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A concept of absolute continuity and a Riemann type integral

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Abstract. We present a descriptive definition of a multidimensional generalized Riemann integral based on a concept of generalized absolute continuity for additive functions of sets of bounded variation.

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In [P_1] (or alternately, in [P_4]), a multidimensional Riemann type integral with a strong geometric appeal was defined on all bounded sets of bounded variation. The aim of this paper is to obtain a descriptive definition of the integral by introducing a notion of *generalized absolute continuity* (abbreviated as ACG^*) for additive functions of sets of bounded variation. We show that in dimension one, ACG^* functions are almost everywhere differentiable, and use this fact to improve the integration by parts results presented in [B-Gi-P, Section 4]. Our ACG^* functions generalize the ACG_δ functions defined in [Go, Definition 1].

1. Notations and definitions.

All functions we consider are real-valued. The algebraic and lattice operations among functions on the same set are defined pointwise.

Throughout this paper, m is a positive integer. The set of all real numbers is denoted by $\mathbf{R}$, and the m-fold Cartesian product of $\mathbf{R}$ is denoted by $\mathbf{R}^m$. For points $x = (\xi_1, \dots, \xi_m)$ and $y = (\eta_1, \dots, \eta_m)$ in $\mathbf{R}^m$ and $\varepsilon > 0$, let $x \cdot y = \sum_{i=1}^m \xi_i \eta_i$, $|x| = \sqrt{x \cdot x}$, and $U_{x,\varepsilon} = \{y \in \mathbf{R}^m : |x - y| < \varepsilon\}$. If $E \subset \mathbf{R}^m$, then $d(E)$, $\mathrm{cl}\, E$, ∂E and $|E|$ denote, respectively, the diameter, closure, boundary, and the Lebesgue outer measure of E. The terms "measure" and "measurable", as well as the expressions "almost all" and "almost everywhere", refer exclusively to the Lebesgue measure in $\mathbf{R}^m$.

Let $E \subset \mathbf{R}^m$. We say that an $x \in \mathbf{R}^m$ is a *nondispersion* point of E whenever

$$\limsup_{\varepsilon \to 0+} \frac{|E \cap U_{x,\varepsilon}|}{|U_{x,\varepsilon}|} > 0\,.$$

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The set of all nondispersion points of E is called the *essential closure* of E, denoted by $\mathrm{cl}^* E$. The *essential boundary* of E is the set $\partial^* E = \mathrm{cl}^* E \cap \mathrm{cl}^*(\mathbf{R}^m - E)$. Clearly, $\mathrm{cl}^* E \subset \mathrm{cl}\, E$ and $\partial^* E \subset \partial E$.

The $(m-1)$-dimensional Hausdorff outer measure $\mathcal{H}$ in $\mathbf{R}^m$ is defined so that it is the counting measure if $m = 1$, and agrees with the Lebesgue outer measure in $\mathbf{R}^{m-1}$ if $m > 1$. A *thin set* is a subset of $\mathbf{R}^m$ whose $\mathcal{H}$ measure is σ-finite.

A bounded set $A \subset \mathbf{R}^m$ is called a BV set (BV for *bounded variation*) whenever the number $\|A\| = \mathcal{H}(\partial^* A)$, called the *perimeter* of A, is finite. By [Fe, Section 2.10.6 and Theorem 4.5.11], the family BV of all BV sets coincides with the collection of all bounded measurable subsets of $\mathbf{R}^m$ whose De Giorgi perimeters in $\mathbf{R}^m$ are finite (see [M-M, Section 2.1.2]). If $E \subset \mathbf{R}^m$ we denote by BV_E the family of all BV subsets of E.

The *regularity* of a BV set A is the number

$$r(A) = \begin{cases} \frac{|A|}{d(A)\|A\|} & \text{if } d(A)\|A\| > 0, \\ 0 & \text{otherwise.} \end{cases}$$

A *partition in* a set $A \subset \mathbf{R}^m$ is a collection (possibly empty)

$$P = \{(A_1, x_1), \dots, (A_p, x_p)\}$$

where $A_1, \dots, A_p$ are disjoint sets from BV_A and $x_i \in \mathrm{cl}^* A_i$ for $i = 1, \dots, p$; the sets $\bigcup_{i=1}^p A_i$ and $\{x_1, \dots, x_p\}$ are called the *body* and *anchor* of P, denoted by $\cup P$ and $\mathrm{an} P$, respectively.

