Linear rescaling of the stochastic process

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Abstract. Discussion on the limits in distribution of processes Y under joint rescaling of space and time is presented in this paper. The results due to Lamperti (1962), Weissman (1975), Hudson & Mason (1982) and Laha & Rohatgi (1982) are improved here.

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1. Introduction.

This paper considers the stochastic processes $X = (X(t),\ t \in R_+^k)$ which are constructed as limits in distribution of linear rescaled processes (1). Rescaling of processes is closely related to the study of critical phenomena in physics. That part of physics is usually referred to as renormalization theory.

The problem was firstly discussed by Lamperti (1962), who considered (1) in the case β is vanishing, X is continuous in probability and $X(1)$ has a nondegenerate distribution. For non-vanishing β, there is a result reached by Weissman (1975) which gives a relation to the extreme value limit distributions as well. A more general case in several dimensions was investigated by Hudson & Mason (1982) and Laha & Rohatgi (1982). All these results lead to a self-similar process (for information and references see Vervaat (1987)) because the process X is assumed to be continuous in probability.

The present paper shows that (1) requires the limit process X to have the property (3) which generalizes self-similarity. Moreover, the further propositions discuss the functions $A,\ B$ used in (3).

2. Main result.

Real-valued stochastic processes $Y = (Y(t),\ t \in R_+^k)$ are considered where R_+ is the set of positive real numbers. The i-th coordinate of $t \in R_+^k$ is denoted by t_i and functions are used coordinatewise, e.g. $st = (s_1t_1, \dots, s_kt_k)$, $t^{-1} = (t_1^{-1}, \dots, t_k^{-1})$, etc. To avoid any misunderstanding, the symbol $\xrightarrow[s\to\infty]{d}$ means the convergence in distribution and always refers to finite-dimensional distributions, the expression $s \to \infty$ denotes $\min\limits_{i=1}^{k} s_i \to +\infty$.

Theorem. *Let $X = (X(t),\ t \in R_+^k)$, $Y = (Y(t),\ t \in R_+^k)$ be real-valued stochastic processes and let $\alpha : R_+^k \to R_+$, $\beta : R_+^k \to R$ be such that*

$$(1) \qquad \big(\alpha(s)\, Y(st) + \beta(s),\ t \in R_+^k\big) \xrightarrow[s\to\infty]{d} X = \big(X(t),\ t \in R_+^k\big)\,.$$

Then the following two cases are possible:

(a) *There exists* $c \in R$ *such that* $X(t) = c$ *a.s.* $\forall\, t \in R_+^k$.

(b) *The finite limits*

$$\begin{aligned} \alpha(s)\ \alpha(as)^{-1} &\xrightarrow[s\to\infty]{} A(a) \in R_+, \\ \beta(s) - \beta(as)\ \alpha(s)\ \alpha(as)^{-1} &\xrightarrow[s\to\infty]{} B(a) \in R \end{aligned} \tag{2}$$

exist and the formula

$$\big(X(at),\ t \in R_+^k\big) \stackrel{d}{=} \big(A(a)\ X(t) + B(a),\ t \in R_+^k\big) \tag{3}$$

holds for every $a \in R_+^k$.

The functions A, B defined in the case (b) of Theorem have special structures which are discussed in the following propositions. Proposition 1 discusses the function A and Proposition 2 describes the function B.

Proposition 1. *Let* $\alpha : R_+^k \to R_+$ *and* $\alpha(s)\ \alpha(as)^{-1} \xrightarrow[s\to\infty]{} A(a) \in R_+$ *for every* $a \in R_+^k$. *Then* $A(a) = A_1(a_1)\dots A_k(a_k)$ *and every partial function* A_i *is determined as follows:*

(I) *If there exists* $\gamma \in R$ *such that*

$$\begin{aligned} &\forall\, t > 2 : \alpha\big(\underbrace{2^n,\dots,2^n}_{i-1}, t^n, \underbrace{2^n,\dots,2^n}_{k-i}\big)\ t^{n\gamma} \xrightarrow[n\to+\infty]{} 0, \\ &\text{or } \forall\, t > 2 : \alpha\big(\underbrace{2^n,\dots,2^n}_{i-1}, t^n, \underbrace{2^n,\dots,2^n}_{k-i}\big)\ t^{n\gamma} \xrightarrow[n\to+\infty]{} +\infty, \\ &\text{then } A_i(t) = t^{H_i} \ \text{ for some } H_i \in R. \end{aligned}$$

(II) *Else,* A_i *is a discontinuous function with the property*

$$\forall\, s,t \in R_+ : A_i(st) = A_i(s)\ A_i(t).$$

Proposition 2. *If* (1), (2) *hold, then the function* A *determines the function* B.

