Convergence of approximating fixed points sets for multivalued nonexpansive mappings

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Abstract. Let K be a closed convex subset of a Hilbert space H and $T : K \multimap K$ a nonexpansive multivalued map with a unique fixed point z such that $\{z\} = T(z)$. It is shown that we can construct a sequence of approximating fixed points sets converging in the sense of Mosco to z.

Keywords: multivalued nonexpansive map, fixed points set, Mosco convergence

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Let H be a Hilbert space, K a closed convex subset of H, T a multivalued nonexpansive map from K in the family of non empty compact subsets of K. It is our object in this paper to show that in a specific case it is possible to construct a sequence of approximant sets converging in the sense of Mosco to a fixed point of T.

Our investigation is prompted by the papers of Browder [1], Reich [2], Singh and Watson [3], in which analogous problems are treated for singlevalued mappings. In particular, in [1] it is shown that: if K is a closed convex bounded subset of a Hilbert space and $T : K \to K$ is a nonexpansive map, then, for any $x_0 \in K$, the sequence $\{x_\lambda\}_{0\le\lambda<1}$ of the fixed points of the contraction maps T_{λ,x_0} defined by $T_{\lambda,x_0}(x) = \lambda T(x) + (1-\lambda)x_0$ converges, as x approaches 1, strongly in K to the fixed point of T in K closest to x_0. The paper [3] extends this result to the case of not self-mappings (but $T(\partial K) \subseteq K$) and K not necessarily bounded (but $T(K)$ bounded).

The following example of multivalued self-map defined on a closed convex bounded subset of a finite-dimensional Hilbert space shows that the recalled results cannot be extended to genuine multivalued case.

Let $H = R^2$, $K = [0,1]\times[0,1]$ and T the nonexpansive map defined by:

$$T(a,b) = \text{triangle whose vertices are } (0,0), (a,0), (0,b),\ \forall\,(a,b) \in K.$$

Thus, for $(x_0, y_0) \in K$ the point $((1-\lambda)x_0, (1-\lambda)y_0)$ is a fixed point of the map $T_{\lambda,(x_0,y_0)}$ for all $\lambda \in [0,1)$ and we have $((1-\lambda)x_0, (1-\lambda)y_0) \to (0,0)$ as λ approaches 1. If $x_0 > y_0$ ($x_0 < y_0$), then the fixed point of T closest to (x_0, y_0) is $(x_0, 0)$ ($(0, y_0)$), but the net of the fixed points sets of $T_{\lambda,(x_0,y_0)}$ does not converge to $(x_0, 0)$ ($(0, y_0)$) even in the weaker convergence of sets, that is, the Kuratowski convergence.

In the setting of Hilbert spaces, our result is formulated for nonexpansive maps T that have a unique fixed point z and this point satisfies $\{z\} = T(z)$. The precise generality of the class of functions satisfying this condition is not known but it has been studied, for example, in [4], [5], [6]. More recently the interest in optimization theory for such type of maps has prompted a corresponding interest in fixed point theory, since in [7] it has been shown that the maximization of a multivalued map T with respect to a cone, which subsumes ordinary and Pareto optimization, is equivalent to a fixed point problem of determining y such that $\{y\} = T(y)$.

Now we introduce some necessary notations and definitions. Let K be a closed convex subset of a Hilbert space H. We denote by $\mathcal{CB}(H)$ the family of non empty closed bounded subsets of H and by $\mathcal{K}(K)$ the family of non empty compact subsets of K.

For $A \in \mathcal{CB}(H)$ we define

$$d(x, A) = \inf\{\|x - y\| : y \in A\}.$$

For any $A, B \in \mathcal{CB}(H)$ we note with $D(A, B)$ the *Hausdorff distance* induced by the norm of H, i.e.

$$D(A, B) = \max\{\sup_{a \in A} d(a, B), \ \sup_{b \in B} d(b, A)\}.$$

Remark. If $B = \{b\}$ and $A \in \mathcal{CB}(H)$, we have that for all $a \in A$

$$\|a - b\| \le D(A, B).$$

We denote by $\to$ and $\rightharpoonup$ the strong and weak convergence, respectively.

