Hereditarity of closure operators and injectivity

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Abstract. A notion of hereditarity of a closure operator with respect to a class of monomorphisms is introduced. Let C be a regular closure operator induced by a subcategory $\mathcal{A}$. It is shown that, if every object of $\mathcal{A}$ is a subobject of an $\mathcal{A}$-object which is injective with respect to a given class of monomorphisms, then the closure operator C is hereditary with respect to that class of monomorphisms.

Keywords: closure operator, hereditary closure operator, injective object, factorization pair

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Introduction.

Let C be a closure operator on a category $\mathcal{X}$ with respect to a class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms. In this paper we introduce the notion of hereditarity of C with respect to a subclass $\mathcal{M}'$ of $\mathcal{M}$. We show that if $\mathcal{M}'$ and $\mathcal{M}''$ are two subclasses of $\mathcal{M}$ which form a factorization pair for $\mathcal{M}$ (cf. Definition 7) then the hereditarity of C with respect to both $\mathcal{M}'$ and $\mathcal{M}''$ implies the hereditarity of C with respect to $\mathcal{M}$.

The main purpose of this paper is to show that hereditarity of a regular closure operator is strongly related to the notion of injectivity. As a matter of fact, let C be a regular closure operator induced by a subcategory $\mathcal{A}$ and let $\mathcal{M}' \subseteq \mathcal{M}$. If $\mathcal{A}$ satisfies the condition that every object of $\mathcal{A}$ is a subobject of an $\mathcal{M}'$-injective object of $\mathcal{A}$, then C is $\mathcal{M}'$-hereditary. Some examples show that in general if C is $\mathcal{M}'$-hereditary, $\mathcal{A}$ need not satisfy the above condition.

We conclude the paper with an example which shows that neither hereditarity nor C-dense hereditarity is preserved under the construction of idempotent hulls.

We use the terminology of [HS] throughout.

Preliminaries.

Throughout, we consider a category $\mathcal{X}$ and a fixed class $\mathcal{M}$ of $\mathcal{X}$-monomorphisms which contains all $\mathcal{X}$-isomorphisms. It is assumed that:

(1) $\mathcal{M}$ is closed under composition.

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(2) Pullbacks of $\mathcal{M}$-morphisms exist and belong to $\mathcal{M}$, and multiple pullbacks of (possibly large) families of $\mathcal{M}$-morphisms with common codomain exist and belong to $\mathcal{M}$.

In addition, we require $\mathcal{X}$ to have equalizers and $\mathcal{M}$ to contain all regular monomorphisms.

One of the consequences of the above assumptions is that there is a uniquely determined class $\mathcal{E}$ of morphisms in $\mathcal{X}$ such that $(\mathcal{E}, \mathcal{M})$ is a factorization structure on $\mathcal{X}$, i.e., each morphism f in $\mathcal{X}$ has a factorization $f = m \circ e$ with $e \in \mathcal{E}$ and $m \in \mathcal{M}$, and if $A \xrightarrow{e} B$, $B \xrightarrow{h} D$, $A \xrightarrow{g} C$ and $C \xrightarrow{m} D$ are $\mathcal{X}$-morphisms with $m \in \mathcal{M}$, and $e \in \mathcal{E}$ such that $m \circ g = h \circ e$, then there exists a unique diagonal, i.e., a morphism $B \xrightarrow{d} C$ such that for each $i \in I$ the both triangles of the diagram

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "g"'] & B \arrow[d, "h"] \arrow[dl, "d"] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

commute (cf. [DG$_1$]).

We regard $\mathcal{M}$ as a full subcategory of the arrow category of $\mathcal{X}$, with the codomain functor from $\mathcal{M}$ to $\mathcal{X}$ denoted by U. Since U is faithful, $\mathcal{M}$ is concrete over $\mathcal{X}$.

Definition 1.

A *closure operator* on $\mathcal{X}$ (with respect to $\mathcal{M}$) is a pair $C = (\gamma, F)$, where F is an endofunctor on $\mathcal{M}$ that satisfies $UF = U$, and γ is a natural transformation from $id_{\mathcal{M}}$ to F that satisfies $(id_U)\gamma = id_U$.

