On embeddings into $C_p(X)$ where X is Lindelöf

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Abstract. A.V. Arkhangel'skii asked that, is it true that every space Y of countable tightness is homeomorphic to a subspace (to a closed subspace) of $C_p(X)$ where X is Lindelöf? $C_p(X)$ denotes the space of all continuous real-valued functions on a space X with the topology of pointwise convergence. In this note we show that the two arrows space is a counterexample for the problem by showing that every separable compact linearly ordered topological space is second countable if it is homeomorphic to a subspace of $C_p(X)$ where X is Lindelöf. Other counterexamples for the problem are also given by making use of the Cantor tree. In addition, we remark that every separable supercompact space is first countable if it is homeomorphic to a subspace of $C_p(X)$ where X is Lindelöf.

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1. Introduction.

In this paper we assume all spaces are Tychonoff topological spaces. N denotes the positive integers. Let ω (resp. ω_1) denote the first infinite (resp. first uncountable) ordinal. As usual, we often regard an ordinal as the set of smaller ordinals. Unexplained notions and terminology are the same as in [2]. We denote by $C_p(X)$ the space of all continuous real-valued functions on a space X with the topology of pointwise convergence. Basic open sets of $C_p(X)$ are of the form $[x_1, x_2, \dots, x_k; U_1, U_2, \dots, U_k] = \{f \in C_p(X) : f(x_i) \in U_i \;\; i = 1, 2, \dots, k\}$, where $k \in N$, $x_i \in X$ and each U_i is an open subset of the real-line R. Canonically $C_p(X)$ is the dense subspace of R^X.

In [1, Problem 16] A.V. Arkhangel'skii asked that, is it true that every space Y of countable tightness is homeomorphic to a subspace of $C_p(X)$ where X is Lindelöf? In this note we give a compact first countable counterexample for the problem. In fact, we show that every separable compact linearly ordered topological space (abbreviated LOTS) is second countable if it is homeomorphic to a subspace of $C_p(X)$ where X is Lindelöf. Hence the two arrows space [2, 3.10. C] is a compact first countable counterexample for the problem. Moreover, we show that a first countable compactification of the Cantor tree and the one-point compactification of the Cantor tree are also counterexamples for the problem. In addition, we remark that every separable supercompact space is first countable if it is homeomorphic to a subspace of $C_p(X)$ where X is Lindelöf (cf. Theorem 2).

2. Results.

The following lemma seems to be well known. As a survey of a LOTS, see [3].

Lemma 1. *Let $(X,<)$ be a separable* LOTS*, where $<$ is an order on X. If the number of points with an immediate successor is countable, then X is second countable.*

Proof: Let D be a countable dense subset of X. Let P (resp. S) be the set of points with an immediate successor (resp. predecessor). Then the collection $\{A_x, B_x : x \in D \cup P \cup S\}$ is a countable open subbase of X, where $A_x = \{y \in X : y < x\}$ and $B_x = \{y \in X : y > x\}$. $\square$

Let Y be a subset of a space X. We set $e(Y,X) = \sup\{|S| : S \subset Y$ and S is closed discrete in $X\}$. If X is Lindelöf, then for every subset Y of X, $e(Y,X)$ is countable. We say that a subset Y of $C_p(X)$ separates points of X if for any distinct $x_1, x_2 \in X$ there is an $f \in Y$ such that $f(x_1) \neq f(x_2)$. For a set X we set $[X]^{\omega_1} = \{Y : Y \subset X$ and $|Y| = \omega_1\}$.

