On the boundedness of the mapping $f \to |f|$ in Besov spaces

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Abstract. For $1 \le p \le \infty$, precise conditions on the parameters are given under which the particular superposition operator $T : f \to |f|$ is a bounded map in the Besov space $B^s_{p,q}(R^1)$. The proofs rely on linear spline approximation theory.

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1. Introduction.

Due to applications in the theory of nonlinear partial differential equations, investigations on mapping properties of superposition (or Nemytzki) operators

$$T_g \ : \ f \ \to \ g(f) \tag{1}$$

where $g : R^1 \to R^1$ is a given function, attracted some attention. We refer to [AZ], and to [S1], [S2] for some recent overview concerning mapping properties in Besov-Sobolev norms. Besides the study of general classes of superposition functions g, a particular interest has been devoted to model cases such as $g(t) = |t|^\alpha$ or $g(t) = t \cdot |t|^{\alpha-1}$, see e.g. [CW], [S2].

In this note we study the boundedness of the mapping

$$T \ : \ f \ \to \ |f| \tag{2}$$

in the scale of Besov spaces $B^s_{p,q}$ on R^1 where $1 \le p, q \le \infty$, and $s > 0$. Using the well-known arguments [MM1], [RS], the results for this one-dimensional situation can be extended to Besov-Sobolev spaces on more general domains in R^n.

It is known (and simple to prove, see Section 2) that T is bounded if $s < 1$. In particular, T is bounded in the Sobolev spaces W^1_p (cf. [MM2], [MM3] for some further references and related results). More recently, a partial extension to the parameters $s > 1$ has been proved in [RS]: if $1 \le s \le 2/p$ $(1 \le p < 2)$ then T maps $B^s_{p,q}$ boundedly into $B^{s-\epsilon}_{p,q}$ for any $\epsilon > 0$. On the other hand, simple examples show that for $s > 1 + 1/p$ the mapping (2) cannot be bounded in Besov-Sobolev spaces.

The following main result of our note completes the picture.

Theorem 1. *Let the parameters p, q, s be as given above. Then the mapping T defined by* (2) *is bounded in $B^s_{p,q}$ if and only if $0 < s < 1 + 1/p$.*

Our proof relies on some tools from approximation theory for linear splines. In Section 2 we give the necessary definitions for the Besov spaces and consider the trivial case $s < 1$. Then, for $p = 1$, the "if"-assertion of Theorem 1 is reduced to an inequality for second order moduli of smoothness (Theorem 2). Moreover, the counterexamples covering the "only if"-part of Theorem 1 are given (Theorem 3). In the concluding Section 3 the proof of Theorem 1 is completed for the case $1 < p < \infty$.

Note that Theorem 1 answers the problem of boundedness of the mapping T also for the closely related scale of Sobolev-Slobodetski spaces W_p^s. Moreover, some extensions to the quasi-normed case $p < 1$ can be given as well.

2. The cases $s < 1$, $p = 1$, and counterexamples.

Recall some definitions. Let

$$\omega_m(t,f)_p = \sup_{0<h\le t} \|\Delta_h^m f\|_{L_p}\,, \quad t > 0\,, \tag{3}$$

be the m-th order modulus of smoothness of $f \in L_p \equiv L_p(R^1)$, $m = 1, 2, \ldots$.

These moduli can be used to give one of the numerous equivalent definitions of the Besov spaces under consideration (cf. [N], [T]): *Let* $1 \le p, q \le \infty$, $s > 0$, *and fix some integer* $m > s$. *Then* $f \in L_p$ *belongs to* $B_{p,q}^s \equiv B_{p,q}^s(R^1)$ *iff*

$$\|f\|_{B_{p,q}^s} \equiv \|f\|_{L_p} + \|2^{ls} \cdot \omega_m(2^{-l}, f)_p\|_{l_q} < \infty\,. \tag{4}$$

Different m lead to the same space (with equivalent norms), as a rule we take the smallest possible m. The l_q norm is defined for bi-infinite sequences as usual. Throughout the paper, by $c, C, \ldots$ we denote positive constants which are independent of the variables in the corresponding formulae but may change from line to line.