Thus, given a closure operator $C = (\gamma, F)$, every member m of $\mathcal{M}$ has a canonical factorization

$$\begin{tikzcd} \bullet \arrow[r, "{]m[_C}"] \arrow[dr, "m"'] & \bullet \arrow[d, "{[m]_C}"] \\ & \bullet \end{tikzcd}$$

where $[m]_C = F(m)$ is called the C-*closure* of m, and $]m[_C$ is the domain of the m-component of γ. The class of all $\mathcal{M}$-morphisms of the form $]m[_C$ ($[m]_C$) will be denoted by $\Delta(C)$ ($\nabla(C)$). In particular, $[\ ]_C$ induces an order-preserving increasing function on the $\mathcal{M}$-subobject lattice of every $\mathcal{X}$-object. Also, these functions are related in the following sense: if p is the pullback of a morphism $m \in \mathcal{M}$ along some $\mathcal{X}$-morphism f, and q is the pullback of $[m]_C$ along f, then $[p]_C \leq q$. Conversely, every family of functions on the $\mathcal{M}$-subobject lattices that has the above properties uniquely determines a closure operator.

Definition 2.

Given a closure operator C, we say that $m \in \mathcal{M}$ is C-*closed* if $]m[_C$ is an isomorphism. An $\mathcal{X}$-morphism f is called C-*dense* if for every $(\mathcal{E}, \mathcal{M})$-factorization (e, m) of f we have that $[m]_C$ is an isomorphism. We call C *idempotent* provided that $[\ ]_C \circ [\ ]_C \simeq [\ ]_C$, i.e., provided that $[m]_C$ is C-closed for every $m \in \mathcal{M}$. C is called *weakly hereditary* if $]m[_C$ is C-dense for every $m \in \mathcal{M}$.

For more background on closure operators see, e.g., [DG1], [DG2], [C], [K] and [DGT].

A special case of an idempotent closure operator arises in the following way. Given any class $\mathcal{A}$ of $\mathcal{X}$-objects and $M \xrightarrow{m} X$ in $\mathcal{M}$, define $[m]_{\mathcal{A}}$ to be the intersection of all equalizers of pairs of $\mathcal{X}$-morphisms r, s from X to some $\mathcal{A}$-object A that satisfy $r \circ m = s \circ m$, and let $]m[_{\mathcal{A}} \in \mathcal{M}$ be the unique $\mathcal{X}$-morphism by which m factors through $[m]_{\mathcal{A}}$. It is easy to see that $(]\ [_{\mathcal{A}}, [\ ]_{\mathcal{A}})$ forms an idempotent closure operator. This generalizes the Salbany construction of closure operators induced by classes of topological spaces, cf. [S]. Such a closure operator was called regular in [DG2]. To simplify the notation, instead of "$[\ ]_{\mathcal{A}}$-dense" we usually write "$\mathcal{A}$-dense".

We denote the collection of all closure operators on $\mathcal{M}$ by $\mathbf{CL}(\mathcal{X}, \mathcal{M})$ pre-ordered as follows: $C \sqsubseteq D$ if $[m]_C \leq [m]_D$ for all $m \in \mathcal{M}$ (where $\leq$ is the usual order on subobjects).

Definition 3.

An $\mathcal{X}$-object I is said to be *injective* with respect to the class of $\mathcal{X}$-morphisms $\mathcal{U}$ (in short $\mathcal{U}$-*injective*) if for each $X \xrightarrow{m} Y$ in $\mathcal{U}$ and $X \xrightarrow{f} I$, there exists $Y \xrightarrow{g} I$ such that $g \circ m = f$. Then g is called an *extension of* f *along* m. $Inj(\mathcal{U})$ will denote the class of all $\mathcal{U}$-injective $\mathcal{X}$-objects.

Let C be a closure operator on $\mathcal{X}$ and let $\mathcal{M}' \subseteq \mathcal{M}$. If $\mathcal{M}'$ is the class of all C-dense $\mathcal{M}$-morphisms (C-closed $\mathcal{M}$-morphisms), then the class of all $\mathcal{M}'$-injective $\mathcal{X}$-objects will be denoted by $Inj_d(C)$ ($Inj_c(C)$).