Theorem 1. *Let $(X,<)$ be a separable compact* LOTS*. If there is a subset Y of $C_p(X)$ which separates points of X with $e(Y, C_p(X)) \leq \omega$, then X is second countable.*

Proof: Let D be a countable dense subset of X. By Lemma 1 we have only to prove that the number of points with an immediate successor is countable. Assume the contrary. Let $\{a_\alpha : \alpha < \omega_1\}$ be a set of points with an immediate successor. Let b_α be the immediate successor of a_α. Since the number of isolated points in X is countable, we may assume that a_α and b_α are not isolated. Moreover, we may assume $D \cap \{a_\alpha, b_\alpha : \alpha < \omega_1\} = \emptyset$. For each $\alpha < \omega_1$ we choose an $f_\alpha \in Y$ such that $f_\alpha(a_\alpha) \neq f_\alpha(b_\alpha)$. Since the real-line R is second countable, there are an $M_0 \in [\omega_1]^{\omega_1}$ and disjoint closed intervals I_0, I_1 in R such that $f_\alpha(a_\alpha) \in \operatorname{Int} I_0$ and $f_\alpha(b_\alpha) \in \operatorname{Int} I_1$ for every $\alpha \in M_0$. For every $\alpha \in M_0$ we choose $c_\alpha, d_\alpha \in D$ such that $c_\alpha < a_\alpha$, $b_\alpha < d_\alpha$, $f_\alpha([c_\alpha, a_\alpha]) \subset \operatorname{Int} I_0$ and $f_\alpha([b_\alpha, d_\alpha]) \subset \operatorname{Int} I_1$. There are an $M_1 \in [M_0]^{\omega_1}$ and $c, d \in D$ such that $c < a_\alpha$, $b_\alpha < d$, $f_\alpha([c, a_\alpha]) \subset \operatorname{Int} I_0$ and $f_\alpha([b_\alpha, d]) \subset \operatorname{Int} I_1$ for every $\alpha \in M_1$.

We put $F = \{f_\alpha : \alpha \in M_1\}$. We would like to show that F is closed discrete in $C_p(X)$. Let π be the projection from $C_p(X)$ onto $C_p([c,d])$. Note that $\pi(f_\alpha) \neq \pi(f_\beta)$ for distinct $\alpha, \beta \in M_1$. We have only to show that $\pi(F)$ is closed discrete in $C_p([c,d])$. Assume $f \in \overline{\pi(F)}$ for an $f \in C_p([c,d])$, then obviously $f([c,d]) \subset I_0 \cup I_1$, $f(c) \in I_0$ and $f(d) \in I_1$. We set $b = \min\{x : f(x) \in I_1\}$ and $a = \max\{x : c \leq x < b\}$, b and a exist because X is compact and f is continuous. $[a, b; R - I_1, R - I_0]$ is a neighborhood of f, but it is easy to see that $|[a, b; R - I_1, R - I_0] \cap \pi(F)| \leq 1$. This means that $\pi(F)$ is closed discrete in $C_p([c,d])$. Consequently $e(Y, C_p(X)) > \omega$. $\square$

Corollary 1. *Let X be a separable compact* LOTS *which is homeomorphic to a subspace of $C_p(X)$. If Y has a dense subset D such that $e(D,Y) \leq \omega$, then X is second countable.*

Proof: Let j be an embedding from X into $C_p(Y)$. We define a map i from Y to $C_p(X)$ by $i(y)(x) = j(x)(y)$. Note that i is continuous and $i(Y)$ separates points of X. Since $i(D)$ is dense in $i(Y)$, $i(D)$ also separates points of X. Therefore, by Theorem 1, X is second countable because of $e(i(D), C_p(X)) \leq \omega$. □

Corollary 2. *Let X be a separable compact* LOTS*. If X is homeomorphic to a subspace of $C_p(Y)$ where Y is Lindelöf, then X is second countable.*

Example 1. Let X be the two arrows space of Alexandroff and Urysohn [2, 3.10. C]. It is known that X is first countable, separable compact LOTS which is not second countable. By virtue of Corollary 2, X is a compact first countable counterexample for Arkhangel'skii's problem.

It is known that every compact LOTS is supercompact [4]. However, the author does not know whether every separable supercompact space is second countable if it is homeomorphic to a subspace of $C_p(X)$ where X is Lindelöf. At least, such a space is first countable, see Theorem 2.