Note that from this definition of the Besov spaces the boundedness of the mapping (2) becomes obvious for $s < 1$: Since

$$|\Delta_h^1(Tf)(x)| \le |\Delta_h^1 f(x)|\,, \quad x \in R^1\,,\ h > 0\,,$$

for any $f \in L_p$, one has

$$\omega_1(t, Tf)_p \le \omega_1(t,f)_p\,, \quad t > 0\,. \tag{5}$$

Together with $\|Tf\|_{L_p} = \|f\|_{L_p}$, this yields

$$\|Tf\|_{B_{p,q}^s} \le \|f\|_{B_{p,q}^s}\,, \quad f \in B_{p,q}^s\,,\ 0 < s < 1\,, \tag{6}$$

if we fix $m = 1$ in (4). Moreover, by the well-known characterization of the Sobolev space W_p^1, $1 < p < \infty$, via first order moduli of smoothness, the boundedness of T follows for these spaces, too.

In order to deal with the case $s \ge 1$, one might try to extend (5) to higher order moduli of smoothness. We present a particular result in this direction.

Theorem 2. *For any $f \in L_1$ we have the inequality*

$$\omega_2(t, Tf)_1 \le C \cdot \omega_2(t, f)_1\,, \quad t > 0. \tag{7}$$

For the proof of Theorem 2 we need a Jackson type estimate for best approximation by linear splines. Let $\pi^{(k)} = \{x_i^{(k)} \equiv i \cdot 2^{-k} : i \in Z\}$ be the bi-infinite uniform partition of R^1 with stepsize 2^{-k}, and denote by $S^{(k)}$ the class of all piecewise linear spline functions $s \in C(R^1)$ with respect to $\pi^{(k)}$, $k \in Z$. The following estimate can be found, e.g., in [Sch], [O1].

Proposition 1. *For any $f \in L_p$, $1 \le p \le \infty$, there exist linear spline functions $s^{(k)} \in S^{(k)}$, $k \in Z$, such that*

$$\|f - s^{(k)}\|_{L_p} \le C \cdot \omega_2(2^{-k}, f)_p\,. \tag{8}$$

Proposition 1 allows us to reduce (7) to a simpler inequality for linear spline functions. Indeed, from (8) we immediately have

$$\begin{aligned} \omega_2(2^{-k}, Tf)_p &\le \omega_2(2^{-k}, Ts^{(k)})_p + 4 \cdot \|Tf - Ts^{(k)}\|_{L_p} \\ &\le \omega_2(2^{-k}, Ts^{(k)})_p + 4 \cdot \|f - s^{(k)}\|_{L_p} \\ &\le \omega_2(2^{-k}, Ts^{(k)})_p + c \cdot \omega_2(2^{-k}, f)_p \end{aligned}$$

and

$$\omega_2(2^{-k}, s^{(k)})_p \le \omega_2(2^{-k}, f)_p + 4 \cdot \|f - s^{(k)}\|_{L_p} \le c \cdot \omega_2(2^{-k}, f)_p$$

for all $1 \le p \le \infty$, especially for $p = 1$. Thus, if we prove

$$\omega_2(2^{-k}, Ts^{(k)})_1 \le C \cdot \omega_2(2^{-k}, s^{(k)})_1\,, \quad k \in Z\,, \tag{9}$$

then (7) holds true.

To prove (9), we derive first a more technical estimate which will be used also in Section 3. Fix some $k \in Z$, and drop for simplicity the upper indices (k) in the notations. Let s be any linear spline over π. On each interval $\Delta_i \equiv [x_{i-1}, x_i]$, $i \in Z$, the spline s vanishes identically or possesses at most one simple zero-crossing. Introduce the set $J \subset Z$ of all those indices i for which Δ_i contains exactly one zero-crossing. If this happens at x_i (resp. at x_{i-1}) then $s \not\equiv 0$ on Δ_{i+1} (resp. on Δ_{i-1}) is assumed. If $i_1 < i_2$ are two *subsequent* indices from J then, as a rule,

$$\Delta s_{i_1} \cdot \Delta s_{i_2} \le 0 \qquad (\Delta s_i \equiv s(x_i) - s(x_{i-1}))\,. \tag{10}$$

If (10) is violated, by the above construction of J there should be at least one index $i_1 < \tilde{i} < i_2$ such that $s \equiv 0$ on $\Delta_{\tilde{i}}$ and, therefore, $\Delta s_{\tilde{i}} = 0$. Including such indices $\tilde{i}$ additionally into J, we may assume that (10) holds for all subsequent indices from J.