Main results.

In what follows, $\hat{C}$ ($\check{C}$) will denote the idempotent hull (weakly hereditary core) of the closure operator C (cf. [DG2]).

Proposition 4.

(a) $Inj_c(C) = Inj_c(\hat{C})$.
(b) *If* $\mathcal{X}$ *is* $\mathcal{M}$*-well powered and* C *is weakly hereditary then* $Inj_d(C) = Inj_d(\hat{C})$.
(c) $Inj_d(C) = Inj_d(\check{C})$.

Proof: (a). It follows from the fact that an $\mathcal{M}$-subobject is C-closed iff it is $\hat{C}$-closed.

(b). Since C-dense always implies $\hat{C}$-dense, we have that $Inj_d(\hat{C}) \subseteq Inj_d(C)$. Now, let $Z \in Inj_d(C)$ and let $M \xrightarrow{m} X$ be a $\hat{C}$-dense $\mathcal{M}$-subobject. Since C is weakly hereditary, $]m[^X_C$ is C-dense. Consequently, for any $\mathcal{X}$-morphism $M \xrightarrow{f} Z$ there exists an $\mathcal{X}$-morphism $[m]^X_C \xrightarrow{g} Z$ such that $g \circ]m[^X_C = f$. Since $\mathcal{X}$ is $\mathcal{M}$-well powered, using transfinite induction we obtain that there exists an $\mathcal{X}$-morphism $[m]^X_{\hat{C}} \xrightarrow{h} Z$ such that $h \circ]m[^X_{\hat{C}} = f$. Since m is $\hat{C}$-dense, $[m]^X_{\hat{C}}$ is an isomorphism and $k = h \circ ([m]^X_{\hat{C}})^{-1}$ is an extension of f along m. Therefore $Z \in Inj_d(\hat{C})$ (cf. [DG2] with $\hat{C} = C^{\infty}$).

(c). It follows from the fact that an $\mathcal{M}$-subobject is C-dense iff it is $\check{C}$-dense. $\square$

The question of whether item (b) of the above proposition might hold without C being weakly hereditary and without $\mathcal{X}$ being $\mathcal{M}$-well powered, remains open.

Since C-closed always implies $\check{C}$-closed, $Inj_c(\check{C}) \subseteq Inj_c(C)$.

Definition 5.

Let $\mathcal{M}' \subseteq \mathcal{M}$ and let C be a closure operator on $\mathcal{X}$ with respect to $\mathcal{M}$. C is called *$\mathcal{M}'$-hereditary* if given two $\mathcal{M}$-subobjects of X, (M,m) and (N,n), with $(M,m) \leq (N,n)$ and $(N,n) \in \mathcal{M}'$, we have that $[M]_C^X \cap N \simeq [M]_C^N$.

Three particularly important cases are *(C-dense)-hereditary*, *(C-closed)-hereditary* and *hereditary* that occur exactly when $\mathcal{M}'$ equals the class of C-dense $\mathcal{M}$-subobjects, the class of C-closed $\mathcal{M}$-subobjects and all of $\mathcal{M}$, respectively.

Notice that $[M]_C^X \cap N$ is isomorphic to the pullback of $([M]_C^X, [m]_C^X)$ along n.

Lemma 6 [DG$_2$]. *An idempotent closure operator C is weakly hereditary iff it is C-closed-hereditary.* $\square$

Definition 7.

Let $\mathcal{M}'$ and $\mathcal{M}''$ be two subclasses of $\mathcal{M}$. We say that $\mathcal{M}$ factors through the pair $(\mathcal{M}', \mathcal{M}'')$ iff every $m \in \mathcal{M}$ can be written as $m = m'' \circ m'$ with $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$. $(\mathcal{M}', \mathcal{M}'')$ will be called a *factorization pair for $\mathcal{M}$.*

Proposition 8. *Let C be a closure operator on $\mathcal{X}$ and let $(\mathcal{M}', \mathcal{M}'')$ be a factorization pair for $\mathcal{M}$. Then, C is hereditary iff C is $\mathcal{M}'$-hereditary and $\mathcal{M}''$-hereditary.*

PROOF: $(\Rightarrow)$. It is obvious.