We give other counterexamples for the problem. Those spaces are given in [4, 1.1.17, 1.1.18] as examples of compact spaces which are not the continuous image of a supercompact space. We start with the Cantor tree. Let ${}^{\omega}2$ be the set of functions from ω to $2 = \{0, 1\}$. We set ${}^{\underset{\sim}{\omega}}2 = \{f \,|\, n : f \in {}^{\omega}2, n \in \omega\}$, where $f \,|\, n$ is the restriction of f to the domain n. In other words, ${}^{\underset{\sim}{\omega}}2$ is the set of finite sequences of 0's and 1's. Then the set $T = {}^{\underset{\sim}{\omega}}2 \cup {}^{\omega}2$ is a tree by the usual partial order $\subset$. Exactly speaking, $f \subset g$ means that g is an extension of f. We equip T with the tree topology. Every point of ${}^{\underset{\sim}{\omega}}2$ is isolated and basic open sets of a point $f \in {}^{\omega}2$ are of the form $\{f\} \cup \{f \,|\, m : m \geq n\}$, where $n \in \omega$. T is first countable and locally compact. Let K be the first countable compactification of T constructed by J. van Mill in [4, 1.1.17]. For the sake of completeness we cite the construction. Let K be $\{0\} \times {}^{\underset{\sim}{\omega}}2 \cup \{1, 2\} \times {}^{\omega}2$. We topologize K by assigning to each $x \in K$ a neighborhood base $\{U(x, n) : n \in \omega\}$. For $\langle i, f\rangle \in K$ define

$$U(\langle i, f\rangle, n) = \begin{cases} \{\langle i, f\rangle\} & \text{if } i = 0 \\ \{\langle i, f\rangle\} \cup \{\langle 0, f \,|\, m\rangle : m \geq n\} & \text{if } i = 1 \\ \{\langle j, g\rangle \in K : j \in 3,\ f \,|\, n \subset g\} - U(\langle 1, f\rangle, 0) & \text{if } i = 2. \end{cases}$$

The space K is separable, compact and first countable, and the open subspace $\{0\} \times {}^{\underset{\sim}{\omega}}2 \cup \{1\} \times {}^{\omega}2$ is homeomorphic to T.

Example 2. If Y is a subset of $C_p(K)$ which separates points of K, then $e(Y, C_p(K)) > \omega$. Hence, by the same argument as the two arrows space, K is not homeomorphic to any subspace of $C_p(X)$ where X is Lindelöf.

Proof: Fix a subset $\{f_\alpha : \alpha \in \omega_1\}$ of ${}^{\omega}2$. For every $\alpha \in \omega_1$ we choose $\varphi_\alpha \in Y$ such that $\varphi_\alpha(\langle 1, f_\alpha\rangle) \neq \varphi_\alpha(\langle 2, f_\alpha\rangle)$. Without loss of generality, we may assume $\varphi_\alpha(\langle 1, f_\alpha\rangle) < \varphi_\alpha(\langle 2, f_\alpha\rangle)$ for every $\alpha \in \omega_1$. We choose rational numbers p_α and q_α with $\varphi_\alpha(\langle 1, f_\alpha\rangle) < p_\alpha < q_\alpha < \varphi_\alpha(\langle 2, f_\alpha\rangle)$. For every $\alpha \in \omega_1$ there is an $n_\alpha \in \omega$ such that $\varphi_\alpha(x) < p_\alpha$ for every $x \in U(\langle 1, f_\alpha\rangle, n_\alpha)$ and $\varphi_\alpha(x) < q_\alpha$ for every $x \in U(\langle 2, f_\alpha\rangle, n_\alpha)$. Since the variations of (p_α, q_α) and n_α are countable, there are an $M_0 \in [\omega_1]^{\omega_1}$, rational numbers p and q, and $k \in \omega$ such that:

(1) $p < q$,
(2) $\varphi_\alpha(x) < p$ for every $\alpha \in M_0$ and $x \in U(\langle 1, f_\alpha\rangle, k)$,
(3) $\varphi_\alpha(x) > q$ for every $\alpha \in M_0$ and $x \in U(\langle 2, f_\alpha\rangle, k)$.

