Operational quantities derived from the norm and generalized Fredholm theory

Manuel Gonzalez, Antonio Martinon

Abstract. We introduce and study some operational quantities associated to a space ideal $\mathbb{A}$. These quantities are used to define generalized semi-Fredholm operators associated to $\mathbb{A}$, and the corresponding perturbation classes which extend the strictly singular and strictly cosingular operators, and we study the generalized Fredholm theory obtained in this way. Finally we present some examples and show that the classes of generalized semi-Fredholm operators are non-trivial for several classical space ideals.

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0. Introduction.

Several operational quantities have been used to characterize the classes of operators of the classical Fredholm theory: upper semi-Fredholm operators, lower semi-Fredholm operators, compact operators, strictly singular operators and strictly cosingular operators. See, for example, [17], [18], [20], [21]. These quantities are defined in terms of the class of finite dimensional spaces.

In this paper we develop a Fredholm theory associated to a space ideal by means of some suitable operational quantities.

In Section 1 we present a survey of the main results of classical Fredholm theory. Then, in Section 2, we introduce and study some operational quantities associated to a space ideal $\mathbb{A}$. These quantities are used in Section 3 to define generalized semi-Fredholm operators associated to $\mathbb{A}$, and the corresponding perturbation classes which extend the strictly singular and strictly cosingular operators, and we study the generalized Fredholm theory obtained in this way. Finally, in Section 4, we present some examples and show that the classes of generalized semi-Fredholm operators are non-trivial for several classical space ideals.

Notations and preliminaries.

X, Y, Z are Banach spaces; X^* the dual space of X; B_X the closed unit ball of X; I_X the identity operator of X; $L(X, Y)$ the space of the operators (linear and

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continuous) between X and Y; $n(T) := \|T\|$ the norm, $N(T)$ the kernel, $R(T)$ the range and T^* the dual of the operator $T \in L(X,Y)$. M, N, U, V are (closed) subspaces; $J_M : M \to X$ the inclusion of M in X; $Q^U : X \to X/U$ the quotient map.

The classes of operators which are used in classical Fredholm theory are defined below.

Definition 1. Let X, Y be Banach spaces and $T \in L(X,Y)$.

(1) T is an injection $\Longleftrightarrow N(T) = \{0\}$ and $R(T)$ is closed.
(2) T is surjection $\Longleftrightarrow R(T) = Y$.
(3) T is upper semi-Fredholm ($T \in SF_+$) $\Longleftrightarrow \dim(N(T)) < \infty$ and $R(T)$ is closed.
(4) T is lower semi-Fredholm ($T \in SF_-$) $\Longleftrightarrow R(T)$ is closed and $\dim(Y/R(T)) < \infty$.
(5) T is strictly singular ($T \in SS$) $\Longleftrightarrow TJ_M$ injection implies $\dim(M) < \infty$.
(6) T is strictly cosingular ($T \in SC$) $\Longleftrightarrow Q^U T$ surjection implies $\dim(Y/U) < \infty$.

1. The classical case.

Let $\mathbb{F}$ denote the class of all finite dimensional Banach spaces. For X an infinite dimensional Banach space, let $S_{\mathbb{F}}(X)$ be the set of subspaces $M \subset X$ such that $M \notin \mathbb{F}$.

If $T \in L(X,Y)$, from the norm, $n(T) := \|T\|$, we can derive the following operational quantities:

$$in_{\mathbb{F}}(T) := \inf\{n(TJ_M) : M \in S_{\mathbb{F}}(X)\},$$
$$sin_{\mathbb{F}}(T) := \sup\{in_{\mathbb{F}}(TJ_M) : M \in S_{\mathbb{F}}(X)\}.$$

The quantity $in_{\mathbb{F}}$ was introduced by B. Gramsch (see [17]) and $sin_{\mathbb{F}}$ by M. Schechter [17].

Dually, for Y an infinite dimensional Banach space, let $S^{\mathbb{F}}(X)$ be the set of subspaces $U \subset X$ such that $X/U \notin \mathbb{F}$. If $T \in L(X,Y)$, we can derive the quantities:

$$in^{\mathbb{F}}(T) := \inf\{n(Q^U T) : U \in S^{\mathbb{F}}(Y)\},$$
$$sin^{\mathbb{F}}(T) := \sup\{in^{\mathbb{F}}(Q^U T) : U \in S^{\mathbb{F}}(Y)\}.$$

The operational quantities $in^{\mathbb{F}}$ and $sin^{\mathbb{F}}$ were introduced by L. Weis [20].

These quantities have been used to characterize the classes SF_+, SF_-, SS and SC, and to obtain some results of perturbation.

Theorem 2. *Let X, Y be Banach spaces and $T \in L(X,Y)$.*

(1) [17] *If* $\dim(X) = \infty$, *then*
(a) $T \in SF_+(X,Y) \Longleftrightarrow in_{\mathbb{F}}(T) > 0$,
(b) $T \in SS(X,Y) \Longleftrightarrow sin_{\mathbb{F}}(T) = 0$.
(2) [20] *If* $\dim(Y) = \infty$, *then*
(a) $T \in SF_-(X,Y) \Longleftrightarrow in^{\mathbb{F}}(T) > 0$,
(b) $T \in SC(X,Y) \Longleftrightarrow sin^{\mathbb{F}}(T) = 0$.