With this notation we will show that

$$(11)\qquad \omega_2(2^{-k-2}, Ts)_p^p \le C \cdot \left\{ \omega_2(2^{-k-2}, s)_p^p + 2^{-k} \cdot \sum_{i\in J} |\Delta s_i|^p \right\}, \ 1 \le p < \infty .$$

To see this, let $0 < h \le 2^{-k-2}$, and denote by E the set of all $x \in R^1$ such that the interval $[x, x+2^{-k-1}]$ contains a simple zero-crossing of s. Since $s \ge 0$ resp. $s \le 0$ on $[x, x+2h]$ whenever $x \in R^1\backslash E$, we get

$$\|\Delta_h^2 Ts\|_{L_p}^p \ = \ \int_{R^1\backslash E} |\Delta_h^2 s|^p \, dx \ + \ \int_E |\Delta_h^2 Ts|^p \, dx .$$

The internal structure of E is very simple: it splits into small intervals I_ν associated with simple zeros ξ_ν resp. a pair $\xi'_\nu < \xi''_\nu$ of subsequent zeros of s satisfying $\xi''_\nu - \xi'_\nu < 2^{-k-1}$ (Figure 1 shows the typical situations).

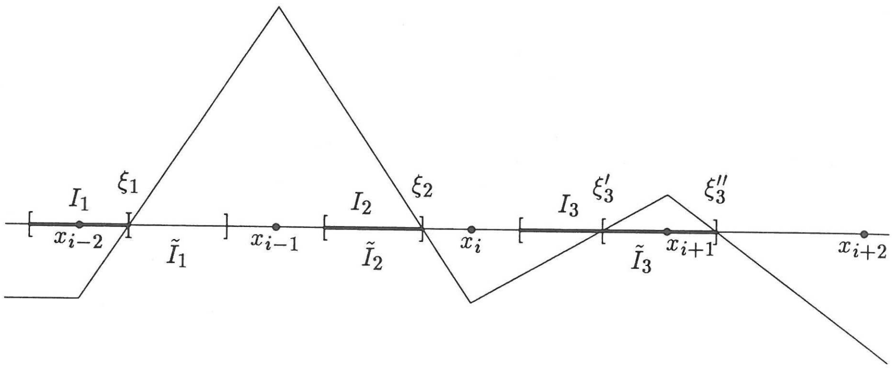

Figure 1.

Obviously, for each such interval we have (with a proper choice of $\tilde{I}_\nu$ as indicated in Figure 1)

$$\int_{I_\nu} |\Delta_h^2 Ts|^p \, dx \le c \cdot \left\{ \int_{I_\nu} |\Delta_h^2 s|^p \, dx + \int_{\tilde{I}_\nu} |s|^p \, dx \right\}$$

and observing that the $\tilde{I}_\nu$ are chosen disjoint and satisfying the inclusion $\cup_\nu \tilde{I}_\nu \subseteq \cup_{i\in J} \Delta_i$, we get

$$\int_E |\Delta_h^2 Ts|^p \, dx \le c \cdot \left\{ \int_E |\Delta_h^2 s|^p \, dx + \sum_{i\in J} \int_{\Delta_i} |s|^p \, dx \right\} .$$

But for $i \in J$ we have

$$\int_{\Delta_i} |s|^p \, dx \le c \cdot 2^{-k} \cdot |\Delta s_i|^p .$$

Putting things together and taking the infimum with respect to h, we arrive at (11).