Moreover, since every n-th level of the tree T is finite, there is an $M_1 \in [M_0]^{\omega_1}$ such that:

(4) $f_\alpha \,|\, k = f_\beta \,|\, k$ for every $\alpha, \beta \in M_1$.

Note that $U(\langle 2, f_\alpha\rangle, k) \cap \{2\} \times {}^\omega 2 = U(\langle 2, f_\beta\rangle, k) \cap \{2\} \times {}^\omega 2$ for every $\alpha, \beta \in M_1$ by (4).

We set $\Phi = \{\varphi_\alpha : \alpha \in M_1\}$. We would like to show that Φ is closed and discrete in $C_p(K)$. It is easy to see that Φ is a discrete subspace of $C_p(K)$, because of $[\langle 1, f_\alpha\rangle; (-\infty, p)] \cap \Phi = \{\varphi_\alpha\}$ for every $\alpha \in M_1$. Assume that there is a $\varphi \in \overline{\Phi} - \Phi$. We set $s_0 = f_\alpha \,|\, k$ ($\alpha \in M_1$), then for every $\alpha \in M_1$ $\varphi_\alpha(\langle 0, s_0\rangle) = \varphi_\alpha(\langle 0, f_\alpha \,|\, k\rangle) < p$ by (2) and (4). Hence $\varphi(\langle 0, s_0\rangle) \le p$. Let s_1 and $\tilde{s}_1$ be the successors of s_0. We put $\Phi(s_1) = \{\varphi_\alpha \in \Phi : f_\alpha \,|\, k+1 = s_1\}$ and $\Phi(\tilde{s}_1) = \{\varphi_\alpha \in \Phi : f_\alpha \,|\, k+1 = \tilde{s}_1\}$, then $\Phi = \Phi(s_1) \cup \Phi(\tilde{s}_1)$. Without loss of generality, we may assume $\varphi \in \overline{\Phi(s_1)}$. For every $\varphi_\alpha \in \Phi(s_1)$ $\varphi_\alpha(\langle 0, s_1\rangle) = \varphi_\alpha(\langle 0, f_\alpha \,|\, k{+}1\rangle) < p$ by (2) and (4). By continuing this operation we obtain a sequence $s_0 \subset s_1 \subset s_2 \subset \dots$ in ${}^{\underset{\sim}{\omega}}2$ with $\varphi(\langle 0, s_i\rangle) \le p$ for every $i \in \omega$. Let $\langle 1, g\rangle \in \{1\} \times {}^\omega 2$ be the limit point of $\{\langle 0, s_i\rangle : i \in \omega\}$, then $\varphi(\langle 1, g\rangle) \le p$. We take an open neighborhood $[\langle 1, g\rangle; (-\infty, q)]$ of φ. It is not difficult to see that $|[\langle 1, g\rangle; (-\infty, q)] \cap \Phi| \le 1$. This is a contradiction. Φ is closed in $C_p(K)$. Consequently $e(Y, C_p(K)) > \omega$. $\square$

Let $T^* = \{\infty\} \cup T$ be the one-point compactification of T. T^* is a separable compact space with countable tightness. We need Šanin's lemma [2, 2.7.10 (c)] to prove the following Example 3.

Example 3. If Y is a subset of $C_p(T^*)$ which separates points of T^*, then $e(Y, C_p(T^*)) > \omega$. Hence T^* is not homeomorphic to any subspace of $C_p(X)$ where X is Lindelöf.

