

On superpositionally measurable semi-Carathéodory multifunctions

WOJCIECH ZYGMUNT

Abstract. For multifunctions $F : T \times X \rightarrow 2^Y$, measurable in the first variable and semi-continuous in the second one, a relation is established between being product measurable and being superpositionally measurable.

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Introduction.

In various problems, one encounters a superposition of the type $F(t, G(t))$, where F and G are, in general, multifunctions and where it is often required that the mentioned superposition is measurable for every measurable multifunction G . A multifunction of such a property is called superpositionally measurable. It is known that under suitable assumptions on the spaces T , X and Y , Carathéodory multifunction $F : T \times X \rightarrow 2^Y$, i.e. measurable in t and continuous in x , is superpositionally measurable (see [1], [6], [8], [11], [12]). Unfortunately, when F is semicontinuous (in some sense) in x , such a multifunction, henceforth called semi-Carathéodory, may not be already superpositionally measurable. In this note we discuss the connection between superpositional measurability and product measurability of semi-Carathéodory multifunctions.

Preliminaries.

Thus, given two arbitrary nonempty sets \mathcal{X} , \mathcal{Y} and denoting by $2^{\mathcal{Y}}$ the family of all subsets of \mathcal{Y} , by a multifunction $\Phi : \mathcal{X} \rightarrow 2^{\mathcal{Y}}$ we mean a mapping Φ of a domain \mathcal{X} and a range contained in $2^{\mathcal{Y}}$. Let Σ be a σ -field of subsets of \mathcal{X} and let \mathcal{Y} be a topological space. A multifunction $\Phi : \mathcal{X} \rightarrow 2^{\mathcal{Y}}$ is said to be Σ -measurable (resp. weakly Σ -measurable) if the set $\Phi^-(A) = \{x \in \mathcal{X} : \Phi(x) \cap A \neq \emptyset\}$ belongs to Σ for every closed (resp. open) set $A \subset \mathcal{Y}$. It is known (see [2], [3], [13]) that when (\mathcal{X}, Σ) is a complete measurable space (i.e. there is a complete σ -finite measure defined on Σ) and \mathcal{Y} is a Polish space (i.e. \mathcal{Y} is separable and metrisable by a complete metric), then these two measurability concepts coincide for a closed values multifunction. Let \mathcal{X} be a topological space, too. A multifunction $\Phi : \mathcal{X} \rightarrow 2^{\mathcal{Y}}$ is said to be lower

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(resp. upper) semicontinuous if the set $\Phi^-(A)$ is open (resp. closed) in \mathcal{X} for every open (resp. closed) set $A \subset \mathcal{Y}$.

Henceforth we use the following notations:

- (T, \mathcal{A}) - is a complete measurable space;
- X - is Polish space;
- $\mathcal{B}(X)$ - is a σ -field of Borel subsets of X ;
- $\mathcal{A} \otimes \mathcal{B}(X)$ - is a product σ -field on $T \times X$ (i.e. the minimal σ -field containing all products $A \times B$, with $A \in \mathcal{A}$, $B \in \mathcal{B}(X)$);
- Y - is a topological space.

Let us consider a multifunction $F : T \times X \rightarrow 2^Y$. F is called product measurable if it is $\mathcal{A} \otimes \mathcal{B}(X)$ -measurable and it is called superpositionally measurable if, for every \mathcal{A} -measurable multifunction $G : T \rightarrow 2^X$ with nonempty closed values, a multifunction $F_G : T \rightarrow 2^Y$ defined by the superposition $F_G(t) = F(t, G(t))$ is \mathcal{A} -measurable, where $F(t, G(t))$ denotes the sum of sets $F(t, x)$ when $x \in G(t)$. Further, we say that F is a lower (resp. upper) semi-Carathéodory multifunction if $F(\cdot, x)$ is \mathcal{A} measurable for each fixed $x \in X$ and $F(t, \cdot)$ is lower (resp. upper) semicontinuous for each fixed $t \in T$.

Main results.

Theorem 1. *Every product measurable multifunction $F : T \times X \rightarrow 2^Y$ is superpositionally measurable.*

PROOF: Let a closed set $A \subset Y$ and an \mathcal{A} -measurable multifunction $G : T \rightarrow 2^X$ with nonempty closed values be given. In view of $\mathcal{A} \otimes \mathcal{B}(X)$ -measurability of F , the set $F^-(A) = \{(t, x) \in T \times X : F(t, x) \cap A \neq \emptyset\}$ belongs to $\mathcal{A} \otimes \mathcal{B}(X)$. On the other hand, the assumptions on multifunction G imply $\mathcal{A} \otimes \mathcal{B}(X)$ -measurability of its graph, i.e. $\text{gr}G = \{(t, x) \in T \times X : x \in G(t)\} \in \mathcal{A} \otimes \mathcal{B}(X)$ (see [3, Theorem 3.5]). Thus,

$$\{(t, x) \in T \times X : F(t, x) \cap A \neq \emptyset, x \in G(t)\} = F^-(A) \cap \text{gr}G \in \mathcal{A} \otimes \mathcal{B}(X).$$