A *gage* in a set $A \subset \mathbf{R}^m$ is a nonnegative function δ defined on $\mathrm{cl}^* A$ and such that its *null set*

$$N_\delta = \{x \in \mathrm{cl}^* A : \delta(x) = 0\}$$

is thin. A *caliber* is a sequence $\eta = \{\eta_j\}_{j=1}^\infty$ of positive real numbers.

Definition 1.1. Let $\varepsilon > 0$, let $\eta = \{\eta_j\}_{j=1}^\infty$ be a caliber, and let δ be a gage in a BV set A. We say that a partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A is:

(1) *ε-regular* if $r(A_i) > \varepsilon$ for $i = 1, \dots, p$;
(2) *δ-fine* if $d(A_i) < \delta(x_i)$ for $i = 1, \dots, p$;
(3) *(ε, η)-approximating* if $A - \cup P$ is the union of disjoint BV sets $B_1, \dots B_k$ such that $\|B_j\| < 1/\varepsilon$ and $|B_j| < \eta_j$ for $j = 1, \dots, k$;
(4) a *partition of* A if $\cup P = A$.

Note. Let η be a caliber and let δ be a gage in a BV set A. In [P_2, Proposition 2.5] we established that a δ-fine ε-regular and (ε, η)-approximating partition in A exists for each positive $\varepsilon < \kappa_m$, where κ_m is a positive constant depending only on the dimension m.

Let $A \subset \mathbf{R}^m$ and let G be a function on BV_A. If f is a function on $\mathrm{cl}^* A$, we set

$$\sigma(f, P; G) = \sum_{i=1}^p f(x_i) G(A_i)$$

for each partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A.

Definition 1.2. Let A be a BV set and let G be a function on BV_A. We say that a function f on $\mathrm{cl}^* A$ is *G-integrable* in A if there is a real number I having the following property: given $\varepsilon > 0$, we can find a gage δ in A and a caliber η so that

$$\bigl|\sigma(f, P; G) - I\bigr| < \varepsilon$$

for each partition P in A which is simultaneously ε-regular, δ-fine, and (ε, η)-approximating.

The family of all G-integrable functions in a BV set A is denoted by $\mathcal{I}(A; G)$. The number I of Definition 1.2 (necessarily unique) is called the *G-integral* of f over A, denoted by $\int_A f\, dG$. For the basic properties of the G-integral we refer to [P_1, Section 2].

The main object of our investigation is the case when $G(A) = |A|$ for each $A \in BV$. We signify this situation by replacing $\sigma(f, P; G)$, $\mathcal{I}(A; G)$, and $\int_A f\, dG$ by the symbols $\sigma(f, P)$, $\mathcal{I}(A)$, and $\int_A f$, respectively, and by using the words "integrability" and "integral" instead of "G-integrability" and "G-integral", respectively. We note that the definition of $\int_A f$ can be extended to functions f defined only almost everywhere in a BV set A ([P_1, Remark 3.3]).

Definition 1.3. Let A be a BV set. A *division* of A is a finite disjoint family of BV sets whose union is A. A function F on BV_A is called:

(1) *additive* if $F(A) = \sum_{D \in \mathcal{D}} F(D)$ for each division $\mathcal{D}$ of A;
(2) *continuous* if given $\varepsilon > 0$, there is an $\alpha > 0$ such that $|F(B)| < \varepsilon$ for each $B \in BV_A$ with $\|B\| < 1/\varepsilon$ and $|B| < \alpha$.

The next proposition is crucial for our exposition. It shows that a continuous additive function satisfying the condition of Henstock's lemma with respect to a point function f ([P_1, Proposition 2.4]) is already an *indefinite* G-integral of f. In particular, we obtain an alternate definition of the G-integral, which does not rely on (ε, η)-approximating partitions.

Proposition 1.4. *Let A be a BV set and let G be a function on BV_A. A function f on $\mathrm{cl}^* A$ is G-integrable in A if and only if there is an additive continuous function F on BV_A having the following property: given $\varepsilon > 0$, we can find a gage δ in A so that*

$$\sum_{i=1}^{p} \bigl|f(x_i)G(A_i) - F(A_i)\bigr| < \varepsilon$$

for each partition P in A which is simultaneously ε-regular and δ-fine. In particular, $\int_B f\, dG = F(B)$ for every $B \in BV_A$.

