Equivalence of certain free topological groups

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Abstract. In this paper we give a complete isomorphical classification of free topological groups $FM(X)$ of locally compact zero-dimensional separable metric spaces X. From this classification we obtain for locally compact zero-dimensional separable metric spaces X and Y that the free topological groups $FM(X)$ and $FM(Y)$ are isomorphic if and only if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.

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1. Introduction.

All spaces and topological groups considered in this paper are assumed to be Tychonov.

In [4] free topological groups in the sense of Graev were introduced. Graev obtained in that paper (Theorem 9 of § 11) a complete isomorphical classification of free topological groups of countable compact spaces (of course two topological groups are said to be isomorphic if and only if there exists a group isomorphism between them which is moreover a topological homeomorphism).

Theorem 1.1 [4]**.** *Two countable compact spaces X and Y have isomorphic free topological groups if and only if one of the following conditions holds:*

1. *X and Y are finite and have the same number of elements.*
2. *There are countable infinite ordinals α and β such that X is homeomorphic to the ordinal space $\alpha + 1$, Y is homeomorphic to the ordinal space $\beta + 1$ and $\max(\alpha, \beta) < [\min(\alpha, \beta)]^\omega$.*

For a space X, $C_p(X)$ denotes the set of all real-valued continuous functions on X endowed with the topology of pointwise convergence. The function spaces $C_p(X)$ are topological vector spaces, hence we can study linear homeomorphisms between two of them. In [2] a similar classification as Theorem 1.1 is given for function spaces $C_p(X)$ of countable compact spaces.

Corollary. *Let X and Y be countable compact spaces. Then the free topological groups of X and Y are isomorphic if and only if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.*

It is always true that if free topological groups are isomorphic, then $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic (cf. [1]). The converse is in general not true. For example if X is the unit interval and Y is the "letter" T, we have by results of

Pavlovskiĭ [8], that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic. However Graev showed in [5] that the free topological groups of X and Y are not isomorphic. Our aim is to prove that for locally compact zero-dimensional separable metric spaces the converse implication also holds, i.e. for such spaces the free topological groups are isomorphic if and only if their function spaces with the topology of pointwise convergence are linearly homeomorphic. The above example shows that the zero-dimensionality assumption is essential.

2. Preliminaries.

Definition. Let X be a space and let $e \in X$ be arbitrary. A topological group $FG(X)$ is a *free topological group in the sense of Graev* whenever the following holds:

- $X \setminus \{e\}$ is an algebraic base for $FG(X)$ and e is the unit element of $FG(X)$.
- X is a subspace of $FG(X)$.
- For each topological group G and each continuous map f from X into G which maps e onto the unit element of G, there is a continuous homomorphism from $FG(X)$ into G extending f.

In [4], Graev proved the existence and uniqueness of such a free topological group and that it makes no difference which point e one chooses. Moreover he showed that X is closed in $FG(X)$. Each element $y \in FG(X) \setminus \{e\}$ can uniquely be written as $y = x_1^{\epsilon_1} \cdots x_n^{\epsilon_n}$, where $x_1, \dots, x_n \in X \setminus \{e\}$, $\epsilon_1, \dots, \epsilon_n \in \{-1, 1\}$ and $n \in \mathbb{N}$. We say that the length of y is equal to n. For convenience let the length of e be 0. In $FG(X)$ the set of all words of length less than or equal to n will be denoted by $F_n(X)$.

In [6], Markov defines another kind of free topological group:

Definition. Let X be a space. A topological group $FM(X)$ is a *free topological group in the sense of Markov* whenever the following holds:

- X is an algebraic base for $FM(X)$.
- X is a subspace of $FM(X)$.
- For each topological group G and each continuous map f from X into G, there is a continuous homomorphism from $FM(X)$ into G extending f.