Theorem 3. *Let X, Y be Banach spaces and $T \in L(X,Y)$.*

(1) [17] *If* $\dim(X) = \infty$, *then*
$T \in SF_+$ *and* $sin_{\mathbb{F}}(S) < in_{\mathbb{F}}(T) \Longrightarrow T + S \in SF_+$.
(2) [20] *If* $\dim(Y) = \infty$, *then*
$T \in SF_-$ *and* $sin^{\mathbb{F}}(S) < in^{\mathbb{F}}(T) \Longrightarrow T + S \in SF_-$.

2. $\mathbb{A}$-operational quantities.

This is a rather technical section which will be basic for the remainder of the paper. We shall introduce and study some operational quantities associated to a space ideal $\mathbb{A}$.

Recall that a space ideal $\mathbb{A}$ in the sense of [15] is a class of Banach spaces bigger than $\mathbb{F}$, which is stable under isomorphisms, finite products and complemented subspaces. $\mathbb{A}$ is injective if every subspace of $X \in \mathbb{A}$ belongs to $\mathbb{A}$, and $\mathbb{A}$ is surjective if every quotient of $X \in \mathbb{A}$ belongs to $\mathbb{A}$.

$S_{\mathbb{A}}(X)$ will be the set of subspaces $M \subset X$ such that $M \notin \mathbb{A}$; and $S^{\mathbb{A}}(X)$ the set of subspaces $U \subset X$ such that $X/U \notin \mathbb{A}$. We shall denote also

$$\mathbb{A}_d = \{X : S_{\mathbb{A}}(X) = \emptyset\} \quad \text{and} \quad \mathbb{A}^d = \{X : S^{\mathbb{A}}(X) = \emptyset\}.$$

Note that in the case $\mathbb{A}$ injective we have $\mathbb{A} = \mathbb{A}_d$, and in the case $\mathbb{A}$ surjective we have $\mathbb{A} = \mathbb{A}^d$.

We shall need the notions of totally incomparable Banach spaces and totally coincomparable Banach spaces.

Definition 4. Let X, Y be Banach spaces.

(1) [16] X and Y are totally incomparable if Banach spaces isomorphic to a subspace of X and to a subspace of Y have finite dimension.
(2) [9] X and Y are totally coincomparable if Banach spaces isomorphic to a quotient of X and to a quotient of Y have finite dimension.

Given a space ideal $\mathbb{A}$, we shall denote by $\mathbb{A}_i(\mathbb{A}^c)$ the class of Banach spaces which are totally incomparable (coincomparable) with every space in $\mathbb{A}$. It is proved in [3] that $\mathbb{A}_i$ is an injective space ideal, $\mathbb{A}^c$ is a surjective space ideal, and they verify $\mathbb{A}_i = \mathbb{A}_{iii}$ and $\mathbb{A}^c = \mathbb{A}^{ccc}$. Moreover, $X \in \mathbb{A}_{ii}(\mathbb{A}^{cc})$ if and only if every infinite dimensional subspace (quotient) of X has an infinite dimensional subspace (quotient) in $\mathbb{A}$. Note that $\mathbb{F} = \mathbb{F}_{ii} = \mathbb{F}^{cc}$.

The $\mathbb{A}$-operational quantities are defined as follows.

Definition 5. Let X, Y be Banach spaces and $T \in L(X,Y)$.

(1) If $X \notin \mathbb{A}_d$, then
$in_{\mathbb{A}}(T) := \inf\{n(TJ_M) : M \in S_{\mathbb{A}}(X)\}$,
$sin_{\mathbb{A}}(T) := \sup\{in_{\mathbb{A}}(TJ_M) : M \in S_{\mathbb{A}}(X)\}$.

(2) If $Y \notin \mathbb{A}^d$, then
$in^{\mathbb{A}}(T) := \inf\{n(Q^U T) : U \in S^{\mathbb{A}}(Y)\}$,
$sin^{\mathbb{A}}(T) := \sup\{in^{\mathbb{A}}(Q^U T) : U \in S^{\mathbb{A}}(Y)\}$.

Next we prove some inequalities between the $\mathbb{A}$-operational quantities.

Proposition 6. *Let X, Y be Banach spaces and $T \in L(X,Y)$.*

(1) *If $X \notin \mathbb{A}_d$, then*
(a) $in_{\mathbb{A}}(T) \leq sin_{\mathbb{A}}(T) \leq n(T)$,
(b) $\mathbb{A} = \mathbb{A}_{ii} \Longrightarrow in_{\mathbb{F}}(T) \leq in_{\mathbb{A}}(T) \leq sin_{\mathbb{A}}(T) \leq sin_{\mathbb{F}}(T)$.
(2) *If $Y \notin \mathbb{A}^d$, then*
(a) $in^{\mathbb{A}}(T) \leq sin^{\mathbb{A}}(T) \leq n(T)$,
(b) $\mathbb{A} = \mathbb{A}^{cc} \Longrightarrow in^{\mathbb{F}}(T) \leq in^{\mathbb{A}}(T) \leq sin^{\mathbb{A}}(T) \leq sin^{\mathbb{F}}(T)$.