With (11) at hand, we can now finish the proof of (9). Let $p = 1$, put $s = s^{(k)}$, and use the notation

$$\Delta s_i^{(k)} = s^{(k)}(x_i^k) - s^{(k)}(x_{i-1}^k)\,, \quad \Delta^2 s_i^{(k)} = \Delta s_{i+1}^{(k)} - \Delta s_i^{(k)}\,, \quad i, k \in Z\,.$$

If $J = \emptyset$ then (9) is straightforward. If J contains only one index i then from $s^{(k)} \in L_1$ we get $\lim_{j\to\infty} \Delta s_j^{(k)} = 0$, and by the identity

$$\Delta s_i^{(k)} = \Delta s_j^{(k)} - \sum_{r=i}^{j-1} \Delta^2 s_r^{(k)}$$

we obtain

$$\begin{aligned} 2^{-k} \cdot \sum_{i\in J} |\Delta s_i^{(k)}| &= 2^{-k} \cdot |\Delta s_i^{(k)}| \le \sum_{r=i}^{\infty} 2^{-k} \cdot |\Delta^2 s_r^{(k)}| \\ &\le c \cdot \omega_2(2^{-k}, s^{(k)})_1\,. \end{aligned}$$

The latter inequality follows from the particular case $p = 1$ of the elementary relation

$$(12) \qquad \omega_2(2^{-k-1}, s^{(k)})_p^p \approx 2^{-k} \cdot \sum_{r\in Z} |\Delta^2 s_r^{(k)}|^p\,, \quad s^{(k)} \in S^{(k)} \cap L_p\,,\ 1 \le p < \infty\,.$$

If $\mathrm{card}(J) > 1$ then, for any pair of subsequent indices $i_1 < i_2$ from J, we have according to (10)

$$|\Delta s_{i_1}^{(k)}| + |\Delta s_{i_2}^{(k)}| = |\Delta s_{i_2}^{(k)} - \Delta s_{i_1}^{(k)}| = \left| \sum_{r=i_1}^{i_2-1} \Delta^2 s_r^{(k)} \right| \le \sum_{r=i_1}^{i_2-1} |\Delta^2 s_r^{(k)}|\,.$$

This gives once again

$$2^{-k} \cdot \sum_{i\in J} |\Delta s_i^{(k)}| \le 2^{-k} \cdot \sum_{r\in Z} |\Delta^2 s_r^{(k)}| \le c \cdot \omega_2(2^{-k}, s^{(k)})_1\,.$$

Substituting into (11) (for $p = 1$) we arrive at (9), and Theorem 2 is completely proved. □

Remark 1. The proof of Theorem 2 carries over to the case $p < 1$ without substantial changes, i.e. we have

$$(13) \qquad \omega_2(t, Tf)_p \le C \cdot \omega_2(t, f)_p\,, \quad t > 0\,,\ f \in L_p\,,\ 0 < p \le 1\,.$$

This considerably improves the results of Section 1.2 in [RS]. Simple examples show that an inequality analogous to (7) resp. (13) can hold neither for $m = 2$ and $p > 1$ nor for $m > 2$ (and arbitrary p).

Theorem 3. (a) *There exists a function* $f_0 \in C_0^\infty$ *such that*

$$Tf_0 \notin B_{p,q}^{1+1/p}\,, \quad 1 \le p \le \infty\,,\; q < \infty\,.$$

(b) *There exists a function* $f_1 \in B_{p,\infty}^{1+1/p}$ *such that*

$$Tf_1 \notin B_{p,\infty}^{1+1/p}\,, \quad 1 \le p \le \infty\,.$$

Proof: The first example is quite obvious: Fix any $f_0 \in C_0^\infty$ such that $f_0(x) \equiv x$ on $[-1,1]$. Then the result in (a) follows from

$$\omega_m(t, Tf_0)_p \approx t^{1+1/p}\,,\; t \to 0\,,\; m = 2,3,\dots\,,\; 1 \le p \le \infty\,.$$

For the part (b), put

$$f_1(x) = -x \cdot \ln(x)\,, \quad x \in [0, 1/e]\,,$$

and extend this function to $[0,\infty)$ such that f_1 vanishes for $x > 1$ and is at least in C^3 on $(0,\infty)$. On $(-\infty,0)$, we define f_1 by a Hestenes type procedure

$$f_1(x) = \frac{5}{2} \cdot f_1(-x) - 15 \cdot f_1(\frac{-x}{3}) + \frac{27}{2} \cdot f_1(\frac{-x}{9})\,, \quad x < 0\,,$$

which is designed to preserve the smoothness of functions up to the differentiability order 3. It is easy to see that f_1 is continuous and vanishes outside $(-9,1)$. Moreover, checking the third order modulus of smoothness of f_1 first with respect to $[0,1/e]$ and then using the properties of the extension procedure, we get

$$\omega_3(t, f_1)_p \approx t^{1+1/p}\,, \quad t \to 0\,, \quad 1 \le p \le \infty\,,$$

which shows that $f_1 \in B_{p,\infty}^{1+1/p}$. The details are left to the reader. Note that functions analogous to f_1 have often been used as counterexamples for Zygmund-Lipschitz classes.