(I) *The case* $A \equiv 1$ *specifies the function* B *in the following form:*

$$\begin{aligned} &\forall\, a \in R_+^k \ \text{ it is } \ B(a) = B_1(a_1) + \dots + B_k(a_k) \\ &\text{and every partial function } B_i \text{ is either of the form} \\ &B_i(t) = Q_i \ln t \ \text{ for some } Q_i \in R, \\ &\text{or } B_i \text{ is a discontinuous function with the property} \\ &\forall\, s,t \in R_+ : B_i(st) = B_i(s) + B_i(t). \end{aligned}$$

(II) *In the case* $A \not\equiv 1$ *there exists* $Q \in R$ *such that*

$$\forall\, a \in R_+^k, \ \ \textit{it is } B(a) = Q(1 - A(a)). \ \textit{Moreover,}$$

(a) *for* $\forall\, a, b \in R_+^k,\ b > 1,\ A(b) > 1$

$$\beta(ab^n) \xrightarrow[n\to+\infty]{} Q \ \ \textit{takes place.}$$

(b) *for* $\forall\, a, b \in R_+^k,\ b > 1,\ A(b) < 1$ *there exists* $\widetilde{Q}(a,b) \in R$ *such that*

$$\frac{\beta(ab^n)}{\alpha(ab^n)} \xrightarrow[n\to+\infty]{} \widetilde{Q}(a,b),$$

$$Y(ab^n) \xrightarrow[n\to+\infty]{d} -\widetilde{Q}(a,b)$$

$$\beta(ab^n) - \widetilde{Q}(a,b)\ \alpha(ab^n) \xrightarrow[n\to+\infty]{} Q.$$

3. Proofs.

Proof of Theorem: If we exclude the case (a), the limit process X is either a non-deterministic function or a non-constant deterministic function.

(i) Assume that $\forall\, t \in R_+^k$ there is $c(t) \in R$ such that $X(t) = c(t)$ a.s. and that there is $\hat{t} \in R_+^k : c(\hat{t}) \neq c(1)$.
Then

$$\alpha(st)\ \big(Y(st\hat{t}) - Y(st)\big) \xrightarrow[s\to\infty]{d} c(\hat{t}) - c(1) \neq 0$$

and

$$\big[\alpha(s)\ \alpha(st)^{-1}\big]\alpha(st)\ \big(Y(st\hat{t}) - Y(st)\big) \xrightarrow[s\to\infty]{d} c(t\hat{t}) - c(t) \in R$$

for every $t \in R_+^k$, according to the assumption.
Consequently, $\alpha(s)\ \alpha(st)^{-1} \xrightarrow[s\to\infty]{} A(t) \in \langle 0, +\infty)$. Thus $\alpha(st)\ \alpha(s)^{-1} = \alpha(st)\ \alpha((st)t^{-1})^{-1} \xrightarrow[s\to\infty]{} A(t^{-1}) \in \langle 0, +\infty)$ as well, and so $A(t) \in R_+$ and $A(t^{-1}) = A(t)^{-1}$.
Consider the equality

$$\begin{aligned}\alpha(s)\ Y(st) + \beta(s) &= \alpha(s)\ \alpha(st)^{-1}\big(\alpha(st)\ Y(st) + \beta(st)\big) + \\ &+ \beta(s) - \alpha(s)\ \alpha(st)^{-1}\beta(st) \ \text{ under } s \text{ tending to infinity.}\end{aligned}$$

We immediately derive

$$\beta(s) - \alpha(s)\ \alpha(st)^{-1}\beta(st) \xrightarrow[s\to\infty]{} c(t) - A(t)\ c(1).$$

Thus (2) is proved in that case.

(ii) For the second possibility, there exists $\hat{t} \in R_+^k$ such that $P(X(\hat{t}) \neq c) > 0$ for every $c \in R$. Let us define, for $s, t \in R_+^k$ and $y \in R$,

$$\begin{aligned} H(s,t,y) &= \frac{y - \beta(s) + \beta(st)\ \alpha(s)\ \alpha(st)^{-1}}{\alpha(s)\ \alpha(st)^{-1}}\,, \\ \underline{H}(t,y) &= \underline{\lim}_{\,s\to\infty} H(s,t,y), \\ \overline{H}(t,y) &= \overline{\lim}_{\,s\to\infty} H(s,t,y). \end{aligned}$$

These functions have the following properties.

(iia) Fix $t \in R_+^k$ and for $y \in R$ select $s_n \in R_+^k$ such that $s_n \xrightarrow[n\to+\infty]{} \infty$ and $H(s_n,t,y) \xrightarrow[n\to+\infty]{} \overline{H}(t,y)$. Then the inequality

$$(*)\qquad \begin{aligned} P\big(X(tq) \le y\big) &\ge \overline{\lim}_{\,s\to\infty} P\big(\alpha(s)\ Y(stq) + \beta(s) \le y\big) = \\ &= \overline{\lim}_{\,s\to\infty} P\big(\alpha(st)\ Y(stq) + \beta(st) \le H(s,t,y)\big) \ge \\ &\ge \overline{\lim}_{\,n\to+\infty} P\big(\alpha(s_n t)\ Y(s_n tq) + \beta(s_n t) < H(s_n,t,y)\big) \ge \\ &\ge P\big(X(q) < \overline{H}(t,y)\big) \end{aligned}$$

holds for every $q \in R_+^k$. Analogically one can verify the inequality

$$(**)\qquad P\big(X(tq) < y\big) \le P\big(X(q) \le \underline{H}(t,y)\big)\,.$$

for every $q \in R_+^k$ and $y \in R$. A consequence of $(*)$, $(**)$ is the relation

