 207–220.

[11] Papadopoulou S., *On the geometry of stable compact convex sets*, Math. Ann. **229** (1977), 193–200.
[12] Suarez–Granero A., *Stable unit balls in Orlicz spaces*, Proc. Amer. Math. Soc. **109**, 1 (1990), 97–104.
[13] Vesterstrøm J., *On open maps, compact convex sets and operator algebras*, J. London Math. Soc. **6** (1973), 289–297.
[14] Wisła M., *Extreme points and stable unit balls in Orlicz sequence spaces*, Archiv der Math. **56** (1991), 482–490.
[15] Wisła M., *Stable unit balls in finite dimensional generalized Orlicz spaces*, Proceedings of the Second Conference "Function Spaces", Poznań, 1989, Teubner Texte zur Mathematik, to appear.
[16] Wisła M., *Continuity of the identity embedding of Musielak–Orlizc sequence spaces*, Proc. of the 14th Winter School on Abstract Analysis, Srní, 1986, Supp. ai Rendiconti del Circolo Mat. di Palermo **14** (1987), 427–437.

Institute of Mathematics, A. Mickiewicz University, Matejki 48/49, 60–769 Poznań, Poland

(*Received* April 11, 1991)