Semigroup formulation of Rothe's method: application to parabolic problems

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Abstract. A semilinear parabolic equation in a Banach space is considered. The purpose of this paper is to show the dependence of an error estimate for Rothe's method on the regularity of initial data. The proofs are done using a semigroup theory and Taylor spectral representation.

Keywords: error estimates, parabolic equation, backward Euler method

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1. Introduction.

The Rothe method (also called method of lines or backward Euler method) is well known as an efficient theoretical tool for solving a wide range of evolution problems. Moreover it has a strong numerical aspect. The aim of this paper is to investigate the Rothe method from the point of view of the semigroup theory.

Let $\mathbb{X}$ be a Banach space with the norm $\|\ \|$. The operator A is assumed to be sectorial in $\mathbb{X}$ (cf. [3, D. 1.3.1]) with the domain $D(A)$, where $\operatorname{Re}\sigma(A) > \delta_0 > 0$. We denote $\mathbb{X}_\alpha = D(A^\alpha)$ for $\alpha \in \mathbb{R}$. The norm in $\mathbb{X}_\alpha$ is defined by $\|v\|_\alpha = \|A^\alpha v\|$. The problem we are considering is the abstract semilinear evolution equation ($t \in \langle 0,T\rangle,\ 0 \le \alpha < 1,\ \beta \ge \alpha$)

$$\begin{aligned} \partial_t u(t) + Au(t) &= f(t,u(t)) \\ u(0) &= v \in \mathbb{X}_\beta\,. \end{aligned} \tag{1.1}$$

The right-hand side $f : \mathbb{R} \times \mathbb{X}_\alpha \to \mathbb{X}$ satisfies

$$\|f(t,x) - f(s,y)\| \le C\, \big(|t-s|\ (1 + \|x\|_\alpha + \|y\|_\alpha) + \|x-y\|_\alpha\big)\,. \tag{1.2$_1$}$$

Further

$$\|f(t,z)\|_\alpha \le C\, \big(1 + \|z\|_{2\alpha}\big) \tag{1.2$_2$}$$

for any $t,s \in \langle 0,T\rangle$, $x,y \in \mathbb{X}_\alpha$ and $z \in \mathbb{X}_{2\alpha}$.

It is easy to find that A generates an analytic semigroup in $\mathbb{X}$ and there exists a global solution of (1.1) which can be described as follows:

$$u(t) = T(t)v + \int_0^t T(t-s) f(s,u(s))\, ds \tag{1.3}$$

where

$$T(t) = \mathrm{e}^{-At} = (2\pi\,\mathrm{i})^{-1} \int_\Gamma \mathrm{e}^{\lambda t} (\lambda + A)^{-1}\, d\lambda$$

and Γ is a curve in $\varrho(-A)$ running from $\infty\,\mathrm{e}^{-\mathrm{i}\phi}$ to $\infty\,\mathrm{e}^{\mathrm{i}\phi}$ for any $\phi \in (\pi/2, \pi)$. Without loss of generality we can put

$$\lambda \in \Gamma \Leftrightarrow \lambda = -\delta - s\ \cos\varphi \pm \mathrm{i}\, s\ \sin\varphi \tag{1.4}$$

for $s \in \langle 0, \infty)$, $\varphi \in (0, \pi/2)$, $\delta = \delta(\delta_0) > 0$.

The backward Euler method is applied to the discretization in time

$$\begin{aligned} (u_i - u_{i-1})\tau^{-1} + Au_i &= f(t_i, u_{i-1}) \\ u_0 &= v \end{aligned} \tag{1.5}$$

for $i \in \mathbb{N}$, τ is a time step, $t_i = i\tau$.

There exists a great number of papers devoted to the study of error estimates for the method of lines applied to (1.1) e.g. [4]–[6], [11], [13], [14], etc. The proof technique demonstrated there needs certain regularity assumptions of initial data (practically $v \in \mathbb{X}_1$) in order to derive some error estimates. For a global Lipschitz continuous right-hand side f and $v \in \mathbb{X}_1$ one can prove

$$\|u(t_i) - u_i\| \le C\ \tau.$$

On the other hand, there exist many papers concerned with nonsmooth data error estimates e.g. [1], [2], [7]–[10], [12], [15], [16], [18], [19], etc. If $\beta = \alpha = 0$ it is known that

$$\|u(t_i) - u_i\|_\gamma \le C\ \left((i-\gamma)^{-1} + \tau^{1-\gamma} \ln \tau^{-1}\right), \quad 0 \le \gamma < 1.$$

We have to point out the fact that any of the articles mentioned above cannot say anything about the error estimate (independent of t) for the backward Euler method applied to (1.1) in the case when $\beta > \alpha > 0$ and β is sufficiently small, i.e. if we consider the initial data of low regularity and f depends on spatial derivatives of u.