Now, observe that $f_1(x) < 0$ in some interval $[-x_0, 0)$ where $x_0 > 0$. This follows from the extension procedure as described above. Thus, for $p < \infty$ and $0 < t < x_0/3$, we get

$$\begin{aligned}
\omega_3(t, Tf_1)_p^p &\ge \int_0^t |Tf_1(x) - 3Tf_1(x-t) + 3Tf_1(x-2t) - Tf_1(x-3t)|^p\,dx \\
&= \int_0^t |2f_1(x) - (f_1(x) - 3f_1(x-t) + 3f_1(x-2t) - f_1(x-3t))|^p\,dx \\
&\ge c \cdot \int_0^t |f_1(x)|^p\,dx - C \cdot \omega_3(t, f_1)_p^p\,.
\end{aligned}$$

But

$$\int_0^t |f_1(x)|^p\,dx = \int_0^t (x \cdot |\ln(x)|)^p\,dx \ge c \cdot t^{p+1} \cdot |\ln(t)|^p\,,\; 0 < t < 1/e\,.$$

This shows that f_1 does not belong to $B_{p,\infty}^{1+1/p}$, $1 \le p < \infty$. The case $p = \infty$ can be dealt with analogously. Theorem 3 is established. ☐

Remark 2. Theorem 3 covers the "only if" part of Theorem 1 while the "if" part is proved till now for $p = 1$ (Theorem 2) and $p = \infty$ (see (6)). The remaining case $1 < p < \infty$ is contained in the next section. It is interesting to note that the mapping T preserves the Lipschitz class

$$\text{Lip } 1 = \{f \in C \,:\, \omega_1(t, f)_\infty = O(t)\,,\; t \to \infty\}$$

(cf. (5)) but does not preserve the Zygmund class $B^1_{\infty,\infty}$.

3. The case $1 < p < \infty$**.**

Throughout this section, let $1 < p < \infty$, $1/p < s < 1 + 1/p$, and $m = 2$ in the definition of the Besov spaces be fixed. Under these assumptions one has the continuous embedding of $B^s_{p,q}$ into C, and the linear splines $I^{(k)}f \in S^{(k)}$, $k \in Z$, interpolating $f \in B^s_{p,q}$ at the knots of $\pi^{(k)}$ (i.e. $I^{(k)}f(x_i^{(k)}) = f(x_i^{(k)})$, $i \in Z$) are well-defined.

From Theorem 2 and Corollary 1 of [O2] (cf. also [O1]), we have

Proposition 2. *If* $1 < p < \infty$, $1 \le q \le \infty$, $1/p < s < 1 + 1/p$, *then the sequence of interpolating splines* $\{I^{(k)}f\}$ *determines an equivalent norm on* $B^s_{p,q}$ *as follows:*

$$\|f\|_{B^s_{p,q}} \approx \|f\|_{L_p} + \|2^{ks} \cdot \|f - I^{(k)}f\|_{L_p}\|_{l_q}\,, \quad f \in B^s_{p,q}\,. \tag{14}$$

With the special choice $s^{(k)} = I^{(k)}f,\ k \in Z$, we can repeat a part of the argumentation in the proof of Theorem 2. Doing so we get

$$\omega_2(2^{-k}, Tf)_p \le \omega_2(2^{-k}, Ts^{(k)})_p + 4 \cdot \|f - s^{(k)}\|_{L_p}\,,$$
$$\omega_2(2^{-k}, s^{(k)})_p \le \omega_2(2^{-k}, f)_p + 4 \cdot \|f - s^{(k)}\|_{L_p}$$

and by (11),

$$\omega_2(2^{-k}, Ts^{(k)})_p^p \le c \cdot \{\omega_2(2^{-k}, s^{(k)})_p^p + 2^{-k} \cdot \sum_{i \in J^{(k)}} |\Delta s_i^{(k)}|^p\}$$

where $J^{(k)}$ denotes the index set J corresponding to $s = s^{(k)}$.