$$(***)\qquad P\big(\underline{H}(t,z) < X(q) < \overline{H}(t,y)\big) = 0$$

for every $q \in R_+^k$, $y, z \in R$, $y < z$. It follows immediately from the observation that

$$\begin{aligned} P\big(X(tq) < z\big) \le P\big(X(q) \le \underline{H}(t,z)\big) \le P\big(X(q) < \overline{H}(t,y)\big) \le \\ \le P\big(X(tq) \le y\big) \le P\big(X(tq) < z\big) \end{aligned}$$

in the case when

$$\underline{H}(t,z) < \overline{H}(t,y).$$

The functions $H(s,t,.)$ are linear and nondecreasing. Therefore the function $\underline{H}(t,.)$ and $\overline{H}(t,.)$ is concave nondecreasing and convex nondecreasing, respectively. Moreover, the domination $\underline{H}(t,y) \le \overline{H}(t,y)$ always holds.

(iib) We will show that there are some $\gamma : R_+^k \to R_+$ and $\delta : R_+^k \to R$ such that $\underline{H}(t,y) = \overline{H}(t,y) = \gamma(t)y + \delta(t)$ for each $y \in R$, $t \in R_+^k$. Let us fix $t \in R_+^k$ and consider all the possibilities.

(α) The case $\underline{H}(t,.) \equiv -\infty$ or $\overline{H}(t,.) \equiv +\infty$ is impossible, because, according to $(*)$, $(**)$, such relations are in contradiction with the assumption that the process X is real-valued.

(β) Suppose that there is a point $\hat{y} \in R$ such that $\underline{H}(t,y) = -\infty$ for every $y < \hat{y}$. Necessarily, $\overline{H}(t,y) = \infty$ for every $y > \hat{y}$ because of the linearity of the functions $H(s,t,.)$. Thus $P(X(\hat{t}) < \hat{y}) = P(X(\hat{t}) > \hat{y}) = 0$ according to $(*)$, $(**)$, for the choice $q = \hat{t}t^{-1}$. But this is in contradiction with our assumption $P(X(\hat{t}) \neq \hat{y}) > 0$.

(γ) The case if there is a point $\hat{y} \in R$ such that $\overline{H}(t,y) < +\infty$ for every $y < \hat{y}$ and $\overline{H}(t,y) = +\infty$ for every $y > \hat{y}$ gives a contradiction by a dual way to (β).

(δ) Now, we know that both the functions $\underline{H}(t,.)$, $\overline{H}(t,.)$ are real-valued and continuous. Hence the relation $(***)$ implies that even

$$P\big(\underline{H}(t,y) < X(\hat{t}) < \overline{H}(t,y)\big) = 0. \tag{$****$}$$

Our aim is to find two points $w, \widetilde{w} \in R$ such that $\underline{H}(t,w) = \overline{H}(t,w)$ and $\underline{H}(t,\widetilde{w}) = \overline{H}(t,\widetilde{w})$.

Assume for this purpose that $\underline{H}(t,y) < \overline{H}(t,y)$ for every $y \in R$. Hence

$$X(\hat{t}) \in \bigcup_{y \in R} \big(\underline{H}(t,y),\ \overline{H}(t,y)\big) \ \text{ a.s. according to } (*),\ (**)$$

and the continuity of $\underline{H}(t,.)$ and $\overline{H}(t,.)$. But this is in contradiction with $(****)$ since one can choose a countable covering. Consequently, there is a point $w \in R$ such that $\underline{H}(t,w) = \overline{H}(t,w)$.

Now, if $\underline{H}(t,y) < \overline{H}(t,y)$ for every $y \in R$, $y \neq w$, then

$$X(\hat{t}) \in \bigcup_{\substack{y \in R \\ y \neq w}} \big(\underline{H}(t,y),\ \overline{H}(t,y)\big) \cup \{\underline{H}(t,w)\} \ \text{ a.s.}$$

By the argument of countable covering, we see that

$$P\big(X(\hat{t}) \in \bigcup_{\substack{y \in R \\ y \neq w}} (\underline{H}(t,y),\ \overline{H}(t,y))\big) = 0 \ \text{ according to } (****)$$

which implies $X(\hat{t}) = \underline{H}(t,w)$ a.s. It is in contradiction with the property of the point $\hat{t}$. Thus, there are $w \neq \widetilde{w}$ such that $\underline{H}(t,w) = \overline{H}(t,w)$, $H(t,\widetilde{w}) = \overline{H}(t,\widetilde{w})$.

Hence, $\underline{H}(t,.) = \overline{H}(t,.)$, and it is a linear function because of the concavity of $\underline{H}(t,.)$ and the convexity of $\overline{H}(t,.)$, as well as the linearity of $H(s,t,.)$.