The aim of this paper is to demonstrate the influence of the parameters α, β, γ on the rate of convergence. Section 2 is devoted to the study of linear homogeneous problem. However some of our results demonstrated there can be derived by a shorter argumentation; we have chosen this way in order to prepare the basic facts for nonhomogeneous case. The main result is formulated in Theorem 1. Section 3 deals with the nonlinear case when the right-hand side f depends on u. There are given the most important results from the practical point of view (cf. Theorem 4). In fact, considering a concrete problem and applying the Sobolev imbedding theorem one can arrive at the error estimate in suitable function space like L_p, W_p^k, C (cf. Appendix).

Our proof-technique is very compact and enables us to deal with the initial data of various regularity (from nonsmooth to smooth) and establish the rate of convergence practically $\tau^{\min(1,\beta-\gamma)}$ uniformly with respect to t.

Sometimes in the literature the Lipschitz condition of f is assumed only locally in $\mathbb{X}$ which generalizes the semilinear parabolic equations covered. But in that case if $f = f(u)$, the solution $u = u(x,t)$ must be uniformly bounded in $\Omega \times \langle 0, T\rangle$. Thus $v \in L_\infty(\Omega)$, whereas we can deal with initial data e.g. $v \in H^{2\beta}(\Omega) \cap \mathring{H}^{\beta}(\Omega)$, $v \notin L_\infty(\Omega)$ and β is a suitable small positive number.

From practical reasons it is important to discuss spatial discretization, too. This will be the subject of a future investigation.

Remark 1. C denotes a generic positive constant independent of τ.

2. Homogeneous problem.

Throughout this section we suppose $f = 0$. Thus according to (1.5) one can deduce

$$u_i = (I + \tau A)^{-i} v. \tag{2.1}$$

Using operational calculus (cf. [17, § 5.6]) we can prolong our approximate solution u_i from time steps into the whole interval $\langle 0, T\rangle$ as follows:

$$T_\tau(t)v = (I + \tau A)^{-t/\tau} v = (2\pi\,\mathrm{i})^{-1} \int_\Gamma (1 - \tau\lambda)^{-t/\tau} (\lambda + A)^{-1} v \, d\lambda \tag{2.2}$$

where Γ is taken from (1.4).

Let us note that the integral in (2.2) is absolutely convergent for every positive t, τ. In spite of this we suppose without loss of generality $\tau < \tau_0 < 1$. $T(t)$ is an analytic semigroup, and the following estimates hold:

$$\|T(t)\| \le C, \quad t \ge 0, \tag{2.3}$$

$$\|A^\delta T(t)\| \le C_\delta t^{-\delta}, \quad \delta \ge 0, \quad t > 0, \tag{2.4}$$

$$\|(T(t) - I)x\| \le \delta^{-1} C_{1-\delta} t^\delta \ \|A^\delta x\|, \quad x \in \mathbb{X}_\delta, \quad 0 < \delta \le 1, \quad t \ge 0, \tag{2.5}$$

$$\|A^\delta x\| \le C \ \|Ax\|^\delta \ \|x\|^{1-\delta}, \quad x \in \mathbb{X}_1, \quad 0 \le \delta \le 1. \tag{2.6}$$

$T_\tau(t)v$, as an approximate solution of (1.1) for $f = 0$, was introduced in [15]. We know that $T_\tau(t)$, $t \ge 0$, is a semigroup for which the smoothing effect takes place. More exactly, we can write (cf. [16, L. 1]

Lemma 1. *Let $\gamma \ge 0$ and $t, \tau > 0$ such that $t > \gamma\tau$. Then $T_\tau(t)x \in \mathbb{X}_\gamma$ for every $x \in \mathbb{X}$.*

Using this fact, we know that both solutions (exact and approximate, $f = 0$) become smoother with increasing time. Hence we can try to establish an error estimate in the norm of $\mathbb{X}_\gamma$. It is easy to see that this must depend on the parameter β, γ and probably on the time t. First, we state or prove some lemmas which play an important part in our proofs.