$(\Leftarrow)$. Let us consider the following commutative diagram

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "t"'] & X \\ N \arrow[r, "n'"'] \arrow[ur, "n"] & N' \arrow[u, "n''"'] \end{tikzcd}$$

$n' \in \mathcal{M}'$ and $n'' \in \mathcal{M}''$. From the commutative diagram

$$\begin{tikzcd} M \arrow[r, "n' \circ t"] \arrow[d, "t"'] & N' \\ N \arrow[ur, "n'"'] & \end{tikzcd}$$

and the fact that C is $\mathcal{M}'$-hereditary, we obtain that $[M]_C^N \simeq N \cap [M]_C^{N'}$. From the commutative diagram

$$\begin{tikzcd} M \arrow[r, "m"] \arrow[d, "n' \circ t"'] & X \\ N' \arrow[ur, "n''"'] & \end{tikzcd}$$

and the fact that C is $\mathcal{M}''$-hereditary, we obtain that $[M]_{\mathcal{C}}^{N'} \simeq N' \cap [M]_{\mathcal{C}}^{X}$. Therefore, $[M]_{\mathcal{C}}^{N} \simeq N \cap [M]_{\mathcal{C}}^{N'} \simeq N \cap N' \cap [M]_{\mathcal{C}}^{X} \simeq N \cap [M]_{\mathcal{C}}^{X}$. Therefore C is hereditary. □

Corollary 9.

(a) *Let C be a closure operator on $\mathcal{X}$. C is hereditary iff it is $\Delta(C)$-hereditary and $\nabla(C)$-hereditary.*

(b) *Let C be an idempotent closure operator on $\mathcal{X}$. C is hereditary iff it is (C-dense)-hereditary and (C-closed)-hereditary.*

Proof: (a) Clearly because $(\Delta(C), \nabla(C))$ always forms a factorization pair for $\mathcal{M}$.

(b) It follows immediately from the fact that if C is idempotent and (C-closed)-hereditary, then (C-dense $\mathcal{M}$-morphisms, C-closed $\mathcal{M}$-morphisms) forms a factorization pair for $\mathcal{M}$ (cf. Lemma 6 and [DG_2]).

□

Proposition 10. *Let $(\mathcal{M}', \mathcal{M}'')$ be a factorization pair for $\mathcal{M}$. Then we have: $Inj(\mathcal{M}') \cap Inj(\mathcal{M}'') = Inj(\mathcal{M})$.*

Proof: We need to prove only one inclusion. Let $X \xrightarrow{m} Y$ be a morphism in $\mathcal{M}$ and let $X \xrightarrow{f} I$ be an $\mathcal{X}$-morphism with $I \in Inj(\mathcal{M}') \cap Inj(\mathcal{M}'')$. By hypothesis, $m = m'' \circ m'$ with $m' \in \mathcal{M}'$ and $m'' \in \mathcal{M}''$. So, there exists an $\mathcal{X}$-morphism g such that $g \circ m' = f$ as well as an $\mathcal{X}$-morphism h such that $h \circ m'' = g$. Therefore $h \circ m = h \circ m'' \circ m' = g \circ m' = f$. Thus, $I \in Inj(\mathcal{M})$. □

Corollary 11.

(a) *Let C be a closure operator on $\mathcal{X}$. Then $Inj(\Delta(C)) \cap Inj(\nabla(C)) = Inj(\mathcal{M})$.*

(b) *Let C be a weakly hereditary and idempotent closure operator on $\mathcal{X}$. Then $Inj_d(C) \cap Inj_c(C) = Inj(\mathcal{M})$.*

Proof: (a) Just notice that $(\Delta(C), \nabla(C))$ always forms a factorization pair for $\mathcal{M}$.

(b) If C is weakly hereditary and idempotent, then (C-dense $\mathcal{M}$-morphisms, C-closed $\mathcal{M}$-morphisms) forms a factorization pair for $\mathcal{M}$ (cf. [DG_2]).

□

For the next few results we assume the additional condition that $\mathcal{X}$ is a regular well-powered category with products.

The following result is well known.