(iic) We have derived that $\underline{H}(t,y) = \overline{H}(t,y) = \gamma(t)y + \delta(t)$ for some $\gamma(t) \geq 0$,

$\delta(t) \in R$. Necessarily, $\gamma(t) > 0$, because the case $\gamma(t) = 0$ gives a contradiction with the assumption $P\big(X(\hat{t}) \neq \delta(t)\big) > 0$. Consequently,
$\alpha(s)\ \alpha(st)^{-1} \xrightarrow[s\to\infty]{} A(t) = \gamma(t)^{-1} \in R_+$ and

$$\beta(s) - \alpha(s)\ \alpha(st)^{-1}\beta(st) \xrightarrow[s\to\infty]{} B(t) = \delta(t)\ \gamma(t)^{-1} \in R.$$

(iii) The relation (3) remains for a proof. Fix $a \in R_+^k$ and consider the equality

$$\big(\alpha(s)\ Y(ast) + \beta(s),\ t \in R_+^k\big) =$$
$$= \Big(\alpha(s)\ \alpha(sa)^{-1} \cdot \big(\alpha(sa)\ Y(ast) + \beta(sa)\big) + \beta(s) - \alpha(s)\ \alpha(sa)^{-1}\beta(as),\ t \in R_+^k\Big).$$

If s tends to infinity, we obtain

$$\big(X(at),\ t \in R_+^k\big) \stackrel{d}{=} \big(A(a)\ X(t) + B(a),\ t \in R_+^k\big).$$

Theorem is completely proved.

□

Lemma 1. *Let $\alpha : R_+^k \to R_+$ and $\alpha(s)\ \alpha(as)^{-1} \xrightarrow[s\to\infty]{} A(a) \in R_+^k$ for every $a \in R_+^k$. Then the value of A at any point $a \in R_+^k$, $a > 1$, satisfies the following*

(a) *if $\forall\, b \in R_+^k\ \alpha(ba^n) \xrightarrow[n\to+\infty]{} 0$ then $A(a) \geq 1$;*

(b) *if $\forall\, b \in R_+^k\ \alpha(ba^n) \xrightarrow[n\to+\infty]{} +\infty$ then $A(a) \leq 1$;*

(c) *otherwise, $A(a) = 1$.*

Proof: Take $a, b \in R_+^k$, $a > 1$ and consider the following four cases

(i) $\alpha(ba^n) \xrightarrow[n\to+\infty]{} +\infty$.
There is a sequence $k_1 < k_2 < \dots$ such that $\alpha(ba^{k_n}) \leq \alpha(ba^{k_n+1})$. Hence $\alpha(ba^{k_n})\ \alpha(ba^{k_n+1})^{-1} \xrightarrow[n\to+\infty]{} A(a) \leq 1$.

(ii) $\alpha(ba^n) \xrightarrow[n\to+\infty]{} 0$.
There is a sequence $k_1 < k_2 < \dots$ such that $\alpha(ba^{k_n}) \geq \alpha(ba^{k_n+1})$. Hence $\alpha(ba^{k_n})\ \alpha(ba^{k_n+1})^{-1} \xrightarrow[n\to+\infty]{} A(a) \geq 1$.

(iii) $\alpha(ba^n) \xrightarrow[n\to+\infty]{} \lambda \in R_+$.
Hence $\alpha(ba^n)\ \alpha(ba^{n+1})^{-1} \xrightarrow[n\to+\infty]{} \lambda^{-1} = 1 = A(a)$.

(iv) $0 \leq \underline{\lim}\, \alpha(ba^n) < \overline{\lim}\, \alpha(ba^n) \leq +\infty$.
There are two sequences $k_1 < k_2 < \dots$, $r_1 < r_2 < \dots$ such that $\alpha(ba^{k_n}) \leq \alpha(ba^{k_n+1})$, $\alpha(ba^{r_n}) \geq \alpha(ba^{r_n+1})$. Therefore,
$\alpha(ba^{k_n})\ \alpha(ba^{k_n+1})^{-1} \xrightarrow[n\to+\infty]{} A(a) \leq 1$, and
$\alpha(ba^{r_n})\ \alpha(ba^{r_n+1})^{-1} \xrightarrow[n\to+\infty]{} A(a) \geq 1$ holds as well. Hence $A(a) = 1$.

□

Lemma 2. *Let* $\alpha : R_+^k \to R_+$, $\beta : R_+^k \to R$ *and let*

$$\alpha(s)\ \alpha(as)^{-1} \xrightarrow[s\to\infty]{} A(a) \in R_+,$$
$$\beta(s) - \beta(as)\ \alpha(s)\ \alpha(as)^{-1} \xrightarrow[s\to\infty]{} B(a) \in R \ \text{ for every } \ a \in R_+^k,$$

Then $A(a) = A_1(a_1)\dots A_k(a_k)$ *for arbitrary* $a \in R_+$ *where each function* A_i *satisfies* $A_i(st) = A_i(s)\ A_i(t)$ *for arbitrary* $s, t \in R_+$. *Moreover, the function* A *determines the function* B *as follows:*

(a) *if* $A \equiv 1$, *then* $B(a) = B_1(a_1) + \dots + B_k(a_k)$ *and each function* B_i *satisfies* $B_i(st) = B_i(s) + B_i(t)$ *for arbitrary* $s, t \in R_+$;

(b) *if* $A \not\equiv 1$, *then* $B(a) = Q(1 - A(a))$ *for some* $Q \in R$ *and arbitrary* $a \in R_+^k$.