Let $\{A_n\}$ be a sequence of closed subsets of H. We define the *inner limit* ($\liminf A_n$) by

$$\liminf A_n = \{x \in H : \exists \ \text{ a sequence } \ \{x_n\}, \ x_n \in A_n \ \text{ such that } \ x_n \to x\}$$

and the *weak-outer limit* ($w - \limsup A_n$) by

$$\begin{aligned} w - \limsup A_n = \{&x \in H : \exists \ \text{ a subsequence } \ \{A_{n_k}\} \ \text{ of } \ \{A_n\} \ \text{ and} \\ &\text{a sequence } \ \{x_{n_k}\}, \ x_{n_k} \in A_{n_k} \ \text{ such that } \ x_{n_k} \rightharpoonup x\}. \end{aligned}$$

We will say that $\{A_n\}$ *converges* to A *in the sense of Mosco* ($A_n \xrightarrow{(M)} A$) if $\liminf A_n = w - \limsup A_n = A$.

A net $\{A_\lambda\}_{\lambda \in [0,1)}$ of closed subsets of H converges to A in the sense defined before if every sequence $\{A_{\lambda_n}\}$, $\lambda_n \to 1$ as $n \to \infty$, converges in such sense to A.

A multivalued map $T : K \to \mathcal{K}(K)$ is said to be *lipschitzian* if

$$D(T(x), T(y)) \le L\|x - y\|$$

for every $x, y \in K$, $L \leq 0$. T is said to be a *contraction* if $L < 1$ and *nonexpansive* if $L = 1$. A map $T : K \to \mathcal{K}(K)$ is said to be *demiclosed* if $x_n \rightharpoonup x$, $y_n \to y$ and $y_n \in T(x_n)$ imply $y \in T(x)$.

Let $T : K \to \mathcal{K}(K)$ be a nonexpansive map. For $x_0 \in K$ and $\lambda \in [0,1)$ we denote by T_{λ,x_0} the contraction map defined by

$$T_{\lambda,x_0}(x) = \lambda T(x) + (1-\lambda)x_0, \ \forall\, x \in K.$$

Finally, we denote by

$$F(T) = \{x \in K : x \in T(X)\}$$

and

$$F(T_{\lambda,x_0}) = \{x \in K; x \in T_{\lambda,x_0}(x)\}$$

the sets of fixed points of T and T_{λ,x_0}, respectively.

Theorem 1. *Let K be a closed convex subset of a Hilbert space H, $T : K \to \mathcal{K}(K)$ a nonexpansive map such that $F(T) = \{z\}$, and let this point z satisfy $T(z) = \{z\}$. Then, for every $x_0 \in K$,*

$$F(T_{\lambda,x_0}) \xrightarrow{(M)} F(T) \ \text{ as } \ \lambda \to 1.$$

Proof: We have to prove that $F(T_{\lambda_n,x_0}) \xrightarrow{(M)} \{z\}$ as $n \to \infty$ for every sequence $\lambda_n \to 1$, $0 \leq \lambda_n < 1$.

Since we have always $\liminf \subseteq w - \limsup$, it remains to prove that $w - \limsup F(T_{\lambda_n,x_0}) \subseteq \{z\}$ and $\{z\} \subseteq \liminf F(T_{\lambda_n,x_0})$.

<u>Step 1.</u> $w - \limsup F(T_{\lambda_n,x_0}) \subset \{z\}$.

Let $x \in w - \limsup F(T_{\lambda_n,x_0})$, then there exist a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ and a sequence $\{y_{\lambda_{n_j}}\}$, $y_{\lambda_{n_j}} \in F(T_{\lambda_{n_j},x_0})$ such that $y_{\lambda_{n_j}} \rightharpoonup x$.

Since $y_{\lambda_{n_j}} \in F(T_{\lambda_{n_j},x_0})$, there exists $w_{\lambda_{n_j}} \in T_{\lambda_{n_j},x_0}(y_{\lambda_{n_j}})$ such that

$$y_{\lambda_{n_j}} = \lambda_{n_j} w_{\lambda_{n_j}} + (1-\lambda_{n_j})x_0.$$

Thus

$$\|y_{\lambda_{n_j}} - w_{\lambda_{n_j}}\| = (1-\lambda_{n_j})\,\|w_{\lambda_{n_j}} 0 x_0\| \to 0 \ \text{ as } \ \lambda_{n_j} \to 1$$

because $\{w_{\lambda_{n_j}}\}$ is bounded. From the demiclosedness of $I - T$ [8] it follows that $0 \in (I-T)(x)$, hence $x = z$.