Lemma 2. *If* $\lambda \in \mathbb{C}$, $\mathrm{Re}\ \lambda < 0$ *and* $t, \tau > 0$ *then*

$$|(1-\tau\lambda)^{-t/\tau} - \mathrm{e}^{\lambda t}| \le |\lambda|^2\ |\mathrm{Re}\ \lambda|^{-2}\ |(1-\tau\,\mathrm{Re}\ \lambda)^{-t/\tau} - \mathrm{e}^{\mathrm{Re}\ \lambda t}|.$$

Proof: See [15]. □

Lemma 3. *Let* $b > 0$ *and*

$$I(x;a,b) = \int_x^\infty \mathrm{e}^{-z}\, z^{-a-1}\big(\mathrm{e}^z(1+b^{-1}z)^{-b} - 1\big)\, dz.$$

Then:

(i) $I(0;a,b) \le b^{-a}\big((1-a)^{-1} + a^{-1}\big)$ *for* $o < a < 1$,
(ii) $I(0;a,b) \le b^{-1}$ *for* $a = 0$,
(iii) $I(\varepsilon;a,b) \le b^{-1}a^{-1}\varepsilon^{1-a}$ *for* $\varepsilon > 0$, $a \ge 1$,
(iv) $I(0;a,b) \le C(a)b^{-a}(b+a)^{-1}$ *for* $0 > a > -b$.

Proof: Let us denote

$$I_{\varepsilon,N} = \int_\varepsilon^N \mathrm{e}^{-z}\, z^{-a-1}\big(\mathrm{e}^z(1+b^{-1}z)^{-b} - 1\big)\, dz.$$

for $b > 0$, $a \in \mathbb{R}$, $N > \varepsilon > 0$. One can easily find $(z > 0)$

$$\partial_z\big(\mathrm{e}^z(1+b^{-1}z)^{-b} - 1\big) = b^{-1}z\,\mathrm{e}^z(1+b^{-1}z)^{-b-1},$$
$$\partial_z \int_N^z \mathrm{e}^{-s}\, s^{-a-1}\, ds = \mathrm{e}^{-z}\, z^{-a-1}.$$

Thus using integration by parts we have

$$I_{\varepsilon,N} = \Big(\int_0^\varepsilon b^{-1}z\,\mathrm{e}^z(1+b^{-1}z)^{-b-1}\, dz\Big)\Big(\int_\varepsilon^N \mathrm{e}^{-z}\, z^{-a-1}\, dz\Big) + \\ + \int_\varepsilon^N \Big(\int_z^N \mathrm{e}^{-s}\, s^{-a-1}\, ds\Big)\, b^{-1}z\,\mathrm{e}^z(1+b^{-1}z)^{-b-1}\, dz. \tag{2.7}$$

We consider the case $0 \le a < 1$, first. We estimate

$$I_{\varepsilon,N} \le \int_0^N \Big(\int_z^N \mathrm{e}^{-s}\, s^{-a-1}\, ds\Big) b^{-1}z\,\mathrm{e}^z(1+b^{-1}z)^{-b-1}\, dz \le \\ \le \int_0^\infty \Big(\int_z^\infty \mathrm{e}^{-s}\, s^{-a-1}\, ds\Big)\, b^{-1}z\,\mathrm{e}^z(1+b^{-1}z)^{-b-1}\, dz.$$

Applying

$$\int_z^\infty \mathrm{e}^{-s}\, s^{-a-1}\, ds \le z^{-a-1} \int_z^\infty \mathrm{e}^{-s}\, ds \le z^{-a-1}\,\mathrm{e}^{-z}$$

we obtain

$$I_{\varepsilon,N} \le \int_0^\infty b^{-1} z^{-a} (1+b^{-1}z)^{-b-1}\, dz = b^{-a} \int_0^\infty w^{-a}(1+w)^{-b-1}\, dw. \tag{2.8}$$

The last estimate yields for $a = 0$

$$I_{\varepsilon,N} \le \int_0^\infty (1+w)^{-b-1}\, dw = b^{-1}$$

from which, taking the limit as $N \to \infty$ and $\varepsilon \to 0$, we prove (ii).

If $0 < a < 1$, then after integration by parts in (2.8), we can write

$$I_{\varepsilon,N} \le b^{-a}(b+1)(1-a)^{-1} \int_0^\infty w^{1-a}(1+w)^{-b-2}\, dw \le b^{-a}\big((1-a)^{-1} + (a+b)^{-1}\big)\,.$$

Now, letting $N \to \infty$ and $\varepsilon \to 0$ in the last inequality, we conclude (i).