Proof: (1a) Obviously $in_{\mathbb{A}}(T) \leq n(T)$. Therefore, for every $M \subset X, M \notin \mathbb{A}$,

$$in_{\mathbb{A}}(T) \leq in_{\mathbb{A}}(TJ_M) \leq n(TJ_M) \leq n(T),$$

and consequently $in_{\mathbb{A}}(T) \leq sin_{\mathbb{A}}(T) \leq n(T)$.

(1b) It is obvious that from $\mathbb{F} \subset \mathbb{A}$ we obtain $in_{\mathbb{F}} \leq in_{\mathbb{A}}$. In (1a) we show that $in_{\mathbb{A}} \leq sin_{\mathbb{A}}$. Moreover, if $M \subset X, M \notin \mathbb{A} = \mathbb{A}_{ii}$, then there exists $N \subset M$ such that $S_{\mathbb{A}}(N) = S_{\mathbb{F}}(N)$; consequently $in_{\mathbb{A}}(TJ_M) \leq in_{\mathbb{A}}(TJ_N) = in_{\mathbb{F}}(TJ_N) \leq sin_{\mathbb{F}}(TJ_M) \leq sin_{\mathbb{F}}(T)$, and taking the supremum over M we have that $sin_{\mathbb{A}}(T) \leq sin_{\mathbb{F}}(T)$.

(2) Analogously to (1). □

Proposition 7. *Let X, Y be Banach spaces and $S, T \in L(X,Y)$.*

(1) *If $X \notin \mathbb{A}_d$, then*
(a) $in_{\mathbb{A}}(T+S) \leq sin_{\mathbb{A}}(T) + in_{\mathbb{A}}(S)$,
(b) $sin_{\mathbb{A}}(S+T) \leq sin_{\mathbb{A}}(S) + sin_{\mathbb{A}}(T)$.
(2) *If $Y \notin \mathbb{A}^d$, then*
(a) $in^{\mathbb{A}}(T+S) \leq sin^{\mathbb{A}}(T) + in^{\mathbb{A}}(S)$,
(b) $sin^{\mathbb{A}}(S+T) \leq sin^{\mathbb{A}}(S) + sin^{\mathbb{A}}(T)$.

Proof: (1) Analogously to (2).

(2a) Because $n(T+S) \leq n(T) + n(S)$, we obtain

$$\begin{aligned} in^{\mathbb{A}}(T+S) = &\inf\{n(Q^U(T+S)) : U \in S^{\mathbb{A}}(Y)\} \leq \\ \leq &\inf\{n(Q^U T) + n(Q^U S) : U \in S^{\mathbb{A}}(Y)\} \leq \\ \leq &\inf\{n(Q^U T) : U \in S^{\mathbb{A}}(Y)\} + \sup\{n(Q^U S) : U \in S^{\mathbb{A}}(Y)\} \leq \\ \leq &\, in^{\mathbb{A}}(T) + n(S). \end{aligned}$$

The last inequality implies, being $U \subset Y$, $Y/U \notin \mathbb{A}$,

$$in^{\mathbb{A}}(T+S) \leq in^{\mathbb{A}}(Q^U(T+S)) \leq in^{\mathbb{A}}(Q^U T) + n(Q^U S) \leq sin^{\mathbb{A}}(T) + n(Q^U S),$$

hence $in^{\mathbb{A}}(T+S) \leq sin^{\mathbb{A}}(T) + in^{\mathbb{A}}(S)$.

(2b) For $U \subset Y$, $Y/U \notin \mathbb{A}$, from (2a) we obtain

$$in^{\mathbb{A}}(Q^U(T+S)) \leq sin^{\mathbb{A}}(Q^U T) + in^{\mathbb{A}}(Q^U S) \leq sin^{\mathbb{A}}(T) + in^{\mathbb{A}}(Q^U S).$$

Taking the supremum over U, we conclude $sin^{\mathbb{A}}(T+S) \leq sin^{\mathbb{A}}(T) + sin^{\mathbb{A}}(S)$. □

3. Generalized Fredholm theory.

In this section we introduce the classes $S\mathbb{A}_+$ and $S\mathbb{A}_-$ of generalized semi-Fredholm operators, and the corresponding perturbation classes $\mathbb{A}SS$ and $\mathbb{A}SC$ which extend the strictly singular and strictly cosingular operators, and we study their properties.

Definition 8.

(1) $S\mathbb{A}_+(X,Y) := L(X,Y)$, if $X \in \mathbb{A}_d$,
$:= \{T \in L(X,Y) : in_{\mathbb{A}}(T) > 0\}$, if $X \notin \mathbb{A}_d$.

(2) $S\mathbb{A}_-(X,Y) := L(X,Y)$, if $Y \in \mathbb{A}^d$,
$:= \{T \in L(X,Y) : in^{\mathbb{A}}(T) > 0\}$, if $Y \notin \mathbb{A}^d$.

Note that $SF_+ = S\mathbb{F}_+$ and $SF_- = S\mathbb{F}_-$. However, the classes $S\mathbb{A}_+$ and $S\mathbb{A}_-$ are empty in some cases.