<u>Step 2.</u> $\{z\} \subseteq \liminf F(T_{\lambda_n,x_0})$.

Let $x_{\lambda_n} \in F(T_{\lambda_n,x_0})$. We prove that $x_{\lambda_n} \to z$. On the contrary, suppose that there exists $\varepsilon_0 > 0$ and a subsequence $\{\lambda_{n_j}\}$ of $\{\lambda_n\}$ such that

$$\|x_{\lambda_{n_j}} - z\| \geq \varepsilon_0. \tag{1}$$

From $x_{\lambda_{n_j}} = \lambda_{n_j} w_{\lambda_{n_j}} + (1-\lambda_{n_j})x_0$, $w_{\lambda_{n_j}} \in T(x_{\lambda_{n_j}})$ it follows that

$$\Big\|\frac{x_{\lambda_{n_j}} - (1-\lambda_{n_j})x_0}{\lambda_{n_j}} - z\Big\| = \|w_{\lambda_{n_j}} - z\|.$$

Furthermore, from the previous Remark, we have

$$\|w_{\lambda_{n_j}} - z\| \leq D(T(x_{\lambda_{n_j}}), T(z))$$

and the nonexpansivity of T yields

$$\|w_{\lambda_{n_j}} - z\| \leq \|x_{\lambda_{n_j}} - z\|.$$

Hence

$$\Big\|\frac{x_{\lambda_{n_j}} - x_0}{\lambda_{n_j}} - (z - x_0)\Big\|^2 \leq \big\|(x_{\lambda_{n_j}} - x_0) + (x_0 - x)\big\|^2,$$

which implies

$$\begin{aligned} \|x_{\lambda_{n_j}} - x_0\|^2 &\leq 2\frac{\lambda_{n_j}}{1+\lambda_{n_j}}\langle x_{\lambda_{n_j}} - x_0, z - x_0\rangle \\ &\leq \langle x_{\lambda_{n_j}} - x_0, z - x_0\rangle \\ &\leq \|x_{\lambda_{n_j}} - x_0\|\ \|z - x_0\|. \end{aligned} \tag{2}$$

If it were $x_{\lambda_{n_j}} = x_0$ for a certain j, we should have

$$\begin{aligned} x_0 &= \lambda_{n_j}, w_{\lambda_{n_j}} + x_0 - \lambda_{n_j} x_0 \\ &= w_{\lambda_{n_j}} \in T(x_{\lambda_{n_j}}) \\ &= T(x_0). \end{aligned}$$

Then $x_0 = z$, contradicting (1).

Thus, from (2) it follows

$$\|x_{\lambda_{n_j}} - x_0\| \leq \|z - x_0\|,$$

which implies that the subsequence $\{x_{\lambda_{n_j}}\}$ is bounded. Hence there exists a subsequence $\{x_{\lambda_{n_{j_k}}}\}$ of $\{x_{\lambda_{n_j}}\}$ such that $x_{\lambda_{n_{j_k}}} \rightharpoonup x$. Proceeding as in the proof of Step 1, we obtain $x = z$. At this point, the well known relation

$$\|z - x_0\| \leq \liminf \|x_{\lambda_{n_{j_k}}} - x_0\|$$

and (see (2))

$$\limsup \|x_{\lambda_{n_{j_k}}} - x_0\| \leq \|z - x_0\|$$

imply

$$\|x_{\lambda_{n_{j_k}}} - x_0\| \to \|z - x_0\|.$$

Hence, we have $x_{\lambda_{n_{j_k}}} \to z$, contradicting (1). $\square$

Remark. In the following example, our theorem works.

Let $H = R$, $K = [0, \infty)$, $T : K \to \mathcal{K}(K)$ be the nonexpansive map defined by $T(x) = [0, \frac{x}{2}]$. Thus $F(T) = \{0\}$, $\{0\} = T(0)$ and the net of fixed point sets of $T_{\lambda,x_0}, F(T_{\lambda,x_0}) = [(1-\lambda)x_0, 2(1-\lambda)x_0]$, converges to $F(T)$ in the sense of Mosco.

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