Hence, using the Projection Theorem, see [2, Theorem III. 23], [10, Theorem 4]), we obtain

$$\begin{aligned} F_G^-(A) &= \{t \in T : F_G(t) \cap A \neq \emptyset\} = \{t \in T : F(t, G(t)) \cap A \neq \emptyset\} = \\ &= \text{proj}_T(F^-(A) \cap \text{gr}G) \in \mathcal{A} \end{aligned}$$

which, in view of the optionality of A and G , means that the multifunction F is superpositionally measurable (here proj_T denotes the projection of $T \times X$ onto T).

In particular, we can see from Theorem 1 that both an upper and a lower semi-Carathéodory product measurable multifunction is superpositionally measurable. The converse implication holds only for the upper semi-Carathéodory multifunction. Namely, we have

Theorem 2. *Every upper semi-Carathéodory superpositionally measurable multifunction is product measurable.*

PROOF: Let us first notice that superpositional measurability implies \mathcal{A} -measurability of sets $\{t \in T : F(t, B) \cap A \neq \emptyset\}$ for each closed $A \subset Y$ and $B \subset X$. Now, for

every $t \in T$ and each closed $A \subset Y$, let us put $\Phi_A(t) = \{x \in X : F(t, x) \cap A \neq \emptyset\}$. In virtue of the upper semicontinuity of $F(t, \cdot)$, the set $\Phi_A(t)$ is closed in X . We claim that thus defined closed valued multifunction $\Phi_A : T \rightarrow 2^X$ is \mathcal{A} -measurable. Indeed, for every closed $B \subset X$, we have:

$$\begin{aligned} \Phi_A^-(B) &= \{t \in T : \Phi_A(t) \cap B \neq \emptyset\} = \{t \in T : \bigvee_{x \in B} x \in \Phi_A(t)\} = \\ &= \{t \in T : \bigvee_{x \in B} F(t, x) \cap A \neq \emptyset\} = \{t \in T : F(t, B) \cap A \neq \emptyset\} \in \mathcal{A}. \end{aligned}$$

Thus, by [3, Theorem 3.5] its graph $\text{gr}\Phi_A$ belongs to the σ -field $\mathcal{A} \otimes \mathcal{B}(X)$. But

$$\begin{aligned} F^-(A) &= \{(t, x) \in T \times X : F(t, x) \cap A \neq \emptyset\} = \{(t, x) \in T \times X : x \in \Phi_A(t)\} = \\ &= \text{gr}\Phi_A \in \mathcal{A} \otimes \mathcal{B}(X) \end{aligned}$$

which completes the proof of product measurability of F . □

Example. In the case of the lower semi-Carathéodory multifunction the superpositional measurability does not generally imply the product measurability. In order to show it, we shall use the multifunction $\Phi : T \times I \rightarrow 2^{\mathbf{R}}$ constructed by A. Kucia in her paper [4, Example]. Let I be the interval $[0, 1]$, \mathcal{A} — the σ -field on I generated by one-point sets, let $(T, \mathcal{A}) = (I, \mathcal{A})$. It is easy to see that (T, \mathcal{A}) is complete measurable space and that a real-valued function $\varphi : T \rightarrow \mathbf{R}$ is measurable if and only if φ is eventually constant, i.e. there exists a countable set $N \subset I$ such that φ is constant on $I \setminus N$. Hence it follows, by “Castaing representation” theorem (see [2, Theorem III. 8]), that a multifunction $G : T \rightarrow 2^{\mathbf{R}}$ with nonempty closed values is \mathcal{A} -measurable if and only if G is eventually constant. The multifunction Φ is defined as follows:

$$\Phi(t, x) = \begin{cases} \{t\} & \text{if } |t - x| = \frac{1}{n} \text{ for some positive integer } n, \text{ or } t = x, \\ I & \text{in the other case.} \end{cases}$$

A. Kucia showed that such a multifunction Φ is lower semi-Carathéodory and is not $\mathcal{A} \otimes \mathcal{B}(I)$ measurable (see [4, p. 240]). Here we shall prove that Φ is superpositionally measurable. To this end, let us first consider an arbitrary but fixed nonempty closed set $B \subset I$. Two cases are possible: 1° B is countable, or 2° B is uncountable. In the second case the following condition holds:

$$(*) \quad \forall t \in I \quad \exists x \in B \quad x \neq t \text{ and } x \neq t \pm \frac{1}{n} \text{ for } n = 1, 2, \dots$$

Indeed, otherwise there exists $\bar{t} \in I$ such that for any $x \in B$ we have $x = \bar{t}$ or $x = \bar{t} + \frac{1}{n}$ or $x = \bar{t} - \frac{1}{n}$ for some $n \in \mathbf{N}$. But then the set B must be countable, which is impossible.