Thus, by Proposition 2,

$$\begin{aligned} \|Tf\|_{B^s_{p,q}} &\le c \cdot \{\|Tf\|_{L_p} + \|2^{ks} \cdot \omega_2(2^{-k}, Tf)_p\|_{l_q}\} \\ &\le c \cdot \{\|f\|_{L_p} + \|2^{ks} \cdot \omega_2(2^{-k}, f)_p\|_{l_q} + \|2^{ks} \cdot \|f - s^{(k)}\|_{L_p}\|_{l_q} \\ &\qquad + \|2^{ks} \cdot (2^{-k} \cdot \sum_{i \in J^{(k)}} |\Delta s_i^{(k)}|^p)^{1/p}\|_{l_q}\} \\ &\le c \cdot \{\|f\|_{B^s_{p,q}} + \|2^{k(s-1/p)} \cdot (\sum_{i \in J^{(k)}} |\Delta s_i^{(k)}|^p)^{1/p}\|_{l_q}\}\,. \end{aligned} \tag{15}$$

For estimating the terms

$$a_k \equiv \sum_{i\in J^{(k)}} |\Delta s_i^{(k)}|^p \,,$$

we use this time a more sophisticated representation of the first order differences $\Delta s_i^{(k)}$ by second order differences $\Delta^2 s_r^{(l)}$, $l \le k$. Fix some $k \in Z$, and introduce for each $i \in J^{(k)}$ the hierarchy of dyadic intervals from the coarser partitions containing $\Delta_i^{(k)}$:

$$\Delta_i^{(k)} \equiv \Delta_{i_0}^{(k)} \subset \Delta_{i_1}^{(k-1)} \subset \Delta_{i_1}^{(k-2)} \subset \dots$$

where the index sequence $\{i_0, i_1, i_2, \dots\}$ depends on i.

By the definition of the spline interpolants we have

$$\begin{aligned} \Delta s_i^{(k)} &= f(x_{i_0}^{(k)}) - f(x_{i_0-1}^{(k)}) \\ &= \begin{cases} \frac12 (f(x_{i_1}^{(k-1)}) - f(x_{i_1-1}^{(k-1)})) + \frac12 (f(x_{i_0}^{(k)}) - 2f(x_{i_0-1}^{(k)}) + (f(x_{i_0-2}^{(k)})) \\ \frac12 (f(x_{i_1}^{(k-1)}) - f(x_{i_1-1}^{(k-1)})) - \frac12 (f(x_{i_0+1}^{(k)}) - 2f(x_{i_0}^{(k)}) + (f(x_{i_0}^{(k)})) \end{cases} \\ &= \frac{1}{2}\cdot (\Delta s_{i_1}^{(k-1)} \pm \Delta^2 s_{r_0}^{(k)}) \end{aligned}$$

in dependence on whether $\Delta_{i_0}^{(k)}$ is the right or the left subinterval of $\Delta_{i_1}^{(k-1)}$. Repeating this consideration, we obtain

$$\Delta s_i^{(k)} = 2^{-l}\cdot \Delta s_{i_l}^{(k-l)} + \sum_{j=0}^{l-1} \pm 2^{-j-1}\cdot \Delta^2 s_{r_j}^{(k-j)} \,, \quad l = 1, 2, \dots \,.$$

Once again, we have three subcases. If $J^{(k)} = \emptyset$ then nothing remains to be estimated. If $\mathrm{card}(J^{(k)}) = 1$ then according to $2^{-l}\cdot \Delta s_{i_l}^{(k-l)} \to 0$ for $l \to \infty$ (which follows from the boundedness of f), we have (cf. also (12))

$$\begin{aligned} a_k = |\Delta s_i^{(k)}|^p &\le \left(\sum_{j=0}^{\infty} 2^{-j-1}\cdot |\Delta^2 s_{r_j}^{(k-j)}| \right)^p \le c\cdot \sum_{j=0}^{\infty} 2^{-j(p-\epsilon)}\cdot |\Delta^2 s_{r_j}^{(k-j)}|^p \\ &\le c\cdot 2^{-k(p-\epsilon)}\cdot \sum_{\nu=-\infty}^{k} 2^{-\nu(1+p-\epsilon)}\cdot \omega_2(2^{-\nu}, s^{(\nu)})_p^p \,, \quad \epsilon > 0 \,. \end{aligned} \tag{16}$$