Proposition 9. *Let X, Y be Banach spaces.*

(1) *If $\mathbb{A} = \mathbb{A}_{ii}, X \notin \mathbb{A}_d$ and $Y \in \mathbb{A}_d$, then $S\mathbb{A}_+(X,Y) = \emptyset$.*
(2) *If $\mathbb{A} = \mathbb{A}^{cc}, X \in \mathbb{A}^d$ and $Y \notin \mathbb{A}^d$, then $S\mathbb{A}_-(X,Y) = \emptyset$.*

Proof: (1) Note that $\mathbb{A}_d = \mathbb{A} = \mathbb{A}_{ii}$. Hence there exists $M \in S_{\mathbb{F}}(X)$ such that $S_{\mathbb{A}}(M) = S_{\mathbb{F}}(M)$.

If $in_{\mathbb{A}}(T) > 0$, then $in_{\mathbb{A}}(TJ_M) = in_{\mathbb{F}}(TJ_M) > 0$; hence TJ_M is an upper semi-Fredholm operator and then there exists $N \in S_{\mathbb{A}}(M)$ such that TJ_N is an injection. Consequently $TN \in S_{\mathbb{A}}(Y)$, hence $S_{\mathbb{A}}(Y) \neq \emptyset$ and $Y \notin \mathbb{A}_d$.

(2) Note that $\mathbb{A}^d = \mathbb{A} = \mathbb{A}^{cc}$. Hence there exists $U \in S^{\mathbb{F}}(Y)$ such that $S^{\mathbb{A}}(Y/U) = S^{\mathbb{F}}(Y/U)$.

If $in^{\mathbb{A}}(T) > 0$, then $in^{\mathbb{A}}(Q^U T) = in^{\mathbb{F}}(Q^U T) > 0$; hence $Q^U T$ is a lower semi-Fredholm operator and then there exists $V \in S^{\mathbb{A}}(Y/U)$ such that $Q^V Q^U T$ is a surjection. Consequently $N(Q^V Q^U T) \in S^{\mathbb{A}}(X)$, hence $S^{\mathbb{A}}(X) \neq \emptyset$ and $X \notin \mathbb{A}^d$.

□

The most important properties of the classes $S\mathbb{A}_+$ and $S\mathbb{A}_-$ are given in the following result.

Theorem 10. *Let X, Y be Banach spaces.*

(1) (a) $SF_+ \subset S\mathbb{A}_+$.

(b) $S\mathbb{A}_+(X,Y)$ *is open in* $L(X,Y)$. *In fact,*

$$T \in S\mathbb{A}_+(X,Y),\ S \in L(X,Y) \ \text{ and } \ \|S\| < in_{\mathbb{A}}(T) \Longrightarrow T + S \in S\mathbb{A}_+.$$

(c) $T \in S\mathbb{A}_+(X,Y) \Longrightarrow N(T) \in \mathbb{A}$.

(2) (a) $SF_- \subset S\mathbb{A}_-$.

(b) $S\mathbb{A}_-(X,Y)$ *is open in* $L(X,Y)$. *In fact,*

$$T \in S\mathbb{A}_-(X,Y),\ S \in L(X,Y) \ \text{ and } \ \|S\| < in^{\mathbb{A}}(T) \Longrightarrow T + S \in S\mathbb{A}_-.$$

(c) $T \in S\mathbb{A}_-(X,Y) \Longrightarrow Y/\overline{R(T)} \in \mathbb{A}$.

Proof: (1a) If $X \in \mathbb{A}_d$, then it is obvious. If $X \notin \mathbb{A}_d$ and $T \in SF_+(X,Y)$, then there exists $M \subset X, \dim(X/M) < \infty$, such that TJ_M is an injection. For every $N \subset X, N \notin \mathbb{A}$, we obtain that $TJ_{N\cap M}$ is an injection, being $N \cap M \notin \mathbb{A}$. If we put

$$\alpha := \inf\{\|Tx\| : x \in M,\ \|x\| = 1\},$$

then we obtain

$$0 < \alpha \le n(TJ_{N\cap M}) \le n(TJ_N),$$

and consequently $in_{\mathbb{A}}(T) \ge \alpha > 0$, hence $T \in S\mathbb{A}_+$.

(1b) If $X \in \mathbb{A}_d$, then it is obvious. Assume $X \notin \mathbb{A}_d$ and $T \in S\mathbb{A}_+(X,Y)$. From Proposition 7 (1) (a) we obtain, for every $S \in L(X,Y)$,

$$in_{\mathbb{A}}(T) \le in_{\mathbb{A}}(S+T) + sin_{\mathbb{A}}(S) \le in_{\mathbb{A}}(S+T) + n(S),$$

and taking $0 < n(S) < in_{\mathbb{A}}(T)$ we have that $in_{\mathbb{A}}(T) > 0$ and $in_{\mathbb{A}}(S+T) > 0$, hence $T \in S\mathbb{A}_+$ and $S + T \in S\mathbb{A}_+$.

(1c) If $X \in \mathbb{A}_d$, then $N(T) \in \mathbb{A}$. Let $X \notin \mathbb{A}_d$ and $N(T) \notin \mathbb{A}$. Then $TJ_{N(T)} = 0$ and $in_{\mathbb{A}}(T) = 0$; hence $T \notin S\mathbb{A}_+$.