Proof: Fix a subset $\{f_\alpha : \alpha \in \omega_1\}$ of ${}^\omega 2$. For every $\alpha \in \omega_1$ we choose $\varphi_\alpha \in Y$ such that $\varphi_\alpha(\infty) \ne \varphi_\alpha(f_\alpha)$. Without loss of generality, we may assume $\varphi_\alpha(\infty) < \varphi_\alpha(f_\alpha)$ for every $\alpha \in \omega_1$. We choose rational numbers p_α and q_α with $\varphi_\alpha(\infty) < p_\alpha < q_\alpha < \varphi_\alpha(f_\alpha)$ and ${}^\omega 2 = \{f \in {}^\omega 2 : \varphi_\alpha(f) < p_\alpha\} \cup \{f \in {}^\omega 2 : \varphi_\alpha(f) > q_\alpha\}$. There are an $M_0 \in [\omega_1]^{\omega_1}$ and rational numbers p and q such that for every $\alpha \in M_0$ $\varphi_\alpha(\infty) < p < q < \varphi_\alpha(f_\alpha)$ and ${}^\omega 2 = \{f \in {}^\omega 2 : \varphi_\alpha(f) < p\} \cup \{f \in {}^\omega 2 : \varphi_\alpha(f) > q\}$. We set $F_\alpha = \{f \in {}^\omega 2 : \varphi_\alpha(f) > q\}$ for every $\alpha \in M_0$, then F_α is finite and $f_\alpha \in F_\alpha$. By Šanin's lemma [2, 2.7.10 (c)], there are an $M_1 \in [M_0]^{\omega_1}$ and a finite set F such that $F = F_\alpha \cap F_\beta$ for every distinct $\alpha, \beta \in M_1$. Since F is finite, without loss of generality we may assume $f_\alpha \in F_\alpha - F$ for every $\alpha \in M_1$. For every $\alpha \in M_1$ we choose $n_\alpha \in \omega$ with $\{f \,|\, n : f \in F,\, n \in \omega\} \cap \{f_\alpha \,|\, n : n \ge n_\alpha\} = \emptyset$ and $\varphi_\alpha(f_\alpha \,|\, n) > q$ for every $n \ge n_\alpha$. Every n-th level of the tree T is finite, therefore there are an $M_2 \in [M_1]^{\omega_1}$, $s_0 \in {}^{\underset{\sim}{\omega}}2$ and $k \in \omega$ such that:

(1) $s_0 = f_\alpha \,|\, k$ for every $\alpha \in M_2$,
(2) $\{f \,|\, n : f \in F,\, n \in \omega\} \cap \{f_\alpha \,|\, n : n \ge k\} = \emptyset$ for every $\alpha \in M_2$,
(3) $\varphi_\alpha(f_\alpha \,|\, n) > q$ for every $\alpha \in M_2$ and $n \ge k$.

We put $\Phi = \{\varphi_\alpha : \alpha \in M_2\}$. We would like to show that Φ is closed and discrete in $C_p(T^*)$. It is easy to see that Φ is a discrete subspace of $C_p(T^*)$, because of $[f_\alpha; (q, \infty)] \cap \Phi = \{\varphi_\alpha\}$ for every $\alpha \in M_2$. Assume that there is a $\varphi \in \overline{\Phi} - \Phi$. By the same argument as Example 2, we obtain a sequence $s_0 \subset s_1 \subset s_2 < \dots$ in ${}^{\omega}_{\sim}2$ with $\varphi(s_i) \geq q$ for every $i \in \omega$. Let $g \in {}^{\omega}2$ be the limit point of $\{s_i : i \in \omega\}$, then $\varphi(g) \geq q$ and $g \in {}^{\omega}2 - F$ by (2). $[g; (p, \infty)]$ is an open neighborhood of φ, it is not difficult to see that $|[g; (p, \infty)] \cap \Phi| \leq 1$. This is a contradiction. Φ is closed in $C_p(T^*)$. Consequently $e(Y, C_p(T^*)) > \omega$. □

In the rest of the paper, we prove Theorem 2.

Let $\mathcal{U}$ be an open collection in a space X. $\mathcal{U}$ is said to have property $(*)$ at $x \in X$ if $\mathcal{U}$ satisfies the following three conditions.

(1) $x \in U$ for every $U \in \mathcal{U}$,
(2) $y \neq x$, then $y \in X - \overline{U}$ for some $U \in \mathcal{U}$,
(3) if $\mathcal{F}$ is a collection of finite subsets in X such that every $U \in \mathcal{U}$ contains some $F \in \mathcal{F}$, then there exists a countable subcollection $\mathcal{H}$ in $\mathcal{F}$ such that every $U \in \mathcal{U}$ contains some $H \in \mathcal{H}$.