Again we have an existence and uniqueness theorem for free topological groups in the sense of Markov. There is a coincidence between the definition 1 and the definition 2. Graev observed in [4] that for a Tychonov space X, the groups $FM(X)$ and $FG(X \cup \{e\})$ are isomorphic, where e is an arbitrary point not in X which is isolated in $X \cup \{e\}$. Moreover he showed that if a Tychonov space X is not connected, say $X = X_1 \oplus X_2$, and if $e \in X_1$ and $f \in X_2$, then $FG(X)$ and $FM(Y)$ are isomorphic, where Y is the subspace $(f \cdot X_1) \cup X_2$ of $FG(X)$ (of course the symbol "$\oplus$" is used for the topological sum of spaces and the symbol "$\cdot$" denotes the group operation). For our purposes we have

Proposition 2.1. *If X is an infinite locally compact zero-dimensional separable metric space, then the free topological groups $FM(X)$ and $FG(X)$ are isomorphic.*

Proof: If X contains infinitely many isolated points, then X and $X \setminus \{e\}$ are homeomorphic, where e is isolated in X. Then $FM(X)$ and $FM(X \setminus \{e\})$ are isomorphic, hence $FM(X)$ and $FG(X)$ are isomorphic. If X contains only finitely many isolated points, then since X is not finite, X contains a clopen copy E of the Cantor set. Write $E = E_1 \cup E_2$, where E_1 and E_2 are disjoint clopen copies of the Cantor set. Find $e \in E_1$ and $f \in E_2$. By the result of Graev we obtain that $FG(X)$ and $FM(Y)$ are isomorphic, where Y is the subspace $(f \cdot E_1) \cup E_2 \cup X \setminus E$ of $FG(X)$. Since $f \cdot E_1$ and E_2 are Cantor sets and $(f \cdot E_1) \cap E_2 = \{f\}$, it follows that $(f \cdot E_1) \cup E_2$ is also a Cantor set. This implies that Y is homeomorphic to X. □

Corollary 2.2. *Let X be an infinite locally compact zero-dimensional separable metric space and let x be any point. Then $FG(X \oplus \{x\})$ and $FG(X)$ are isomorphic.*

In the sequel we need the following four theorems.

Theorem 2.3 [7]**.** *Let X be a Tychonov space and A a compact retract of X. Let Y be the quotient space obtained from X by identifying A to one point. Let p be an arbitrary point not in X. Then the free topological groups $FM(X \oplus \{p\})$ and $FM(Y \oplus A)$ are isomorphic.*

Theorem 2.4 [4]**.** *Let X be a compact space and let Y be a closed subspace of $FG(X)$ such that there are $n, m \in \mathbb{N}$ such that X is a subset of $F_n(Y)$, Y is a subset of $F_m(X)$ and such that $Y \setminus \{e\}$ is an algebraic base for $FG(X)$, then $FG(X)$ and $FG(Y)$ are isomorphic.*

Theorem 2.5 [1]**.** *Let X and Y be Tychonov spaces. If $FG(X)$ and $FG(Y)$ are isomorphic, then $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.*

Theorem 2.6 [2]**.** *Let X and Y be locally compact zero-dimensional separable metric spaces. Then $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic if and only if one of the following conditions holds:*

1. *X and Y are finite and have the same number of elements.*
2. *There are countable infinite ordinals α and β such that X is homeomorphic to the ordinal space $\alpha + 1$, Y is homeomorphic to the ordinal space $\beta + 1$ and $\max(\alpha, \beta) < [\min(\alpha, \beta)]^\omega$.*
3. *X and Y are uncountable and compact.*
4. *X and Y are not compact and there are compacta X_i and Y_i ($i \in \mathbb{N}$) such that $X = \bigoplus_{i=1}^\infty X_i$, $Y = \bigoplus_{i=1}^\infty Y_i$, and for each $i \in \mathbb{N}$, $C_p(X_i)$ and $C_p(Y_i)$ are linearly homeomorphic.*

3. A classification of free topological groups.

We aim at proving the following

Theorem 3.1. *Let X and Y be locally compact zero-dimensional separable metric spaces. Then $FG(X)$ and $FG(Y)$ are isomorphic if and only if one of the following conditions holds:*