(2a) If $Y \in \mathbb{A}^d$, then it is obvious. If $Y \notin \mathbb{A}^d$ and $T \in SF_-(X,Y)$, then there exists $U \subset Y, \dim(U) < \infty$ such that $Q^U T$ is a surjection. For every $V \subset Y$, $Y/V \notin \mathbb{A}$, we obtain that $Q^{U+V}T$ is a surjection, being $Y/(U+V) \notin \mathbb{A}$. If we put

$$\alpha := \sup\{\varepsilon > 0 : \varepsilon B_{Y/U} \subset Q^U T B_X\},$$

then we obtain

$$0 < \alpha \le n(Q^{U+V}T) \le n(Q^V T),$$

and consequently $in^{\mathbb{A}}(T) \ge \alpha > 0$, hence $T \in S\mathbb{A}_-$.

(2b) If $Y \in \mathbb{A}^d$, then it is obvious. Assume $Y \notin \mathbb{A}^d$ and $T \in S\mathbb{A}_-(X,Y)$. From Proposition 7 (2) (a) we obtain, for every $S \in L(X,Y)$,

$$in^{\mathbb{A}}(T) \le in^{\mathbb{A}}(S+T) + sin^{\mathbb{A}}(S) \le in^{\mathbb{A}}(S+T) + n(S),$$

and taking $0 < n(S) < in^{\mathbb{A}}(T)$ we have that $in^{\mathbb{A}}(S+T) > 0$ and consequently $S + T \in S\mathbb{A}_-$.

(2c) If $Y \in \mathbb{A}^d$, then $Y/\overline{R(T)} \in \mathbb{A}$. Let $Y \notin \mathbb{A}^d$ and $Y/\overline{R(T)} \notin \mathbb{A}$. Then $Q^{\overline{R(T)}}T = 0$ and $in^{\mathbb{A}}(T) = 0$; hence $T \notin S\mathbb{A}_-$. □

We have the following algebraic properties.

Proposition 11. *Let X, Y, Z be Banach spaces and $T \in (X, Y), S \in L(Y, Z)$.*

(1) $ST \in S\mathbb{A}_+ \Longrightarrow T \in S\mathbb{A}_+$.
(2) $ST \in S\mathbb{A}_- \Longrightarrow S \in S\mathbb{A}_-$.

Proof: (1) If $X \in \mathbb{A}_d$, it is obvious. If $X \notin \mathbb{A}_d$, then we obtain

$$0 < in_{\mathbb{A}}(ST) \leq n(STJ_M) \leq n(S)\; n(TJ_M),$$

hence

$$0 < in_{\mathbb{A}}(ST) < n(S)\; in_{\mathbb{A}}(T)$$

and we obtain the result.

(2) Analogously to (1). □

Remark 12. We do not know if the following implications are true:

$$S, T \in S\mathbb{A}_+ \Longrightarrow ST \in S\mathbb{A}_+,$$
$$S, T \in S\mathbb{A}_- \Longrightarrow ST \in S\mathbb{A}_-;$$

that is, if the classes $S\mathbb{A}_+$ and $S\mathbb{A}_-$ are semigroups.

Next we introduce the perturbation classes $\mathbb{A}SS$ and $\mathbb{A}SC$ which extend the strictly singular operators and strictly cosingular operators, respectively.

Definition 13. Let X, Y be Banach spaces.

(1) $\mathbb{A}SS(X, Y) := L(X, Y)$, if $X \in \mathbb{A}_d$,
$\qquad := \{T \in L(X, Y) : sin_{\mathbb{A}}(T) = 0\}$, if $X \notin \mathbb{A}_d$.
(2) $\mathbb{A}SC(X, Y) := L(X, Y)$, if $Y \in \mathbb{A}^d$,
$\qquad := \{T \in L(X, Y) : sin^{\mathbb{A}}(T) = 0\}$, if $Y \notin \mathbb{A}^d$.

Note that $\mathbb{F}SS = SS$ and $\mathbb{F}SC = SC$. In the following theorem, we give the most important properties of the classes $\mathbb{A}SS$ and $\mathbb{A}SC$.

Theorem 14. *Let X, Y be Banach spaces.*

(1) (a) *$\mathbb{A}SS(X, Y)$ is closed in $L(X, Y)$.*
(b) *$I_X \in \mathbb{A}SS \Longleftrightarrow X \in \mathbb{A}_d$.*
(c) *If $\mathbb{A} = \mathbb{A}_{ii}$, then $\mathbb{A}SS(X, Y) = \{T \in L(X, Y) : TJ_M$ injection $\Longrightarrow M \in \mathbb{A}_d\}$ and $\mathbb{A}SS$ is an operator ideal that includes the strictly singular operators.*

(2) (a) *$\mathbb{A}SC(X, Y)$ is closed in $L(X, Y)$.*
(b) *$I_X \in \mathbb{A}SC \Longleftrightarrow X \in \mathbb{A}^d$.*
(c) *If $\mathbb{A} = \mathbb{A}^{cc}$, then $\mathbb{A}SC(X, Y) = \{T \in L(X, Y) : Q^U T$ surjection $\Longrightarrow Y/U \in \mathbb{A}^d\}$ and $\mathbb{A}SC$ is an operator ideal that includes the strictly cosingular operators.*

Proof: (1a) We show that $L(X,Y) \setminus \mathbb{A}SS(X,Y)$ is open in $L(X,Y)$. If $X \in \mathbb{A}_d$, then it is obvious. Let $X \notin \mathbb{A}_d$ and $T \notin \mathbb{A}SS(X,Y)$; from Proposition 7 (1) (b) we have that

$$0 < sin_{\mathbb{A}}(T) \leq sin_{\mathbb{A}}(S+T) + sin_{\mathbb{A}}(S) \leq sin_{\mathbb{A}}(S+T) + n(S),$$

and taking $0 < n(S) < sin_{\mathbb{A}}(T)$ we obtain $sin_{\mathbb{A}}(S+T) > 0$. Hence $S+T \notin \mathbb{A}SS$.