Proof: (i) The equality

$$\alpha(s)\ \alpha(sab)^{-1} = \alpha(s)\ \alpha(sa)^{-1}\alpha(sa)\ \alpha(sab)^{-1}$$

holds for every $s, a, b \in R_+^k$. Thus the limit under s tending to infinity gives

$$A(ab) = A(a)\ A(b).$$

Using this formula, one obtains

$$A(a) = A_1(a_1)\dots A_k(a_k), \ \text{ where}$$
$$A_i(t) = A(\underbrace{1,\dots,1}_{i-1}, t, \underbrace{1,\dots,1}_{k-i}).$$

(ii) This part deals with the form of the function B. The following equality holds:

$$\beta(s) - \beta(sab)\ \alpha(s)\ \alpha(sab)^{-1} =$$
$$= \alpha(s)\ \alpha(as)^{-1}\big(\beta(as) - \beta(sab)\ \alpha(sa)\ \alpha(sab)^{-1}\big) +$$
$$+ \beta(s) - \beta(sa)\ \alpha(s)\ \alpha(sa)^{-1}$$

and the limit under s tending to infinity gives

$$B(ab) = A(a)\ B(b) + B(a).$$

(iia) Let $A \equiv 1$. Then $B(a) = B_1(a_1) + \dots + B_k(a_k)$, where $B_i(t) = B(\underbrace{1,\dots,1}_{i-1}, t, \underbrace{1,\dots,1}_{k-i})$. The function B_i satisfies $B_i(st) = B_i(s) + B_i(t)$ for every $s, t \in R_+$.

(iib) Let $A \not\equiv 1$. Take $a \in R_+^k$, $n \in N$; then the equality

$$
\begin{aligned}
B(a^n) &= A(a)\ B(a^{n-1}) + B(a) = \\
&= B(a)\ \big(1 + A(a) + \cdots + A(a)^{n-1}\big) \ \text{ holds.}
\end{aligned}
$$

If $A(a) \neq 1$, then $B(a^n) = \frac{B(a)}{1-A(a)}\big(1-A(a)^n\big)$. Let us denote $Q(a) = \frac{B(a)}{1-A(a)}$. If $A(a) = 1$, then $B(a^n) = B(a)n$. We will prove in the sequel that $Q(a)$ does not depend on a and $B(a)$ is vanishing if $A(a) = 1$.

Fix a point $\hat{a} \in R_+^k$ such that $A(\hat{a}) < 1$. Note that such a point always exists because $A(a^{-1}) = (A(a))^{-1}$ and $A \not\equiv 1$. Hence, $B(\hat{a}^n) = Q(\hat{a})(1 - A(\hat{a})^n)$. Let $b \in R_+^k$ be another point.

(iiba) Let $A(b) \neq 1$. Then there exists $\ell \in N$ such that $A(\hat{a}^\ell b) = A(b)\ A(\hat{a})^\ell < 1$. Taking into account that

$$
\begin{aligned}
&B(b^n) = Q(b)\big(1 - A(b^n)\big) \ \text{ and} \\
&B\big((\hat{a}^\ell b)^n\big) = Q(\hat{a}^\ell b)\big(1 - A(\hat{a}^\ell b)^n\big),
\end{aligned}
$$

we have

$$
\begin{aligned}
B\big((b\hat{a}^\ell)^n\big) &= A(\hat{a}^{\ell n})\ B(b^n) + B(\hat{a}^{\ell n}) = \\
&= A(\hat{a})^{\ell n} Q(b)\big(1 - A(b)^n\big) + Q(\hat{a})\big(1 - A(\hat{a})^{\ell n}\big),
\end{aligned}
$$

and, consequently,

$$
Q(\hat{a}^\ell b)\big(1 - A(b\hat{a}^\ell)^n\big) = Q(b)\big(A(\hat{a})^{\ell n} - A(b\hat{a}^\ell)^n\big) + Q(\hat{a})\big(1 - A(\hat{a})^{\ell n}\big).
$$

The limit under n tending to infinity gives $Q(\hat{a}^\ell b) = Q(\hat{a})$. Hence, it is $(Q(\hat{a}) - Q(b))\ A(b^n) = Q(\hat{a}) - Q(b)$. Then $Q(b) = Q(\hat{a})$ since $A(b) \neq 1$.

(iibb) Let $A(b) = 1$. Then, $B(b^n) = B(b)n$ and

$$
B\big((\hat{a}b)^n\big) = Q(\hat{a}b)\big(1 - A(\hat{a}b)^n\big) = Q(\hat{a}b)\big(1 - A(\hat{a})^n\big).
$$

Further, $B\big((\hat{a}b)^n\big) = A(\hat{a}^n)\ B(b^n) + B(\hat{a}^n)$ and hence,

$$
Q(\hat{a}b)\big(1 - A(\hat{a}b)^n\big) = nB(b)\ A(\hat{a})^n + Q(\hat{a})\big(1 - A(\hat{a})^n\big)
$$

and the limit under n tending to infinity yields $Q(\hat{a}b) = Q(\hat{a})$. Consequently $B(b) = 0$.

The function B is of the form $B(a) = Q(\hat{a})(1 - A(a))$ for every $a \in R_+^k$. □

Proof of Proposition 1: According to Lemma 2, the function A_i may be either a continuous or discontinuous function with the property

$$
\forall\, s,t \in R_+ : A_i(st) = A_i(s)\ A_i(t).
$$

The equivalences between the following assertions will be proved.