Lemma 12. *Let $\mathcal{M}' \subseteq \mathcal{M}$. $Inj(\mathcal{M}')$ is closed under products.* □

Theorem 13. *Let $\mathcal{A}$ be a class of $\mathcal{X}$-objects and let $\mathcal{M}' \subseteq \mathcal{M}$. Suppose that for each $A \in \mathcal{A}$, there is an $\mathcal{X}$-monomorphism $A \xrightarrow{k} A'$ with $A' \in \mathcal{A}$ being $\mathcal{M}'$-injective. Then the $\mathcal{A}$-closure is $\mathcal{M}'$-hereditary.*

Proof: Let $\Pi(Inj(\mathcal{M}') \cap \mathcal{A})$ denote the family of all possible products of the objects of $Inj(\mathcal{M}') \cap \mathcal{A}$. Let $M \xrightarrow{m} X$ be an $\mathcal{M}$-subobject of X and let $X \underset{g}{\overset{f}{\rightrightarrows}} A$ be two $\mathcal{X}$-morphisms with $A \in \mathcal{A}$ and $f \circ m = g \circ m$. If $A \xrightarrow{k} A'$ is an $\mathcal{X}$-monomorphism with $A' \in Inj(\mathcal{M}') \cap \mathcal{A}$, then it is easy to see that $equ(f,g) \simeq equ(k \circ f, k \circ g)$. Therefore, the $\mathcal{A}$-closure agrees with the regular closure operator induced by the family $Inj(\mathcal{M}') \cap \mathcal{A}$ as well as with the one induced by $\Pi(Inj(\mathcal{M}') \cap \mathcal{A})$ (cf. [C, Proposition 1.4] and [G]).

Let us consider the commutative diagram

$$\begin{tikzcd}
M \arrow[r, "m"] \arrow[d, "p"'] \arrow[dr, "t"] & X \\
{[M]^N_{\mathcal{A}}} \arrow[r, "{[t]^N_{\mathcal{A}}}"'] & N \arrow[u, "n"']
\end{tikzcd}$$

with $m \in \mathcal{M}$ and $n \in \mathcal{M}'$. Consider two morphisms r and s with domain N and codomain in $\Pi(Inj(\mathcal{M}') \cap \mathcal{A})$, such that $[t]^N_{\mathcal{A}} = equ(r,s)$ (cf. [C, Proposition 1.6]). Since every $Y \in \Pi(Inj(\mathcal{M}') \cap \mathcal{A})$ is $\mathcal{M}'$-injective (cf. Lemma 12), we get that there exist two morphisms h and k such that $h \circ n = r$ and $k \circ n = s$. Now, $r \circ t = s \circ t$ implies that $h \circ m = h \circ n \circ t = r \circ t = s \circ t = k \circ n \circ t = k \circ m$. Therefore $h \circ [m]^X_{\mathcal{A}} = k \circ [m]^X_{\mathcal{A}}$.

Let us consider the diagram

$$\begin{tikzcd}
{[M]^X_{\mathcal{A}} \cap N} \arrow[rrr, "\beta"] \arrow[ddd, "\alpha"'] \arrow[dr, "\alpha"] & & & {[M]^X_{\mathcal{A}}} \arrow[ddd, "{[m]^X_{\mathcal{A}}}"] \\
& N \arrow[ddl, "id"] & {[M]^N_{\mathcal{A}}} \arrow[l, "{[t]^N_{\mathcal{A}}}"'] \arrow[ur, "\mu"] \arrow[ddr, "{m \circ [t]^N_{\mathcal{A}}}"'] & \\
& & & \\
N \arrow[rrr, "n"'] & & & X
\end{tikzcd}$$