If a point $x \in X$ is a G_δ-point, then x has an open collection in X which has property $(*)$ at x. If X is compact, then the condition (2) implies that $\mathcal{U}$ is an open subbase at x, "open subbase at x" means that the collection of finite intersections from $\mathcal{U}$ is an open neighborhood base at x. We remark that an open neighborhood base at $x \in X$ has property $(*)$ at x iff $p(x, X) = \omega$, where $p(x, X)$ is the supertightness of x in X. For the definition of supertightness, see [5].

For a space X we set $\mathcal{U}_p(X) = \{[x; W_n] : x \in X, n \in N\}$, where $W_n = (-1/n, 1/n)$ and $[x; W_n] = \{f \in C_p(X) : f(x) \in W_n\}$. $\mathcal{U}_p(X)$ is the canonical open subbase at f_0 in $C_p(X)$, where f_0 is the constant function to 0.

Lemma 2. *$\mathcal{U}_p(X)$ has property $(*)$ at f_0 if X is Lindelöf.*

Proof: We have only to examine the condition (3). Let $\mathcal{F}$ be a collection of finite subsets of $C_p(X)$ such that every $[x; W_n] \in \mathcal{U}_p(X)$ contains some $F^n_x \in \mathcal{F}$. For $x \in X$ and $n \in N$, we set $U^n_x = \bigcap\{f^{\leftarrow}(W_n) : f \in F^n_x\}$. U^n_x is an open subset of X which contains x. Since $\mathcal{U}_n = \{U^n_x : x \in X\}$ is an open cover of X and X is Lindelöf, there exists a countable subset A_n of X such that $\mathcal{V}_n = \{U^n_x : x \in A_n\}$ is a subcover of $\mathcal{U}_n$. Put $\mathcal{H} = \{F^n_x : x \in \bigcup_{n \in N} A_n, n \in N\}$. Obviously $\mathcal{H}$ is a countable subset of $\mathcal{F}$. Let $[y; W_n]$ be an arbitrary element in $\mathcal{U}_p(X)$. Then we can find $x \in A_n$ with $y \in U^n_x$. This means $F^n_x \subset [y; W_n]$. □

Corollary 3. *Let Y be a subspace of $C_p(X)$ where X is Lindelöf. Then, for every $y \in Y$ there exists an open subbase at y in Y which has property $(*)$ at y.*

Proof: We may assume $y = f_0$, because $C_p(X)$ is homogeneous. The collection $\mathcal{U} = \{Y \cap U : U \in \mathcal{U}_p(X)\}$ is an open subbase at y in Y which has property $(*)$ at y. □

A closed subbase for a space is said to be binary if any of its linked (= every two of its members meet) subcollections has nonvoid intersection. A space is said to be supercompact if it has a closed subbase which is binary [4]. Let $\mathcal{S}$ be a binary subbase for a space X. For $A \subset X$, define $I(A) \subset X$ by $I(A) = \bigcap\{S \in \mathcal{S} : A \subset S\}$.

Lemma 3 [5, Lemma 2.1]. *Let $\mathcal{S}$ be a binary subbase for X and let $p \in X$. If U is a neighborhood of p and if A is a subset of X with $p \in \overline{A}$, then there is a subset B of A with $p \in \overline{B}$ and $I(B) \subset U$.*

Formally the following Lemma 4 is a slight generalization of Theorem 2.2 in [5], but the proof is quite similar to the proof of Mill and Mills. For the sake of completeness, we cite the proof.