1. *X and Y are finite and have the same number of elements.*
2. *There are countable infinite ordinals α and β such that X is homeomorphic to the ordinal space $\alpha + 1$, Y is homeomorphic to the ordinal space $\beta + 1$ and $\max(\alpha, \beta) < [\min(\alpha, \beta)]^\omega$.*

3. *X and Y are uncountable and compact.*
4. *X and Y are not compact and there are compacta X_i and Y_i ($i \in \mathbb{N}$) such that $X = \bigoplus_{i=1}^{\infty} X_i$, $Y = \bigoplus_{i=1}^{\infty} Y_i$, and for each $i \in \mathbb{N}$, $FG(X_i)$ and $FG(Y_i)$ are isomorphic.*

First we consider spaces satisfying the condition 3 in this theorem. Let C denote the Cantor set

Lemma 3.2. *Let X be a compact zero-dimensional metric space. Then $FG(C)$ and $FG(X \oplus C)$ are isomorphic.*

Proof: First suppose that X contains only finitely many isolated points. Then X is finite or X is the union of a Cantor set and a finite set. In both cases we are done by Corollary 2.2. So from now on we assume that X contains infinitely many isolated points.

Enumerate the set of isolated points of X by $\{x_1, x_2, \cdots\}$. We assume that $e \in C$. Let $\{U_n : n \in \mathbb{N}\}$ be a clopen decreasing base for $e \in C$. We assume that for every $n \in \mathbb{N}$, $U_n \setminus U_{n+1}$ is a Cantor set and $U_1 = C$. For every $n \in \mathbb{N}$, let $V_n = U_n \setminus U_{n+1}$. Decompose $\mathbb{N}$ into infinitely many disjoint copies of $\mathbb{N}$, say $\mathbb{N} = \bigcup_{m=1}^{\infty} N_m$, such that for each $m \in \mathbb{N}$, $N_m \subset \{n \in \mathbb{N} : n \geq m\}$. For every $n \in \mathbb{N}$, let

$$A_n = \{e\} \cup \bigcup_{m \in N_n} V_m.$$

Then for $n \neq m$ we have $A_n \cap A_m = \{e\}$, and for each $n \in \mathbb{N}$, $A_n \subset U_n$ and A_n is a Cantor set. Furthermore $\bigcup_{n=1}^{\infty} A_n = C$. Let

$$Z = \{A_n \cdot x_n : n \in \mathbb{N}\} \cup X \cup \{e\}.$$

For convenience let $Y = X \oplus C$. Observe that $Z \subset F_2(Y)$ and that $F_2(Y)$ is a compact metric subspace of $FG(Y)$, hence Z is a metric space (note that the map $h : Y \times Y \to F_2(Y)$ is a continuous surjection). Moreover $Z \setminus \{e\}$ is an algebraic base for $FG(Y)$. We claim that Z is closed in $FG(Y)$. Indeed let $(z_m)_{m \in \mathbb{N}}$ be a sequence in Z which converges to a point $z \in FG(Y)$. If a subsequence is contained in X, then $z \in X \subset Z$ since X is closed in $FG(Y)$. If a subsequence is contained in some $A_n \cdot x_n$, then $z \in A_n \cdot x_n \subset Z$, since $A_n \cdot x_n$ is a Cantor set. So we may assume that for each $m \in \mathbb{N}$ $z_m \in A_m \cdot x_m$, say $z_m = a_m \cdot x_m$ with $a_m \in A_m$. Since $z_m \to z$ and $a_m \to e$ (here we use that each $A_m \subset U_m$), $a_m^{-1} \cdot z_m \to z$. This gives $x_m \to z$. Since X is closed in $FG(Y)$, $z \in X \subset Z$. We conclude that Z is closed in $FG(Y)$, so Z being a subspace of $F_2(Y)$ is compact. Since X is a subset of $F_3(Z)$ we have by Theorem 2.4 that $FG(Z)$ and $FG(Y)$ are isomorphic. By Theorem 2.5, we obtain that $C_p(Z)$ and $C_p(Y)$ are linearly homeomorphic. Since Z is a countable union of zero-dimensional closed subsets, Z is zero-dimensional. We shall prove that $Z \setminus \{e\}$ contains no isolated points. Let $z \in Z \setminus \{e\}$. If $z \in A_n \cdot x_n$ for some $n \in \mathbb{N}$, then z is certainly not isolated in Z. If $z \in X$ we may assume $z = x_1$. But then $z = e \cdot x_1 \in A_1 \cdot x_1$, so again z is not isolated in Z. A careful look at the above proof that Z is closed in $FG(X)$ gives that e is isolated in Z. This implies that $Z \setminus \{e\}$ is a Cantor set and by Corollary 2.2 we have that $FG(Z \setminus \{e\})$ and $FG(Y)$ are isomorphic. $\square$