$h \circ [m]^X_{\mathcal{A}} = k \circ [m]^X_{\mathcal{A}}$ implies that $h \circ [m]^X_{\mathcal{A}} \circ \beta = k \circ [m]^X_{\mathcal{A}} \circ \beta$. From $[m]^X_{\mathcal{A}} \circ \beta = n \circ \alpha$, we get that $r \circ \alpha = h \circ n \circ \alpha = h \circ [m]^X_{\mathcal{A}} \circ \beta = k \circ [m]^X_{\mathcal{A}} \circ \beta = k \circ n \circ \alpha = s \circ \alpha$. Since $[t]^N_{\mathcal{A}} = equ(r,s)$, there exists a morphism $[M]^X_{\mathcal{A}} \cap N \xrightarrow{\gamma} [M]^N_{\mathcal{A}}$ such that $[t]^N_{\mathcal{A}} \circ \gamma = \alpha$. $[\mathrm{M}]^N_{\mathcal{A}}$ is an $\mathcal{M}$-subobject of N and by functoriality of $[\ ]_{\mathcal{A}}$, it is also an $\mathcal{M}$-subobject of $[M]^X_{\mathcal{A}}$. So, there exists a morphism $[M]^N_{\mathcal{A}} \xrightarrow{c} [M]^X_{\mathcal{A}} \cap N$ such that $\alpha \circ c = [t]^N_{\mathcal{A}}$. Now $\alpha \circ c \circ \gamma = \alpha$ implies that $c \circ \gamma = id$, since α is a monomorphism. Thus, c is an isomorphism, since it is a monomorphism and a retraction. $\square$

Corollary 14.

(a) *If $\mathcal{A}$ has enough $\mathcal{M}'$-injectives, (i.e., for every $A \in \mathcal{A}$, there is a monomorphism $A \xrightarrow{k} A'$ with $k \in \mathcal{M}'$ and with $A' \in \mathcal{A}$ being $\mathcal{M}'$-injective), then the $\mathcal{A}$-closure is $\mathcal{M}'$-hereditary.*

(b) *If $\mathcal{A}$ is epireflective in $\mathcal{X}$ and admits a system of $\mathcal{M}'$-injective cogenerators, then the $\mathcal{A}$-closure is $\mathcal{M}'$-hereditary.*

PROOF: (a) We just observe that $\mathcal{M}' \subseteq \mathcal{M} \subseteq \mathcal{X}$-monomorphisms.

(b) It just follows from the fact that every $A \in \mathcal{A}$ is an extremal subobject of a product of $\mathcal{M}'$-injective objects of $\mathcal{A}$. □

Notice that Lemma 6, Corollary 9 and the above corollary yield the following interesting special cases.

(a) Hereditarity of the $\mathcal{A}$-closure is implied by $\mathcal{A}$ having enough $\mathcal{M}$-injectives.
(b) Weakly hereditarity of the $\mathcal{A}$-closure (=($\mathcal{A}$-closed)-hereditarity) is implied by $\mathcal{A}$ having enough ($\mathcal{A}$-closed)-injectives.
(c) ($\mathcal{A}$-dense)-hereditarity of the $\mathcal{A}$-closure is implied by $\mathcal{A}$ having enough ($\mathcal{A}$-dense)-injectives.
(d) The $\mathcal{A}$-closure is hereditary iff $\mathcal{A}$ has enough ($\mathcal{A}$-closed)-injectives and enough ($\mathcal{A}$-dense)-injectives.

Examples 17 and 18 below show that the implications in the items (a)–(c) cannot be reversed in general. Example 19 provides a case in which the item (a) becomes a characterization.

Remark 15. For any idempotent closure operator C, its weakly hereditary core $\check{C}$ is hereditary iff C is C-dense hereditary. As a matter of fact, since every closure operator C and its weakly hereditary core, $\check{C}$, determine the same dense morphisms (i.e., C-dense $=$ $\check{C}$-dense), if C is C-dense-hereditary, so is $\check{C}$ and if C is idempotent, so is $\check{C}$ (cf. [DG$_2$, Theorem 4.2 (3)]). Therefore from Corollary 9 and Lemma 6, we get that $\check{C}$ is hereditary iff C is C-dense-hereditary.

In all of the following examples $\mathcal{M}$ will be the class of embeddings.

Example 16. If $\mathcal{X} = \mathbf{TOP}$ and $\mathcal{A} = \mathbf{TOP}_0$, then the Sierpinski space S, which is a cogenerator for $\mathbf{TOP}_0$, is trivially injective. Thus the $\mathbf{TOP}_0$-closure (b-closure, [Sk]) is a hereditary operator.