(α) A_i is a continuous function;
(β) $A_i(t) = t^{H_i}$ for every $t \in R_+$ and some $H_i \in R$;
(γ) there exists $\gamma \in R$ such that $\forall\, t > 2$:

$$\alpha(\underbrace{2^n, \dots, 2^n}_{i-1}, t^n, \underbrace{2^n, \dots, 2^n}_{k-i})t^{n\gamma} \xrightarrow[n\to+\infty]{} 0,$$

(δ) there exists $\gamma \in R$ such that $\forall\, t > 2$:

$$\alpha(\underbrace{2^n, \dots, 2^n}_{i-1}, t^n, \underbrace{2^n, \dots, 2^n}_{k-i})t^{n\gamma} \xrightarrow[n\to+\infty]{} +\infty.$$

(α) $\Longleftrightarrow$ (β)
Consider the function $f(t) = \ln A_i(\exp t)$. Obviously f fulfils the equations $f(t+s) = f(t) + f(s)$ for each real $t,\ s$. Hence by Aczél (Chapter 2.1) or Jarník (Chapter V, § 13) we have

$$f \text{ is continuous iff } f(t) = \alpha\ t \text{ for some real } \alpha.$$

Consequently

$$A_i \text{ is continuous iff } A_i(t) = t^{H_i} \text{ for some real } H_i.$$

(β) $\Longrightarrow$ (γ)
Let $A_i(t) = t^{H_i}$ and $\gamma \in R$. Put $\widetilde{\alpha}(a) = \alpha(a)\ A_i^{\gamma}$ and look at the limit

$$\widetilde{\alpha}(s)\ \widetilde{\alpha}(sa)^{-1} = \alpha(s)\ \alpha(sa)^{-1} s_i^{\gamma}(s_i a_i)^{-\gamma} \xrightarrow[s\to\infty]{} A(a)\ a_i^{-\gamma} = \widetilde{A}(a).$$

It is $\widetilde{A}(a) = A_1(a_1)\dots A_{i-1}(a_{i-1}) \cdot A_{i+1}(a_{i+1})\dots A_k(a_k)\cdot a_i^{H_i-\gamma}$. If γ is taken such that $H_i - \gamma \geq 0$ and

$$A_1(2)\dots A_{i-1}(2)\cdot A_{i+1}(2)\dots A_k(2)\cdot 2^{H_i-\gamma} > 1,$$

then

$$\forall\, t > 2 : \widetilde{A}(\underbrace{2, \dots, 2}_{i-1}, t, \underbrace{2, \dots, 2}_{k-i}) > 1.$$

Consequently, $\forall\, t > 2$,

$$\widetilde{\alpha}(\underbrace{2^n, \dots, 2^n}_{i-1}, t^n, \underbrace{2^n, \dots, 2^n}_{k-i}) \xrightarrow[n\to+\infty]{} 0,$$

according to Lemma 1.

$(\gamma) \Longrightarrow (\beta)$
Let $\gamma \in R$ such that $\alpha\big(\underbrace{2^n,\dots,2^n}_{i-1},t^n,\underbrace{2^n,\dots,2^n}_{k-i}\big)t^{n\gamma} \xrightarrow[n\to+\infty]{} 0$ for every $t \in R_+$, $t > 2$. Put $\widetilde{\alpha}(a) = \alpha(a)a_i^{\gamma}$ and consider the following facts that

$$\widetilde{\alpha}(s)\ \widetilde{\alpha}(sa)^{-1} \xrightarrow[n\to+\infty]{} A(a)a_i^{-\gamma} = \widetilde{A}(a)$$

and

$$\widetilde{\alpha}\big(\underbrace{2^n,\dots,2^n}_{i-1},t^n,\underbrace{2^n,\dots,2^n}_{k-i}\big) \xrightarrow[n\to+\infty]{} 0.$$

Lemma 1 shows

$$\widetilde{A}\big(\underbrace{2,\dots,2}_{i-1},t,\underbrace{2,\dots,2}_{k-i}\big) \geq 1 \quad \text{for every} \ \ t > 2.$$

Therefore, $A_i(t) = \widetilde{A}_i(t)t^{\gamma} \geq W > 0$ for each $2 < t < 3$ and some $W > 0$. Consider the function $f(t) = \ln A_i(\exp t)$. Obviously f fulfils the equation $f(t+s) = f(t)+f(s)$ and moreover $f(t) \geq \ln W$ for each $2 < t < 3$. Consequently $f(t) = \alpha\, t$ for some real α according to Aczél (Chapter 2.1) or Jarník (Chapter V, § 13). Therefore $A_i = t^{H_i}$ for some real H_i.
$(\beta) \Longleftrightarrow (\delta)$
This remaining implication can be proved by aid of the transformation $\widetilde{\alpha}(t) = \alpha(t)^{-1}$. □

Lemma 3. *Let $f : R_+ \to R$ satisfy the property $f(st) = f(s) + f(t)$. Then f is continuous iff $f(t) = Q \ln t$ for some $Q \in R$.*

Proof: See Aczél (Chapter 2.1) or Jarník (Chapter V, § 13). □

Lemma 4. *Let $\alpha : R_+^k \to R_+$, $\beta : R_+^k \to R$ and*

$$\alpha(s)\ \alpha(as)^{-1} \xrightarrow[s\to\infty]{} A(a) \in R_+,$$
$$\beta(s) - \beta(as)\ \alpha(s)\ \alpha(as)^{-1} \xrightarrow[s\to\infty]{} B(a) \in R \ \ \textit{for every} \ \ a \in R_+^k.$$

Then there exists $\lim_{n\to+\infty} \beta(ab^n) \in \langle -\infty, +\infty\rangle$ for every $a, b \in R_+^k$. $A(b) \neq 1$, $b > 1$.