Proof: The necessity follows from [P_1, Propositions 2.3 and 2.4]; the sufficiency is established by an argument analogous to that employed in the proof of [P_2, Theorem 3.3]. □

Let $A \in BV$ and $x \in \mathrm{cl}^* A$. We say that a function F on BV_A is *derivable* at x if there exists a finite

$$\lim \frac{F(B_n)}{|B_n|}$$

for each sequence $\{B_n\}$ in BV_A such that $x \in \mathrm{cl}^* B_n$ for $n = 1, 2, \dots$, $\lim d(B_n) = 0$, and $\inf r(B_n) > 0$. When all these limits exist they have the same value, denoted by $F'(x)$.

Proposition 1.5. *Let $A \in BV$, $f \in \mathcal{I}(A)$, and let $F(B) = \int_B f$ for each $B \in BV_A$. For almost all $x \in \mathrm{cl}^* A$, the function F is derivable at x and $F'(x) = f(x)$.*

The proof of this proposition is given in [P_3, Section 2].

2. Generalized absolute continuity.

Let $A \subset \mathbf{R}^m$ and let F be an additive function on BV_A. We say that F is AC^* in a set $E \subset \mathrm{cl}^* A$ if given $\varepsilon > 0$, there are an $\alpha > 0$ and a gage δ in A such that $|F(\cup P)| < \varepsilon$ for each ε-regular δ-fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A with $\mathrm{an}P \subset E$ and $|\cup P| < \alpha$. It is easy to see that the condition $|F(\cup P)| < \varepsilon$ can be replaced by $\sum_{i=1}^p |F(A_i)| < \varepsilon$; in particular, if F is AC^* in E, it is also AC^* in *every subset* of E. We note that F is *automatically* AC^* in any set $E \subset \mathrm{cl}^* A$ that is thin.

Definition 2.1. Let A be a BV set and let F be an additive continuous function on BV_A. We say that F is ACG^* whenever there are sets $E_n \subset \mathrm{cl}^* A$, $n = 1, 2, \dots$, such that the set $\mathrm{cl}^* A = \bigcup_{n=1}^\infty E_n$ and F is AC^* in each E_n.

In view of the preceding remarks, in Definition 2.1 we may assume that the sets E_n are *disjoint*. The next lemma shows that each ACG^* function is an AC_{BV} function defined in [P_3, Definition 2.5] (cf. [Go, Lemma 2]).

Lemma 2.2. *Let A be a BV set and let F be an ACG^* function on BV_A. Given a set $E \subset \mathrm{cl}^* A$ with $|E| = 0$ and $\varepsilon > 0$, there is a gage δ in A such that*

$$\sum_{i=1}^p \bigl|F(A_i)\bigr| < \varepsilon$$

for each ε-regular δ-fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A with $\mathrm{an}P \subset E$.

Proof: Choose disjoint sets $E_n \subset \mathrm{cl}^* A$, $n = 1, 2, \dots$, so that $\mathrm{cl}^* A = \bigcup_{n=1}^\infty E_n$ and F is AC^* in each E_n. Let $E \subset \mathrm{cl}^* A$, $|E| = 0$, and $\varepsilon > 0$. For $n = 1, 2, \dots$, there are $\alpha_n > 0$ and gages δ_n in A such that $|F(\cup Q)| < \varepsilon/2^n$ for each $(\varepsilon/2^n)$-regular δ_n-fine partition Q in A with $\mathrm{an}Q \subset E_n$ and $|\cup Q| < \alpha_n$. Find an open set U_n containing $E \cap E_n$ for which $|U_n| < \alpha_n$, and for each $x \in U_n$ denote by $\rho_n(x)$ the distance from x to ∂U_n. Now define a gage δ in A by letting

$$\delta(x) = \begin{cases} \min\{\delta_n(x), \rho_n(x)\} & \text{if } x \in E \cap E_n,\ n = 1, 2, \dots, \\ 1 & \text{if } x \in \mathrm{cl}^* A - E. \end{cases}$$

If P is an ε-regular δ-fine partition in A with $\mathrm{an}P \subset E$, then $P_n = \{(B, x) \in P : x \in E \cap E_n\}$, $n = 1, 2, \dots$, is an $(\varepsilon/2^n)$-regular δ_n-fine partition in A with $\mathrm{an}P_n \subset E_n$ and $|\cup P_n| \le |U_n| < \alpha_n$. Consequently

$$\bigl|F(\cup P)\bigr| \le \sum_{n=1}^\infty \bigl|F(\cup P_n)\bigr| < \sum_{n=1}^\infty \frac{\varepsilon}{2^n} = \varepsilon\,.$$