(1b) If $X \notin \mathbb{A}_d$, then $sin_{\mathbb{A}}(I_X) = 1$, hence $I_X \notin \mathbb{A}SS$. Moreover, if $X \in \mathbb{A}_d$, then $\mathbb{A}SS(X,X) = L(X,X)$, hence $I_X \in \mathbb{A}SS$.

(1c) If $X \in \mathbb{A}_d$, then it is obvious. Suppose $X \notin \mathbb{A}_d$. If there exists a subspace $M \subset X, M \notin \mathbb{A}$, such that TJ_M is an injection, then TJ_M is an upper semi-Fredholm operator, hence

$$0 < in_{\mathbb{F}}(TJ_M) \leq in_{\mathbb{A}}(TJ_M),$$

and consequently $sin_{\mathbb{A}}(T) > 0$. Hence $T \notin \mathbb{A}SS$.

Conversely, if $sin_{\mathbb{A}}(T) > 0$, then there exists a subspace $M \subset X, M \notin \mathbb{A}$, such that $in_{\mathbb{A}}(TJ_M) > 0$. From $\mathbb{A} = \mathbb{A}_{ii}$ we have that there exists $N \subset M$ such that $S_{\mathbb{A}}(N) = S_{\mathbb{F}}(N)$ and, consequently,

$$0 < in_{\mathbb{A}}(TJ_M) \leq in_{\mathbb{A}}(TJ_N) = in_{\mathbb{F}}(TJ_N);$$

hence TJ_N is an upper semi-Fredholm operator and TJ_P is an injection, for some $P \in S_{\mathbb{A}}(N)$. From $sin_{\mathbb{A}} \leq sin_{\mathbb{F}}$ (Proposition 6 (1) (b)), we obtain $SS \subset \mathbb{A}SS$.

(2a) We show that $L(X,Y) \setminus \mathbb{A}SC(X,Y)$ is open in $L(X,Y)$. If $Y \in \mathbb{A}^d$, then it is obvious. Let $Y \notin \mathbb{A}^d$ and $T \notin \mathbb{A}SC(X,Y)$; from Proposition 7 (2) (b) we have that

$$0 < sin^{\mathbb{A}}(T) \leq sin^{\mathbb{A}}(S+T) + sin^{\mathbb{A}}(S) \leq sin^{\mathbb{A}}(S+T) + n(S),$$

and taking $0 < n(S) < sin^{\mathbb{A}}(T)$ we obtain $sin^{\mathbb{A}}(S+T) > 0$. Hence $S+T \notin \mathbb{A}SC$.

(2b) If $X \notin \mathbb{A}^d$, then $sin^{\mathbb{A}}(I_X) = 1$, hence $I_X \notin \mathbb{A}SC$. Moreover, if $X \in \mathbb{A}^d$, then $\mathbb{A}SC(X,X) = L(X,X)$, hence $I_X \in \mathbb{A}SC$.

(2c) If $Y \in \mathbb{A}^d$, then it is obvious. Suppose $Y \notin \mathbb{A}^d$. If there exists a subspace $U \subset Y, Y/U \notin \mathbb{A}$, such that $Q^U T$ is a surjection, then $Q^U T$ is a lower semi-Fredholm operator, hence

$$0 < in^{\mathbb{F}}(Q^U T) \leq in^{\mathbb{A}}(Q^U T),$$

and consequently $sin^{\mathbb{A}}(T) > 0$. Hence $T \notin \mathbb{A}SC$.

Conversely, if $sin^{\mathbb{A}}(T) > 0$, then there exists a subspace $U \subset Y$, $Y/U \notin \mathbb{A}$, such that $in^{\mathbb{A}}(Q^U T) > 0$. From $\mathbb{A} = \mathbb{A}^{cc}$ we have that there exists $V \supset U$ such that $S^{\mathbb{A}}(Y/V) = S^{\mathbb{F}}(Y/V)$ and, consequently,

$$0 < in^{\mathbb{A}}(Q^U T) \leq in^{\mathbb{A}}(Q^V T) = in^{\mathbb{F}}(Q^V T);$$

hence $Q^V T$ is a lower semi-Fredholm operator and $Q^W T$ is a surjection, for some $W \supset V$, $Y/W \notin \mathbb{A}$. From $sin^{\mathbb{A}} \leq sin^{\mathbb{F}}$ (Proposition 6 (2) (b)), we obtain $SC \subset \mathbb{A}SC$. $\square$

Remark 15. In the cases $\mathbb{A} = \mathbb{A}_{ii}$ and $\mathbb{A} = \mathbb{A}^{cc}$, the classes $\mathbb{A}SS$ and $\mathbb{A}SC$ coincide with classes considered by several authors. Taking the equalities of Theorem 14 (1) (c), (2) (c) as definitions, the classes $\mathbb{A}SS$ and $\mathbb{A}SC$ have been studied in [4], [19]; it is showed in [4], [14] that if $\mathbb{A} = \mathbb{A}_{ii}$, then $\mathbb{A}SS$ is an operator ideal, and if $\mathbb{A} = \mathbb{A}^{cc}$, then $\mathbb{A}SC$ is an operator ideal.