Let us consider $a \ge 1$. Using (2.7) one can find

$$I_{\varepsilon,N} \le \int_0^\varepsilon b^{-1}(1+b^{-1}z)^{-b-1}\, dz\ \varepsilon\, \mathrm{e}^\varepsilon \int_\varepsilon^\infty \mathrm{e}^{-z}\, z^{-a-1}\, dz + \\ + \int_\varepsilon^\infty \Big(\int_z^\infty \mathrm{e}^{-s}\, s^{-a-1}\, ds \Big) b^{-1} z\, \mathrm{e}^z (1+b^{-1}z)^{-b-1}\, dz.$$

Applying

$$\int_z^\infty \mathrm{e}^{-s}\, s^{-a-1}\, ds \le \mathrm{e}^{-z} \int_z^\infty s^{-a-1}\, ds = a^{-1} z^{-a}\, \mathrm{e}^{-z}$$

we have

$$I_{\varepsilon,N} \le \int_0^\varepsilon b^{-1}(1+b^{-1}z)^{-b-1}\, dz\ a^{-1}\varepsilon^{1-a} + \\ + \int_\varepsilon^\infty a^{-1} z^{1-a} b^{-1} (1+b^{-1}z)^{-b-1}\, dz \le b^{-1}a^{-1}\varepsilon^{1-a}\,,$$

from which, taking the limit as $N \to \infty$, we prove (iii).

So we have to prove (iv) to conclude the proof. Analogously as for $0 \le a < 1$ one can find

$$I_{\varepsilon,N} \le \int_0^\infty \int_z^\infty \mathrm{e}^{-s}\, s^{-a-1}\, ds\ b^{-1} z\, \mathrm{e}^z (1+b^{-1}z)^{-b-1}\, dz. \tag{2.9}$$

Let us denote

$$x = [x] + \{x\}, \quad x \in \mathbb{R} \tag{2.10}$$

where $[x] \in \mathbb{Z}$, $\{x\} \in \langle 0,1)$. Further we set

$$(x-1)_k = \prod_{i=1}^{k} (x-i), \quad k \geq 1 \text{ and } (x-1)_0 = 1.$$

Using integration by parts one can write

$$\int_z^\infty \mathrm{e}^{-s}\, s^{-a-1}\, ds =$$
$$= \mathrm{e}^{-z}\, z^{-1} \sum_{k=0}^{[-a]-1} (-a-1)_k z^{-a-k} + (-a-1)_{[-a]} \int_z^\infty \mathrm{e}^{-s}\, s^{\{-a\}-1}\, ds$$

and taking into account

$$\int_z^\infty \mathrm{e}^{-s}\, s^{\{-a\}-1}\, ds \leq \mathrm{e}^{-z}\, z^{\{-a\}-1}$$

we get

$$\int_z^\infty \mathrm{e}^{-s}\, s^{-a-1}\, ds \leq \mathrm{e}^{-z}\, z^{-1} \sum_{k=0}^{[-a]} (-a-1)_k z^{-a-k}\,. \tag{2.11}$$

Here, (2.9) and (2.11) yield for $b > -a > 0$

$$I_{\varepsilon,N} \leq \sum_{k=0}^{[-a]} (-a-1)_k b^{-a-k} \int_0^\infty w^{-a-k}(1+w)^{-b-1}\, dw \leq$$
$$\leq \sum_{k=0}^{[-a]} (-a-1)_k b^{-a-k} \int_0^\infty (1+w)^{-a-b-k-1}\, dw \leq \sum_{k=0}^{[-a]} (-a-1)_k b^{-a}(a+b)^{-1}\,.$$

Hence we have proved $(b > -a > 0)$

$$I_{\varepsilon,N} \leq C(a) b^{-a}(a+b)^{-1}$$

and letting $\varepsilon \to 0$, $N \to \infty$ we conclude (iv). $\square$

Now we are ready to derive the main result at this paragraph.

Theorem 1. *Let A be a sectorial operator in a Banach space $\mathbb{X}$ where* $\operatorname{Re} \sigma(A) > \delta_0 > 0$. *Then* $(\nu > 0)$

(i) $\|T_\tau(t) - T(t)\| \leq C\ \tau\ t^{-1}$ *for* $t > 0$,
(ii) $\|T_\tau(t) - T(t)\|_{-\nu} \leq C\ \tau^{\min(1,\nu)}$ *for* $t > 0$,
(iii) $\|T_\tau(t) - T(t)\|_\nu \leq C\ \tau^{1-\nu}(t-\nu\tau)^{-1}$ *for* $t > \nu\tau$.