Now let an \mathcal{A} -measurable multifunction $G : T \rightarrow 2^I$ with nonempty closed values be given. There exist a nonempty closed set $B \subset I$ and a countable set $N \subset I$ such

that $G(t) = B$ for $t \in I \setminus N$. For every closed $A \subset \mathbf{R}$ let us denote $N_A = \{t \in N : \Phi_G(t) \cap A \neq \emptyset\} = \{t \in N : \Phi(t, G(t)) \cap A \neq \emptyset\}$. It is obvious that $N_A \in \mathcal{A}$.

Now, if $B = \{b_1, b_2, \dots\}$ then for every closed $A \subset \mathbf{R}$ we have

$$\begin{aligned} \Phi_G^-(A) &= \{t \in T : \Phi_G(t) \cap A \neq \emptyset\} = \{t \in I \setminus N : \Phi_G(t) \cap A \neq \emptyset\} \cup N_A = \\ &= \{t \in I \setminus N : \Phi(t, B) \cap A \neq \emptyset\} \cup N_A = \\ &= \bigcup_{n=1}^{\infty} \{t \in I \setminus N : \Phi(t, b_n) \cap A \neq \emptyset\} \cup N_A \in \mathcal{A}. \end{aligned}$$

If B is uncountable then from (*) and the definition of Φ we get $\Phi(t, B) = I$ for every $t \in I \setminus N$. Hence for every closed $A \subset \mathbf{R}$ we have

$$\begin{aligned} \Phi_G^-(A) &= \{t \in T : \Phi_G(t) \cap A \neq \emptyset\} = \{t \in I \setminus N : \Phi_G(t) \cap A \neq \emptyset\} \cup N_A = \\ &= \{t \in I \setminus N : \Phi(t, B) \cap A \neq \emptyset\} \cup N_A = \{t \in I \setminus N : I \cap A \neq \emptyset\} \cup N_A \in \mathcal{A} \end{aligned}$$

because

$$\{t \in I \setminus N : I \cap A \neq \emptyset\} = \begin{cases} \emptyset & \text{if } I \cap A = \emptyset, \\ T \setminus N & \text{if } I \cap A \neq \emptyset. \end{cases}$$

Finally we can see that $\Phi_G^-(A)$ belongs to \mathcal{A} for every \mathcal{A} -measurable multifunction $G : T \rightarrow 2^I$ with nonempty closed values and each closed set $A \subset \mathbf{R}$, what completes the proof of the superpositional measurability of the multifunction $\Phi : T \times X \rightarrow 2^{\mathbf{R}}$.

Conclusion.

It is known that many multifunctions $F : T \times X \rightarrow 2^Y$ which describe the right hand of differential inclusions are exactly semi-Carathéodory multifunctions. Hence, it would also be useful to know if such multifunctions are superpositionally measurable. From **Theorem 1** it follows that a multifunction $F : T \times X \rightarrow 2^Y$ is superpositionally measurable, provided it is product measurable. However, in general, the semi-Carathéodory multifunction is not product measurable (see, for instance, [9, p. 31]). Below we give three most often recurring cases when the semi-Carathéodory multifunction is product measurable.

1. ([8, Theorem 3.3], [7, Proposition 2.3])
 (T, \mathcal{A}) is a measurable space, X is a separable metric space, Y is a metric space, $F : T \times X \rightarrow 2^Y$ is an upper semi-Carathéodory multifunction with nonempty closed values and such that $F(t, \cdot)$ is lower semi-continuous with respect to a Hausdorff topology.
2. ([8, Theorem 3.4])
 (T, \mathcal{A}) is a complete measurable space, $X = Y$ is a separable reflexive Banach space, $F : T \times X \rightarrow 2^X$ is a lower semi-Carathéodory multifunction with nonempty closed convex values and such that $F(t, \cdot)$ is upper semi-continuous from X to X_ω , where X_ω denotes space X with weak topology.
3. ([14, Theorems 3 and 4])
 (T, \mathcal{A}, μ) is a measure space with a Hausdorff compact metric space T and

a Borel σ -finite, regular and complete measure μ defined on \mathcal{A} , X is a Polish space, Y — a separable metric space, $F : T \times X \rightarrow 2^Y$ is a lower (resp. upper) semi-Carathéodory multifunction with nonempty closed values and such that the following condition — due to Scorza-Drăgăni — is satisfied:

“for every $\varepsilon > 0$ there exists a closed subset T_ε of T , with $\mu(T \setminus T_\varepsilon) < \varepsilon$, such that $F|_{T_\varepsilon \times X}$ is lower (resp. upper) semi-continuous.”

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INSTYTUT MATEMATYKI UMCS, PL. M.CURIE-SKŁODOWSKIEJ 1, 20–031 LUBLIN, POLAND

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