Remark 16. It is not known in general if the classes $\mathbb{A}SS$ and $\mathbb{A}SC$ are operator ideals. Moreover, we do not know if for every space ideal $\mathbb{A}$ it is verified $SS \subset \mathbb{A}SS$ and $SC \subset \mathbb{A}SC$. But it is true that the class of the compact operators C_0 verifies $C_0 \subset \mathbb{A}SS$ and $C_0 \subset \mathbb{A}SC$, for every space ideal $\mathbb{A}$. In fact, assume $X \notin \mathbb{A}_d$ and $T \in C_0(X,Y)$; it results for every $M \in S_{\mathbb{A}}(X)$ that $TJ_M \in C_0(M,Y)$ and consequently for every $\varepsilon > 0$ there exists a finite codimensional subspace N of M (hence $N \in S_{\mathbb{A}}(X)$) such that $n(TJ_N) < \varepsilon$ [6, III. 2.3]; hence $in_{\mathbb{A}}(TJ_M) = 0$ for every $M \in S_{\mathbb{A}}(X)$, and $sin_{\mathbb{A}}(T) = 0$. Analogously, we can prove that $C_0 \subset \mathbb{A}SC$ based on the following fact: if $\dim(Y) = \infty$, $T \in C_0(X,Y)$ and $\varepsilon > 0$, then there exists a finite dimensional subspace U of Y such that $n(Q^U T) < \varepsilon$ [5, 21].

Finally, we give two properties about the perturbation of the classes $S\mathbb{A}_+$ and $S\mathbb{A}_-$ by the classes $\mathbb{A}SS$ and $\mathbb{A}SC$.

Theorem 17.

(1) $S\mathbb{A}_+$ *is invariant by* $\mathbb{A}SS : T \in S\mathbb{A}_+$ *and* $S \in \mathbb{A}SS \Longrightarrow T + S \in S\mathbb{A}_+$.
(2) $S\mathbb{A}_-$ *is invariant by* $\mathbb{A}SC : T \in S\mathbb{A}_-$ *and* $S \in \mathbb{A}SC \Longrightarrow T + S \in S\mathbb{A}_-$.

Proof: From Proposition 6 (1) (a), (2) (a) we obtain the following inequalities:

$$in_{\mathbb{A}}(T) \leq in_{\mathbb{A}}(S+T) + sin_{\mathbb{A}}(S), \quad in^{\mathbb{A}}(T) \leq in^{\mathbb{A}}(S+T) + sin^{\mathbb{A}}(S).$$

It is immediate to prove the statements. □

4. Examples and remarks.

Let M, N be subspaces of X. The gap between M and N (see [13], for example) is defined by

$$\delta(M,N) := \sup\{\operatorname{dist}(m,N) : m \in M,\ \|m\| = 1\}.$$

We shall need also the following concept of stability.

Definition 18 [1]**.** Let $\mathbb{A}$ be a space ideal.

$$\mathbb{A} \text{ is stable } \Leftrightarrow \Big(\text{ there exists } \alpha > 0 \text{ such that } N \in \mathbb{A} \text{ and } \delta(M,N) < \alpha \Rightarrow M \in \mathbb{A}\Big).$$

It is proved in [1] that the stability can be characterized in terms of quotients:

$$\mathbb{A} \text{ is stable } \Leftrightarrow \Big(\text{ there exists } \alpha > 0 \text{ such that } X/M \in \mathbb{A} \text{ and } \delta(M,N) < \alpha \Rightarrow X/N \in \mathbb{A}\Big).$$

Theorem 19. *Assume $\mathbb{A}$ is stable. Let X, Y be Banach spaces and $T \in L(X,Y)$ such that $R(T)$ is closed. Then*

(1) $N(T) \in \mathbb{A} \Longrightarrow T \in S\mathbb{A}_+$,
(2) $Y/R(T) \in \mathbb{A} \Longrightarrow T \in S\mathbb{A}_-$.

Proof: Because $R(T)$ is closed, T defines an isomorphism of $X/N(T)$ onto $R(T)$. Hence there exists $c > 0$ such that, for every $x \in X$,

$$c \operatorname{dist}(x, N(T)) \leq \|Tx\|.$$

(1) If $M \notin \mathbb{A}$, then $\delta(M, N(T)) \geq \alpha$ for some $\alpha > 0$, and there exists $m \in M, \|m\| = 1$, such that $\operatorname{dist}(m, N(T)) \geq \alpha$; hence $\|Tm\| \geq c\alpha$. From this we obtain $n(TJ_M) \geq c\alpha$, hence $in_{\mathbb{A}}(T) \geq c\alpha > 0$.