Lemma 4. *Let Y be a separable space which is a continuous image of a supercompact space X. If there exists an open collection $\mathcal{U}$ in Y which has property $(*)$ at $y_0 \in Y$, then the character of y_0 in Y is countable.*

PROOF: Let $\mathcal{S}$ be a binary subbase for X which is closed under arbitrary intersections and let $f : X \to Y$ be a continuous surjection. Let $\{d_n : n \in \omega\}$ be a dense subspace of Y. For every $U \in \mathcal{U}$ we can find a finite subset $J(U)$ of $\mathcal{S}$ such that $\bigcup J(U) \subset f^{\leftarrow}(U)$ and $\bigcup J(U)$ is a neighborhood of $f^{\leftarrow}(y_0)$. Put $J(U) = \{S_0^U, S_1^U, \dots, S_{n(U)}^U\}$. Fix $c_n \in X$ with $f(c_n) = d_n$. For every $k \in \omega$, $U \in \mathcal{U}$ and $i \in \{0, 1, \dots n(U)\}$, we can choose a point $e_i^k(U) \in S_i^U \cap \left(\bigcap\{I(\{s, c_k\}) : s \in S_i^U\}\right)$, because $\mathcal{S}$ is binary. Put $E^k(U) = \{e_0^k(U), e_1^k(U), \dots, e_{n(U)}^k(U)\}$. Since every $U \in \mathcal{U}$ contains $f(E^k(U))$ and $\mathcal{U}$ has property $(*)$ at y_0, for every $k \in \omega$ there exists a countable subcollection $\mathcal{U}_k$ of $\mathcal{U}$ such that every $U \in \mathcal{U}$ contains $f(E^k(V))$ for some $V \in \mathcal{U}_k$. We claim that

$$\bigcap\{\bigcup J(U) : U \in \bigcup_{k\in\omega} \mathcal{U}_k\} \cap \overline{\{c_n : n \in \omega\}} = f^{\leftarrow}(y_0) \cap \overline{\{c_n : n \in \omega\}}\,.$$

This means that the character of y_0 in Y is countable. Assume that

$$x \in \bigcap\{\bigcup J(U) : U \in \bigcup_{k\in\omega} \mathcal{U}_k\} \cap \overline{\{c_n : n \in \omega\}} - f^{\leftarrow}(y_0) \cap \overline{\{c_n : n \in \omega\}}\,.$$

Select an $U_0 \in \mathcal{U}$ with $f(x) \in Y - \overline{U}_0$. By Lemma 3 there exists a subset C_0 of $\{c_n : n \in \omega\}$ such that $x \in \overline{C}_0$ and $I(C_0) \subset X - f^{\leftarrow}(\overline{U}_0)$. Note $x \in I(C_0)$. Fix an arbitrary $c_k \in C_0$. U_0 contains $f(E^k(V))$ for some $V \in \mathcal{U}_k$. We select an $S_{i_0}^V \in J(V)$ with $x \in S_{i_0}^V$. Then,

$$e_{i_0}^k(V) \in S_{i_0}^V \cap \left(\bigcap\{I(\{s, c_k\}) : s \in S_{i_0}^V\}\right) \subset I(\{x, c_k\}) \subset I(C_0) \subset X - f^{\leftarrow}(\overline{U}_0).$$

This is a contradiction, because of $e_{i_0}^k(V) \in f^{\leftarrow}(U_0)$. □

From Lemma 4 and Corollary 3, we obtain Theorem 2.

Theorem 2. *Let Y be a separable space which is a continuous image of a supercompact space. If Y is a subspace of $C_p(X)$ where X is Lindelöf, then Y is first countable.*

References

[1] Arkhangel'skii A.V., *Problems in C_p-theory*, in: J. van Mill and G.M. Reed, Eds., *Open Problems in Topology*, North-Holland, 1990, 601–615.
[2] Engelking R., *General Topology*, Sigma Series in Pure Math. 6, Helderman Verlag, Berlin, 1989.
[3] Lutzer D.J., *On generalized ordered spaces*, Dissertationes Math. **89** (1971).
[4] Mill J. van, *Supercompactness and Wallman spaces*, Mathematical Centre Tracts **85** (1977).
[5] Mill J. van, Mills C.F., *On the character of supercompact spaces*, Top. Proceed. **3** (1978), 227–236.

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