$\square$

Proposition 2.3. *Let A be a BV set and let $f \in \mathcal{I}(A)$. Then the function $F : B \mapsto \int_B f$ on BV_A is ACG^*.*

PROOF: For $n = 1, 2, \dots$, let $E_n = \{x \in \mathrm{cl}^* A : |f(x)| \leq n\}$. Then $\mathrm{cl}^* A = \bigcup_{n=1}^{\infty} E_n$ and we show that F is AC^* on each E_n. To this end, fix an integer $n \geq 1$, choose an $\varepsilon > 0$, and let $\alpha = \varepsilon/(2n)$. By Proposition 1.4, there is a gage δ in A such that

$$\sum_{i=1}^{p} \bigl| f(x_i)|A_i| - F(A_i) \bigr| < \frac{\varepsilon}{2}$$

for each ε-regular δ-fine partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. Suppose that, in addition, an$P \subset E_n$ and $|\cup P| < \alpha$. Then

$$\sum_{i=1}^{p} \bigl|F(A_i)\bigr| \leq \sum_{i=1}^{p} \bigl| f(x_i)|A_i| - F(A_i) \bigr| + \sum_{i=1}^{p} \bigl|f(x_i)\bigr| \cdot |A_i| < \frac{\varepsilon}{2} + n\alpha = \varepsilon\,.$$

□

Theorem 2.4. *Let $A \in BV$ and let f be a function on $\mathrm{cl}^* A$. Then $f \in \mathcal{I}(A)$ if and only if there is an ACG^* function F on BV_A such that for almost all $x \in \mathrm{cl}^* A$ the function F is derivable at x and $F'(x) = f(x)$. In particular, $\int_B f = F(B)$ for each $B \in BV_A$.*

PROOF: The necessity follows from Propositions 2.3 and 1.5. The sufficiency is a consequence of Lemma 2.2 and [P_3, Theorem 2.6]. □

Corollary 2.5. *Let $A \in BV$ and let G be an ACG^* function on BV_A that is derivable almost everywhere in $\mathrm{cl}^* A$. A function f on $\mathrm{cl}^* A$ belongs to $\mathcal{I}(A; G)$ if and only if there is an ACG^* function F on BV_A such that for almost all $x \in \mathrm{cl}^* A$ the function F is derivable at x and $F'(x) = f(x)G'(x)$. In particular, $\int_B f\, dG = F(B)$ for each $B \in BV_A$.*

PROOF: By Theorem 2.4, $G' \in \mathcal{I}(A)$ and $G(B) = \int_B G'$ for each $B \in BV_A$. The corollary follows from [P_1, Proposition 2.7] by another application of Theorem 2.4.

□

Question 2.6. If A is a BV set, is each ACG^* function on BV_A derivable almost everywhere in $\mathrm{cl}^* A$?

For the one-dimensional case, the affirmative answer to Question 2.6 is given in Corollary 3.3.

3. In the real line.

Each equivalence class of BV sets modulo the sets of measure zero contains a unique *essentially closed* set, i.e., a set equal to its essential closure. In dimension one such a set is a finite union of nondegenerate compact intervals (see [V, Section 6]). This observation enables us to view the one-dimensional BV sets as figures defined in [B-Gi-P, Section 1].

Throughout this section we assume that $m = 1$. A *segment* is a nondegenerate compact subinterval of $\mathbf{R}$, i.e., an interval $[a, b]$ where $a, b \in \mathbf{R}$ and $a < b$. A *figure* is a finite union of segments, and the family of all figures is denoted by $\mathcal{F}$. For $E \subset \mathbf{R}$ we let $\mathcal{F}_E = \{A \in \mathcal{F} : A \subset E\}$. We say that figures A and B *overlap* whenever $|A \cap B| > 0$.

Let g be a function on a figure A. If the connected components of $B \in \mathcal{F}_A$ are the segments $[a_k, b_k]$, $k = 1, \dots, n$, we let

$$g(B) = \sum_{k=1}^{n} \bigl[g(b_k) - g(a_k)\bigr] .$$

Since, up to an order, the segments $[a_k, b_k]$ are determined uniquely by the figure B, the function g on A induces a function on $\mathcal{F}_A$, also denoted by g, which is additive with respect to nonoverlapping figures. It is easy to verify that the function $g : \mathcal{F}_A \to \mathbf{R}$ has a unique extension to an additive function $G : BV_A \to \mathbf{R}$, and that G is continuous if and only if the function $g : A \to \mathbf{R}$ is continuous. We say that $g : A \to \mathbf{R}$ is AC^* in a set $E \subset A$ whenever G is AC^* in E; similarly, we say that $g : A \to \mathbf{R}$ is ACG^* whenever G is ACG^*.