For $\mathrm{card}(J^{(k)}) > 1$, consider any pair of subsequent indices $i < i'$ from $J^{(k)}$. According to (10), we have $\Delta s_i^{(k)} \cdot \Delta s_{i'}^{(k)} \le 0$. Obviously, there exists a smallest $l \ge 1$ such that $i_l = i'_l$ (the only exception occurs in the case $i \le 0 < i'$ which will be dealt with seperately). With this l we can estimate as follows :

$$\begin{aligned}
|\Delta s_i^{(k)}|^p + |\Delta s_{i'}^{(k)}|^p &\le |\Delta s_i^{(k)} - \Delta s_{i'}^{(k)}|^p \le \\
&\le \left| \left(\sum_{j=0}^{l-1} \pm 2^{-j-1} \cdot |\Delta^2 s_{r_j}^{(k-j)}| \right) - \left(\sum_{j=0}^{l-1} \pm 2^{-j-1} \cdot |\Delta^2 s_{r'_j}^{(k-j)}| \right) \right|^p \\
&\le c \cdot \sum_{j=0}^{l-1} 2^{-j(p-\epsilon)} \cdot \left(|\Delta^2 s_{r_j}^{(k-j)}|^p + |\Delta^2 s_{r'_j}^{(k-j)}|^p \right) .
\end{aligned}$$

In the exceptional case $i \le 0 < i'$ there exists a smallest l such that $i_l = 0$ and $i'_l = 1$. Running the same estimations, we have to add only one more term $2^{-l(p-\epsilon)} \cdot |\Delta^2 s_0^{(k-l)}|^p$ to the above sum.

A simple monotonicity argument shows that $r_j \le r'_j$ for any pair of subsequent indices and any $j = 0, \ldots, l-1$, with equality only for $j = l-1$ in the nonexceptional case. From this fact and the construction of the hierarchies $\{\Delta_{i_j}^{(k-j)}\}$, one easily observes that if we take the sum of the above estimates with respect to all pairs of subsequent indices $i < i'$ from $J^{(k)}$, any index r_j will not be repeated more than four times. Thus, we arrive once again at (16).

It remains to substitute (16) into (15). Then, fixing some $\epsilon < p(1 + 1/p - s)$, we obtain

$$\begin{aligned}
&\|Tf\|_{B^s_{p,q}} \le \\
&\le \; c \cdot \{\|f\|_{B^s_{p,q}} + \|2^{k(\epsilon - p(1+1/p-s))/p} \cdot (\sum_{\nu=-\infty}^{k} 2^{\nu(1+p-\epsilon)} \cdot \omega_2(2^{-\nu}, s^{(\nu)})_p^p)^{1/p} \|_{l_q}\} \\
&\le \; c \cdot \{\|f\|_{B^s_{p,q}} + \|2^{k(\epsilon - p(1+1/p-s))/p} \cdot \sum_{\nu=-\infty}^{k} 2^{\nu(1+p-\epsilon)/p} \cdot \omega_2(2^{-\nu}, s^{(\nu)})_p \|_{l_q}\} \\
&\le \; c \cdot \{\|f\|_{B^s_{p,q}} + \|2^{\nu s} \cdot \omega_2(2^{-\nu}, s^{(\nu)})_p\|_{l_q}\} \le c \cdot \|f\|_{B^s_{p,q}} ,
\end{aligned}$$

where in the last step the same inequalities have been used that already led to (15). This completes the proof of Theorem 1. □

Remark 3. Though the methods used for the proof of our Theorem 1 do not automatically generalize to other superposition functions g, the result itself indicates that one can expect some assertions for smoothness parameters $s > 1$ for larger classes of g (cf. [MM2], [MM3], [S1], [S2] for more information and references).

Note added in proof. After submitting the paper, we have been informed that the same result has been obtained by G. Bourdaud, Y. Meyer in [BM] using a completely different method.

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