On a condition for the pseudo radiality of a product

A. Bella, J. Gerlits

Abstract. A sufficient condition for the pseudo radiality of the product of two compact Hausdorff spaces is given.

Keywords: compact, product, semi radial, pseudo radial

Classification: 54A25, 54B10, 54D55

It was recently shown by I. Juhàsz and Z. Szentmiklossy [6] that the assumption $2^\omega \leq \omega_2$ implies that pseudo radiality is finitely productive in the class of compact Hausdorff spaces. In ZFC similar results have been obtained only for special subclasses of pseudo radial spaces (see [4] and [5]). In the present note we give a sufficient condition for the pseudo radiality of a finite product. In particular our result generalizes both those in [4] and [5].

Recall that a subset A of a topological space X is said to be κ-closed ($<\kappa$-closed) provided that $\overline{B} \subset A$ for any set $B \subset A$ with $|B| \leq \kappa(|B| < \kappa)$.

A space X is pseudo radial if for any non closed subset A of X there exists a sequence $\{x_\xi : \xi \in \kappa\} \subset A$ which converges outside A.

A space X is radial if for any set $A \subset X$ and any $x \in \overline{A}$ there exists a sequence in A which converges to x.

We say that the sequence $\{x_\xi : \xi \in \kappa\}$ strictly converges to x if in addition κ is regular and $x \notin \overline{\{x_\xi : \xi \in \nu\}}$ for any $\nu \in \kappa$.

A space X is almost radial if for any non closed set $A \subset X$ there exists a sequence in A which strictly converges to a point outside A.

In this paper we consider a particular subclass of pseudo radial spaces. Precisely we look at the following condition:

(*) for any non κ-closed set $A \subset X$ there exists a sequence $\{x_\xi : \xi \in \lambda\} \subset A$, with $\lambda \leq \kappa$, which converges outside A.

For short, a space satisfying (*) will be called semi radial.

Lemma 1. *Radial $\Longrightarrow$ semi radial $\Longrightarrow$ almost radial $\Longrightarrow$ pseudo radial.*

Proof: We need only to show that semi radial $\Longrightarrow$ almost radial. Thus let X be a semi radial space and A a non closed subset of X. Let κ be the smallest cardinal such that A is not κ-closed. Fix a sequence $\{x_\xi : \xi \in \lambda\} \subset A$, with $\lambda \leq \kappa$ which converges to a point $x \in \overline{A} \setminus A$. Now it is enough to observe that, being A $<\kappa$-closed, we must have $\lambda = \kappa$, κ regular and $x \notin \overline{\{x_\xi : \xi \in \nu\}}$ for any $\nu \in \kappa$.

□

Theorem. *The product of two compact Hausdorff pseudo radial spaces is pseudo radial provided that one of them is semi radial.*

Proof: Assume by contradiction that there exist a compact Hausdorff semi radial space X and a compact Hausdorff pseudo radial space Y such that the product $Z = X \times Y$ is not pseudo radial. Then there is a chain closed set $A \subset Z$ which is not closed. Chain closed means that the set contains the limit points of all converging sequences contained in it. Let κ be the minimum cardinal such that the set A is not κ-closed and choose a set $B \subset A$ satisfying $|B| = \kappa$ and $\overline{B} \setminus A \neq \emptyset$. Select a point $(x, y) \in \overline{B} \setminus A$. As $\{x\} \times Y$ is pseudo radial and $A \cap \{x\} \times Y$ is chain closed, there exists a closed neighbourhood V of (x, y) in Z such that $V \cap A \cap \overline{\{x\} \times Y} = \emptyset$. Changing A with $A \cap V$, we can assume that $x \notin \pi_X(A)$. Since $x \in \overline{\pi_X(B)}$, it follows that $\pi_X(A)$ is not κ-closed. Now, being X semi radial, we can fix a sequence $\{x_\xi : \xi \in \lambda\} \subset \pi_X(A)$ which converges to a point $\hat{x} \in X \setminus \pi_X(A)$. Observe that the set $\pi_X(A)$ is $< \kappa$-closed and consequently $\lambda = \kappa$ and κ is a regular cardinal. For any $\xi \in \kappa$, choose y_ξ such that $(x_\xi, y_\xi) \in A$. Next select a complete accumulation point $p \in Y$ of the set $\{y_\xi : \xi \in \kappa\}$. Since the point $(\hat{x}, p) \notin A$, we can assume as before that $p \notin \pi_Y(A)$. For any $\xi \in \kappa$, denote by C_ξ the closure in Y of the set $\{y_\nu : \nu \in \xi\}$ and put $C = \bigcup_{\xi \in \kappa} C_\xi$. As $\pi_Y(A)$ is $< \kappa$-closed, it follows that $C \subset \pi_Y(A)$. Moreover, since $p \in \overline{C} \setminus \pi_Y(A)$, it follows that C is not closed in Y. Thus there exists a regular cardinal λ and a sequence $\{y'_\xi : \xi \in \lambda\} \subset C$ which converges to a point $\hat{y} \notin C$. Both $\lambda < \kappa$ and $\lambda > \kappa$ cannot occur and so we have $\lambda = \kappa$. By taking a subsequence, we can also assume that $y'_\xi \notin C_\xi$ for any $\xi \in \kappa$. Fix a function $f : \kappa \to \kappa$ such that $y'_\xi \in \overline{\{y_\nu : \xi \in \nu \in f(\xi)\}}$ for any ξ. Using now the fact that A is $< \kappa$-closed, we can select a point $x'_\xi \in \overline{\{x_\nu : \xi \in \nu \in f(\xi)\}}$ in such a way that $(x'_\xi, y'_\xi) \in A$ for any $\xi \in \kappa$. To finish observe that the sequence $\{(x'_\xi, y'_\xi) : \xi \in \kappa\}$ must converge to the point $(\hat{x}, \hat{y}) \notin A$ — a contradiction. □

A consequence of the previous Theorem is the following result obtained by Z. Frolík and G. Tironi:

Corollary 1 (see [5]). *The product of compact Hausdorff radial space and a compact Hausdorff pseudo radial space is pseudo radial.*

Recall that the chain character of a pseudo radial space X, denoted by $\sigma_c(X)$, is the smallest cardinal κ such that for any non closed set $A \subset X$ there exists a sequence $\{x_\xi : \xi \in \lambda\} \subset A$, with $\lambda \leq \kappa$, which converges outside A.

In [4], for a pseudo radial space X, the following property was considered:

(**) for any set $A \subset X$ $\sigma_c(\overline{A}) \leq |A|$.

It was also observed in [2] that every quasi monolithic (in particular monolithic) compact Hausdorff space is a pseudo radial space with the property (**).

Lemma 2. *A pseudo radial space with the property* (**) *is semi radial.*

Taking into account Lemma 2, another consequence of the Theorem is the following result proved in [4]:

Corollary 2. *The product of two compact Hausdorff pseudo radial spaces is pseudo radial provided that one of them has the property* (**).

Notice that there are compact semi radial spaces which are neither radial nor satisfying (**).

We do not know any example of a compact almost radial space which is not semi radial. In view of the preceding Theorem, it should be actually very interesting to prove that such an example cannot exist. It is in fact still unknown whether the product of two compact almost radial spaces is pseudo radial.

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Department of Mathematics, University of Messina, Contrada Papardo – Salita Sperone, Messina, Italy

Mathematical Research Institute of the Hungarian Academy of Sciences, 1364 Budapest, PF127 Hungary

(*Received* November 20, 1991)