As the elements and subsets of a figure A are denoted by lower and upper case letters, respectively, denoting the point function $A \to \mathbf{R}$ and the induced set function $\mathcal{F}_A \to \mathbf{R}$ by the same symbol will cause no confusion.

An *$\mathcal{F}$-partition in $A \subset \mathbf{R}$* is a collection (possibly empty) $\{(A_1, x_1), \dots, (A_p, x_p)\}$ such that $A_1, \dots, A_p$ are nonoverlapping figures in $\mathcal{F}_A$ and $x_i \in A_i$ for $i = 1, \dots, p$. The *ε-regular* and *δ-fine* $\mathcal{F}$-partitions, as well as an *$\mathcal{F}$-partition of* a figure A, are defined in the obvious way. For an $\mathcal{F}$-partition P, the meaning the symbols $\cup P$, $\mathrm{an}P$, $\sigma(f, P; g)$, and $\sigma(f, P)$ is also obvious.

Combining [P$_2$, Proposition 3.5] and [B-Gi-P, Proposition 3.6], we obtain the following theorem.

Theorem 3.1. *Let f and g be functions on a figure A. Then $f \in \mathcal{I}(A; g)$ if and only if either of the following conditions are satisfied.*

(i) *There is a real number I having the following property: given $\varepsilon > 0$, we can find a* **positive** *gage δ in A so that*

$$\bigl|\sigma(f, P; g) - I\bigr| < \varepsilon$$

for each ε-regular δ-fine $\mathcal{F}$-partition P **of** *A. In particular, $\int_A f\,dg = I$.*

(ii) *There is a continuous function F on A having the following property: given $\varepsilon > 0$, we can find a* **positive** *gage δ in A so that*

$$\sum_{i=1}^{p} \bigl|f(x_i)g(A_i) - F(A_i)\bigr| < \varepsilon$$

for each ε-regular δ-fine $\mathcal{F}$-partition $\{(A_1, x_1), \dots, (A_p, x_p)\}$ in A. In particular, $\int_B f\,dg = F(B)$ for every $B \in \mathcal{F}_A$.

Proposition 3.2. *A continuous function F on a figure A is AC^* in a set $E \subset A$ if and only if given $\varepsilon > 0$, there are an $\alpha > 0$ and a* **positive** *gage δ in A such that $|F(\cup P)| < \varepsilon$ for each ε-regular δ-fine $\mathcal{F}$-partition $P = \{(A_1, x_1), \dots, (A_p, x_p)\}$ in A with an$P \subset E$ and $|\cup P| < \alpha$. The condition $|F(\cup P)| < \varepsilon$ can be replaced by $\sum_{i=1}^{p} |F(A_i)| < \varepsilon$.*

Proof: Observe that thin subsets of $\mathbf{R}$ are countable. Using this fact and the continuity of F, it is easy to replace an arbitrary gage in A by a positive gage. □

Corollary 3.3. *Each ACG^* function on a figure A is an ACG_* function on A in the usual sense* (see [S, Chapter VII, Section 8]). *In particular, each ACG^* function on A is derivable almost everywhere in A.*

Proof: Since the regularity of any segment equals 1/2, it follows from Proposition 3.2 that each ACG^* function on A is ACG_δ on A in the sense of [Go, Definition 1]. Now it suffices to apply [Go, Theorem 6].

□

Theorem 3.4. *Let g be an ACG^* function on a figure A. A function f on A belongs to $\mathcal{I}(A; g)$ if and only if there is an ACG^* function F on A such that $F' = fg'$ almost everywhere in A. In particular, $\int_B f\, dg = F(B)$ for each $B \in \mathcal{F}_A$.*

The theorem follows directly from Corollaries 2.5 and 3.3.

Corollary 3.5. *The class of all ACG^* functions on the unit segment $A = [0, 1]$ is* **properly** *contained in the class of all ACG_* functions on A.*

Proof: By [B-Gi-P, Example 3.8], there is a Denjoy-Perron integrable function f on A which is not integrable in A. Let $F(0) = 0$, and for each $x \in (0, 1]$ let $F(x)$ be the Denjoy-Perron integral of f over the segment $[0, x]$. Then F is ACG_* on A and $F' = f$ almost everywhere in A. As f in not integrable in A, it follows from Theorem 3.4 that F is not ACG^* on A. □

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