Proposition 3.3. *If X and Y are uncountable zero-dimensional compact metric spaces, then $FG(X)$ and $FG(Y)$ are isomorphic.*

Proof: It suffices to prove that $FG(X)$ and $FG(C)$ are isomorphic. Since X is an uncountable zero-dimensional compact metric space, X contains a copy D of the Cantor set. Moreover D is a retract of X (cf. [3]). By Theorem 2.3 we get that $FM(X \oplus \{p\})$ and $FM(Y \oplus D)$ are isomorphic, where p is any point not in X and Y is the quotient space obtained from X by identifying D to one point. By Lemma 3.2, $FG(Y \oplus D)$ is isomorphic to $FG(C)$. So by Proposition 2.1 and Corollary 2.2, $FG(X)$ and $FG(C)$ are isomorphic. $\square$

Secondly, we consider spaces X and Y satisfying the condition 4 in Theorem 3.1.

Proposition 3.4. *Let X and Y be zero-dimensional separable metric spaces. Suppose there are compacta X_i and Y_i ($i \in \mathbb{N}$) such that $X = \bigoplus_{i=1}^{\infty} X_i$, $Y = \bigoplus_{i=1}^{\infty} Y_i$, and such that for each $i \in \mathbb{N}$, $FG(X_i)$ and $FG(Y_i)$ are isomorphic. Then $FG(X)$ and $FG(Y)$ are isomorphic.*

Proof: Since each X_i is clopen in X, it is easily seen that the subgroup of $FM(X)$ generated by X_i is the free topological group of X_i in the sense of Markov. Hence we may assume that $FM(X_i) \subset FM(X)$. Similarly $FM(Y_i) \subset FM(Y)$. For each $i \in N$, let $\phi_i : FM(X_i) \to FM(Y_i)$ be an isomorphism. Define $f : X \to FM(Y)$ by $f(x) = \phi_i(x)$ for $x \in X_i$. Then f is continuous. Let ψ be the continuous homomorphism from $FM(X)$ into $FM(Y)$ extending f. By defining in a similar way the inverse of ψ we obtain that ψ is an isomorphism. The result now follows from Proposition 2.1. $\square$

We are now in a position to give the proof of Theorem 3.1:

Proof: If the spaces X and Y satisfy the condition 1 or 2, then by Theorem 9 in [4] we have that $FG(X)$ and $FG(Y)$ are isomorphic. By Propositions 3.3 and 3.4 we get the same result for spaces satisfying the condition 3 or 4. Now let X and Y be locally compact zero-dimensional separable metric spaces such that $FG(X)$ and $FG(Y)$ are isomorphic. By Theorem 2.5 we have that $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic, hence X and Y satisfy one of the conditions in Theorem 2.6. If X and Y satisfy the condition 1, 2 or 3 in Theorem 2.6, we are done. So suppose X and Y satisfy the condition 4 in Theorem 2.6. Then the proof is finished because we have by the preceding cases that for zero-dimensional compact metric spaces free topological groups are isomorphic if and only if the corresponding function spaces with the topology of pointwise convergence are linearly homeomorphic. $\square$

Corollary 3.5. *Let X and Y be locally compact zero-dimensional separable metric spaces. Then $FG(X)$ and $FG(Y)$ are isomorphic if and only if $C_p(X)$ and $C_p(Y)$ are linearly homeomorphic.*

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