Proof: Fix $a, b \in R_+^k$, $b > 1$, $A(b) \neq 1$. Assume $-\infty \leq \underline{\lim}\,\beta(ab^n) < \overline{\lim}\,\beta(ab^n) \leq +\infty$. Hence there exist sequences $k_1 < k_2 < \dots$, $r_1 < r_2 < \dots$ such that

$$\beta\big(ab^{k_n}\big) \leq \beta\big(ab^{k_n+1}\big) \xrightarrow[n\to+\infty]{} \overline{\lim}\,\beta(ab^n),$$
$$\beta\big(ab^{r_n+1}\big) \leq \beta\big(ab^{r_n}\big) \xrightarrow[n\to+\infty]{} \overline{\lim}\,\beta(ab^n).$$

Then

$$\begin{aligned}
&\beta\big(ab^{k_n}\big) - \beta\big(ab^{k_n+1}\big)\ \alpha\big(ab^{k_n}\big)\ \alpha\big(ab^{k_n+1}\big)^{-1} \le \\
&\le \beta\big(ab^{k_n+1}\big)\ \big(1-\alpha(ab^{k_n})\ \alpha(ab^{k_n+1})^{-1}\big), \\
&\beta\big(ab^{r_n}\big) - \beta\big(ab^{r_n+1}\big)\ \alpha\big(ab^{r_n}\big)\ \alpha\big(ab^{r_n+1}\big)^{-1} \ge \\
&\ge \beta\big(ab^{r_n}\big)\ \big(1-\alpha(ab^{r_n})\ \alpha(ab^{r_n+1})^{-1}\big)
\end{aligned}$$

and taking the limit under n tending to infinity we get

$$\overline{\lim}\,\beta(ab^n)\ (1-A(b)) \le B(b) \le \overline{\lim}\,\beta(ab^n)\ (1-A(b)).$$

Consequently, $\overline{\lim}\,\beta(ab^n)(1-A(b)) = B(b)$. In the similar way, it can be shown

$$\underline{\lim}\,\beta(ab^n)\ (1-A(b)) = B(b).$$

These two results imply the existence of the limit, which is a contradiction. So $\lim\beta(ab^n) \in \langle -\infty, +\infty\rangle$ always exists. □

Lemma 5. *Let $\alpha : R_+^k \to R_+$, $\beta : R_+^k \to R$ and*

$$\begin{gathered}
\alpha(s)\ \alpha(as)^{-1} \xrightarrow[s\to\infty]{} A(a) \in R_+, \\
\beta(s) - \beta(as)\ \alpha(s)\ \alpha(as)^{-1} \xrightarrow[s\to\infty]{} B(a) \in R, \\
B(a) = Q(1-A(a)) \ \ \text{for every} \ \ a \in R_+^k \ \ \text{and some} \ \ Q \in R.
\end{gathered}$$

Take $a, b \in R_+^k$, $b > 1$.

(a) *If $A(b) > 1$, then $\lim_{n\to+\infty} \beta(ab^n) = Q$.*

(b) *If $A(b) < 1$, then $\lim_{n\to+\infty} \frac{\beta(ab^n)}{\alpha(ab^n)} = \widetilde{Q}(a,b) \in R$ and $\lim_{n\to+\infty} \big(\beta(ab^n) - \widetilde{Q}(a,b)\ \alpha(ab^n)\big) = Q$.*

Proof: Lemma 4 shows that $\lim_{n\to+\infty} \beta(ab^n) \in \langle -\infty, +\infty\rangle$ exists.

(i) Let $A(b) > 1$. Assume $\lim\beta(ab^n) = +\infty$. Then there exists a sequence $k_1 < k_2 < \dots$ such that $\beta\big(ab^{k_n}\big) \le \beta\big(ab^{k_n+1}\big) \xrightarrow[n\to+\infty]{} +\infty$. Therefore,

$$\begin{aligned}
\beta\big(ab^{k_n}\big) - \beta\big(ab^{k_n+1}\big)\ \alpha\big(ab^{k_n}\big)\ \alpha\big(ab^{k_n+1}\big) \le& \\
\le \beta\big(ab^{k_n+1}\big)\ \big(1-\alpha\big(ab^{k_n}\big)\ \alpha\big(ab^{k_n+1}\big)\big)&
\end{aligned}$$

and taking the limit we get $B(b) \le -\infty$, which is impossible since $B(b) \in R$.

The assumption $\lim\beta(ab^n) = -\infty$ leads to a contradiction as well. We conclude $\lim\beta(ab^n) \in R$ and

$$B(b) = \lim\beta(ab^n)\ (1-A(b)),$$

which yields $\lim\beta(ab^n) = Q$.