(2) If $Y/U \notin \mathbb{A}$, then $\delta(R(T), U) \geq \alpha$ for some $\alpha > 0$, and there exists $y \in R(T)$, $\|y\| = 1$, such that $\operatorname{dist}(y, U) \geq \alpha$. Let x such that $Tx = y$. We obtain

$$c \operatorname{dist}(x, N(T)) \leq \|y\| = 1.$$

We take x such that $\|x\|$ and $1/c$ are sufficiently near and $\|Q^U Tx\| \geq \alpha/2$. Hence $\|Q^U T\| \geq c\alpha$ and consequently $in^{\mathbb{A}}(T) \geq c\alpha > 0$. □

As examples of stable space ideals $\mathbb{A}$ for which the above theorem holds we have: reflexive Banach spaces, spaces containing no copies of l_1, separable spaces, superreflexive spaces, B-convex spaces, quasireflexive spaces (see [1]). However, for the class of all Banach spaces such that every infinite dimensional subspace contains a copy of $l_2, \mathbb{A} = (l_2)_{ii}$, the above result is not true:

Proposition 20. *Suppose $\mathbb{A} = (l_2)_{ii}$. There exists a Banach space X containing a subspace U isometric to l_2 such that the associated quotient map $Q^U : X \to X/U$ does not belong to $S\mathbb{A}_+$.*

Proof: The following construction is based on [11]. Given $1 < p < \infty$, we define the (nonlinear map)

$$A_p : (x_i) \in l_2 \longrightarrow \Big(|x_i|^{2/p} \operatorname{sgn}(x_i) \Big) \in l_p.$$

A_p is an isometric bijection.

Taking $1/p + 1/q = 1$, it is not difficult to show that the space

$$\begin{aligned} &X_p := l_p \times l_2 \quad \text{with the norm} \\ &\|(y,x)\| := \sup\{(A_q z)(y) + z(x) : z \in l_2,\ \|z\| = 1\} \end{aligned}$$

is a Banach space, and the subspaces $U_p := \{0\} \times l_2$ and $M_p := l_p \times \{0\}$ are isometric to l_2 and l_p, respectively. Moreover, if J_p is the inclusion of M_p into X_p and Q_p is the quotient map from X_p onto X_p/U_p, then

$$\|Q_p J_p\| \leq 2(2-p)/p.$$

We take a sequence $1 < p_1 < \cdots < p_n < \cdots < 2$ with $\lim p_n = 2$. Then we define

$$X := (X_{p_1} \oplus \cdots \oplus X_{p_n} \oplus \dots)_{l_2}$$

and we consider the subspaces

$$U := (U_{p_1} \oplus \cdots \oplus U_{p_n} \oplus \dots)_{l_2}$$

and

$$M_n := (\oplus\{0\} \oplus \cdots \oplus \{0\} \oplus M_{p_n} \oplus \{0\} \oplus \dots)_{l_2}.$$

Then, U and M_n are isometric to l_2 and l_{p_n}, respectively. Moreover,

$$\|Q_U J_{M_n}\| \leq 2(2-p_n)/p_n. \text{ Consequently } \lim \|Q_U J_{M_n}\| = 0.$$

Now, since $M_n \notin l_2^{ii} = \mathbb{A}$, we conclude $in_{\mathbb{A}}(Q_U) = 0$; i.e. $Q_U \notin S\mathbb{A}_+$. ☐

Finally, we show the relation between the tauberian operators and the class $S\mathbb{R}_+$, where $\mathbb{R}$ is the class of all reflexive spaces.

Recall that $T \in L(X,Y)$ is tauberian [12] when $T^{**^{-1}} J_Y(Y) = J_X(X)$, where J_X is the canonical inclusion of X in X^{**}.

Proposition 21. *The class $S\mathbb{R}_+$ is properly contained in the class of tauberian operators.*

Proof: Let $T \in S\mathbb{R}_+(X,Y)$. For every compact operator $T \in L(X,Y)$ we have that $T + K$ belongs to $S\mathbb{R}_+$ (Remark 15 and Proposition 16), hence $N(T+K)$ is reflexive; consequently T is tauberian [10]. Moreover, $S\mathbb{R}_+$ is open, but the class of tauberian operators is not [2]. ☐

Remark 22. If we consider for $T \in L(X,Y)$ the quantities [15]

$$j(T) := \inf\{\|Tx\| : \|x\| = 1\},$$

the injection modulus of T, and

$$q(T) := \sup\{\varepsilon > 0 : \varepsilon B_Y \subset TB_X\},$$

the surjection modulus of T, we have that $j(T) > 0$ if and only if T is an injection and $q(T) > 0$ if and only if T is a surjection [15].

From j and q several authors have derived some operational quantities: $sj_{\mathbb{F}}$ [17], $sq^{\mathbb{F}}$ [21], $isj_{\mathbb{F}}$ and $isq^{\mathbb{F}}$ [14]. These quantities characterize the classes of Fredholm theory in a similar manner as the quantities considered in Section 2.

Analogously, we could define the operational quantities $isj_{\mathbb{A}}, isq^{\mathbb{A}}, sj_{\mathbb{A}}$ and $sq^{\mathbb{A}}$. The associated classes of operators have been studied in [8], [14].

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Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cantabria, 39071 Santander, Spain

Departamento de Análisis Matemático, Facultad de Matemáticas, Universidad de La Laguna, 38271 La Laguna, Tenerife, Spain

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