Example 17. (a) Let $\mathcal{X} = \mathcal{A}$ be any epireflective non bireflective subcategory of $\mathbf{TOP}$ different from $\mathbf{TOP}_0$ and from $\mathbf{Sgl}$ (spaces with at most one point). Then, $\mathcal{A} \subseteq \mathbf{TOP}_1$ (cf. [G]) and the injective objects with respect to embeddings are the spaces with exactly one point. As a matter of fact, by assumption $\mathcal{A}$ contains a discrete two-point space, so it also contains any 0-dimensional Hausdorff space. In particular it contains the one-point compactification of the discrete space of natural numbers, $\mathcal{N}_\infty$. Now, suppose that $I \in \mathcal{A}$ has at least two points, say $I = \{0, 1\}$, and let $\mathcal{N} \xrightarrow{f} I$ be the continuous map defined by $f(n) = 0$ for n odd and $f(n) = 1$ for n even. Now, if we take the embedding $\mathcal{N} \xrightarrow{e} \mathcal{N}_\infty$, there is no extension of f along e.

(b) If $\mathcal{A}$ is one of the categories **Haus**, **Tych** or **0-Dim**, the morphism f of the item (a) is $\mathcal{A}$-dense (= dense cf. [DG$_1$]). So, in these cases, the injective objects with respect to the dense embeddings are the spaces with exactly one point.

Example 18. For $\mathcal{A} = \mathbf{Tych}$, the cogenerator $[0,1]$ is not closed injective. In fact, if X is a Tychonoff not normal space, we know from Tietze's Theorem that there exist a closed subset F of X and a continuous function $F \xrightarrow{f} [0,1]$ that cannot be extended to all of X. Since every cogenerator of **Tych** must contain a copy of the unit interval $[0,1]$, it is easy to conclude that **Tych** does not have a **Tych**-closed-injective cogenerator. This proves that the implications in Corollary 14 cannot be reversed in general. As a matter of fact, if $\mathcal{A} = \mathbf{Tych}$, then the $\mathcal{A}$-closure in **Tych** is the ordinary closure (cf. [DG$_1$]), which is hereditary.

Example 19. For a fixed ring R with unity, let $\mathcal{X}$ be the category R-**Mod** of left R-modules, let $\mathcal{M}$ be the class of monomorphisms in R-**Mod** and let $(\mathcal{T}, \mathcal{F})$ be a torsion theory. $(\mathcal{T}, \mathcal{F})$ is hereditary iff $\mathcal{F}$ is simply cogenerated by an injective module (cf. [DG$_3$] and [L]). Thus $[\ ]_{\mathcal{F}}$ is hereditary iff $\mathcal{F}$ is simply cogenerated by an injective object. This shows that in the category R-**Mod**, the item (a) of Corollary 14 can be reversed.

Neither hereditarity nor dense-hereditarity is preserved under the construction of idempotent hulls, as the following example shows.

Example 20. Let us consider the sets: $M = \{(m,n) : m,n \in \mathcal{N}\}$, $X = M \cup \{\infty_1, \infty_2, \dots\} \cup \{\infty\}$ and $N = M \cup \{\infty\}$. We consider in X the pretopological structure in which every point of the form (m,n) is isolated, a basic nbhd of ∞_i is of the form $\{(i,m) : \bar{m} \le m \text{ for some } \bar{m} \in \mathcal{N}\} \cup \{\infty_i\}$ and a basic nbhd of ∞ is of the form $\{\infty_j, \infty_{j+1}, \dots\} \cup \{\infty\}$ for some $j \in \mathcal{N}$. Let $\hat{K}$ be the idempotent hull of the closure operator K induced by the pretopology in **PrTOP** (cf. [DG$_4$]). Clearly $\hat{K}_X(N) = X$, i.e., N is $\hat{K}$-dense. Now, $\hat{K}_X(M) = X$, so $\hat{K}_X(M) \cap N = N$, but $\hat{K}_N(M) = M$, since N is discrete as a pretopological subspace. Thus $\hat{K}$ is $\hat{K}$-closed-hereditary but not $\hat{K}$-dense-hereditary and therefore is not hereditary, although K is.

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