(ii) Let $A(b) <\triangle< 1$. Take $n_0 \in N$ such that $\forall\, n \geq n_0 : \alpha(ab^n)\ \alpha(ab^{n+1})^{-1} <\triangle$ and

$$|\beta(ab^n) - \beta(ab^{n+1})\ \alpha(ab^n)\ \alpha(ab^{n+1})^{-1}| \leq |Q|.$$

Hence

$$\Big|\frac{\beta(ab^n)}{\alpha(ab^n)} - \frac{\beta(ab^{n+1})}{\alpha(ab^{n+1})}\Big| =$$

$$= \frac{1}{\alpha(ab^n)}\big|\beta(ab^n) - \beta(ab^{n+1})\ \alpha(ab^n)\ \alpha(ab^{n+1})^{-1}\big| \leq$$

$$\leq \frac{|Q|}{\alpha(ab^n)} \leq \frac{\triangle\ |Q|}{\alpha(ab^{n-1})} \leq \frac{|Q|}{\alpha(ab^{n_0})}\ \triangle^{n-n_0}\ .$$

Then $\frac{\beta(ab^n)}{\alpha(ab^n)} \xrightarrow[n\to+\infty]{} \widetilde{Q}(a,b) \in R$ since $\sum_{n=0}^{+\infty} \triangle^n = \frac{1}{1-\triangle} < +\infty$. Put $\widetilde{\beta}(n) = \beta(ab^n) - \widetilde{Q}(a,b)\ \alpha(ab^n)$ and note that for every $n \geq n_0$, it holds

$$\Big|\frac{\widetilde{\beta}(n+1)}{\alpha(ab^{n+1})}\Big| \geq \Big|\frac{\widetilde{\beta}(n)}{\alpha(ab^n)}\Big| -$$

$$- \frac{1}{\alpha(ab^n)}\Big|\widetilde{\beta}(n) - \widetilde{\beta}(n+1)\ \alpha(ab^n)\ \alpha(ab^{n+1})^{-1}\Big| =$$

$$= \Big|\frac{\widetilde{\beta}(n)}{\alpha(ab^n)}\Big| - \frac{1}{\alpha(ab^n)}\Big|\beta(ab^n) - \beta(ab^{n+1})\ \alpha(ab^n)\ \alpha(ab^{n+1})^{-1}\Big| \geq$$

$$\geq \frac{|\widetilde{\beta}(n)|}{\alpha(ab^n)} - \frac{|Q|}{\alpha(ab^n)}\ .$$

Therefore, for every $j \in N$ the estimate

$$\Big|\frac{\widetilde{\beta}(n+j)}{\alpha(ab^{n+j})}\Big| \geq \Big|\frac{\widetilde{\beta}(n)}{\alpha(ab^n)}\Big| - \frac{|Q|}{\alpha(ab^n)} \sum_{k=0}^{+\infty} \triangle^k, \text{ i.e.}$$

$$\Big|\frac{\widetilde{\beta}(n+j)}{\alpha(ab^{n+j})}\Big| \geq \frac{1}{\alpha(ab^n)}\big(|\widetilde{\beta}(n)| - \frac{|Q|}{1-\triangle}\big) \text{ holds.}$$

Taking the limit under j tending to infinity we derive

$$0 \geq \frac{1}{\alpha(ab^n)}\big(|\widetilde{\beta}(n)| - \frac{|Q|}{1-\triangle}\big),$$

since

$$\frac{\widetilde{\beta}(n)}{\alpha(ab^n)} = \frac{\beta(ab^n)}{\alpha(ab^n)} - \widetilde{Q}(a,b) \xrightarrow[n\to+\infty]{} 0.$$

Consequently

$$|\widetilde{\beta}(n)| \leq \frac{|Q|}{1-\triangle} \quad \text{and then} \quad \lim \widetilde{\beta}(n) \in R.$$

The conclusion is
$$\big(\beta(ab^n) - \widetilde{Q}(a,b)\ \alpha(ab^n)\big) \xrightarrow[n\to+\infty]{} Q.$$

□

Proof of Proposition 2: Proposition 2 sums up the results of Lemmas 2, 3, 5. It remains to consider $A(b) < 1$ and to show that
$$Y(ab^n) \xrightarrow[n\to+\infty]{d} \widetilde{Q}(a,b).$$

As $\alpha(ab^n)\ Y(ab^n) + \beta(ab^n) \xrightarrow[n\to+\infty]{d} X(1)$, we have
$$\alpha(ab^n)\big(Y(ab^n) + \widetilde{Q}(a,b)\big) + \beta(ab^n) - \widetilde{Q}(a,b)\ \alpha(ab^n) \xrightarrow[n\to+\infty]{d} X(1),$$

and consequently
$$\alpha(ab^n)\ \big(Y(ab^n) + \widetilde{Q}(a,b)\big) \xrightarrow[n\to+\infty]{d} X(1) - Q.$$

If $A(b) < 1$, then $\alpha(ab^n) \xrightarrow[n\to+\infty]{} +\infty$, which implies the final conclusion
$$Y(ab^n) \xrightarrow[n\to+\infty]{d} -\widetilde{Q}(a,b).$$

□

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