Proof: (ii) Using the spectral representation of $T_\tau(t)$, $T(t)$ and

$$A(\lambda + A)^{-1} = I - \lambda(\lambda + A)^{-1}, \quad \lambda \in \Gamma, \tag{2.12}$$

one can write

$$A^{-\nu}\big(T_\tau(t) - T(t)\big) = \\ = (-1)^{[\nu]+1}(2\pi\,\mathrm{i})^{-1} \int_\Gamma \big((1-\tau\lambda)^{-t/\tau} - \mathrm{e}^{\lambda t}\big)\lambda^{-[\nu]-1} A^{1-\{\nu\}}(\lambda + A)^{-1}\, d\lambda$$

where [ ], { } are defined by (2.10). Thus taking into account

$$\|A^\xi(\lambda + A)^{-1}\| \le C\,|\lambda|^{\xi - 1} \quad \lambda \in \Gamma, \quad 0 \le \xi \le 1 \tag{2.13}$$

and Lemma 2, we have

$$\|T_\tau(t) - T(t)\|_{-\nu} \le C \int_\Gamma \Big|(1-\tau\lambda)^{-t/\tau} - \mathrm{e}^{\lambda t}\Big| |\lambda|^{-\nu-1}\,|d\lambda| \le \\ \le C\, t^\nu \int_{\delta t}^\infty \mathrm{e}^{-z}\, z^{-\nu-1}\big(\mathrm{e}^z(1+\tau t^{-1}z)^{-t/\tau} - 1\big)\,dz.$$

Putting $a = \nu$ and $b = t\tau^{-1}$, in virtue of Lemma 3, we conclude (ii).

The assertions (i), (iii) can be proved in the same way. □

3. Nonhomogeneous problem.

Throughout this paragraph we suppose that the function f satisfies (1.2) i.e. $(1.2)_1$ and $(1.2)_2$. Considering the discretization scheme (1.5) one can easily find

$$u_i = T_\tau(t_i)v + \sum_{k=0}^{i-1} T_\tau(t_i - t_k) f(t_{k+1}, u_k)\tau. \tag{3.1}$$

Applying the semigroup theory we know that if

$$0 \le \alpha < 1, \quad \alpha \le \beta < 1 \tag{3.2}$$

then there exists a global unique solution u(t) of (1.1) defined by (1.3).

Let us prove some discrete analogs of Gronwall's lemma, first.

Lemma 4. *Let $\{A_i\}$, $\{a_i\}$ be sequences of nonnegative real numbers satisfying* ($q \ge 0$)

$$a_i \le A_i + \sum_{j=1}^{i-1} a_j q, \quad i \in \mathbb{N}. \tag{3.3}$$

Then

$$a_i \le A_i + \mathrm{e}^{qi} \sum_{j=1}^{i-1} A_j q, \quad i \in \mathbb{N}.$$

Proof: Let us fix $i \ge 2$. We define

$$\begin{aligned}
S(n,i) &= \sum_{j=1}^{i-1} \sum_{k_1=1}^{j-1} \sum_{k_2=1}^{k_1-1} \cdots \sum_{k_n=1}^{k_{n-1}-1} A_{k_n} \quad \text{for} \quad 0 < n \le i-2,\\
S(n,i) &= \sum_{j=1}^{i-1} A_j \quad \text{for} \quad n = 0,\\
S(n,i) &= 0 \quad \text{for} \quad n > i-2.
\end{aligned}$$

Iterating (3.3) $(i-1)$-times one can find

$$a_i \le A_i + q \sum_{k=0}^{i-1} q^k S(k,i). \tag{3.4}$$

Using mathematical induction with respect to $n \ge 0$ and taking into account the fact that $\binom{x}{y} = 0$ for $x < y$ one can prove

$$S(n,i) = \sum_{j=1}^{i-1} A_j \binom{i-1-j}{n}, \quad n \ge 0. \tag{3.5}$$

Hence, (3.4) and (3.5) yield

$$\begin{aligned}
a_i &\le A_i + q \sum_{j=1}^{i-1} A_j \Big\{ 1 + \sum_{k=1}^{i-j-1} q^k \binom{i-1-j}{k} \Big\} =\\
&= A_i + q \sum_{j=1}^{i-1} A_j \Big\{ 1 + \sum_{k=1}^{i-j-1} \frac{q^k (i-j)^k}{k!} \prod_{l=1}^{k} \Big(1 - \frac{l}{i-j}\Big) \Big\} \le\\
&\le A_i + q \sum_{j=1}^{i-1} A_j \, \mathrm{e}^{q(i-j)} \le A_i + \mathrm{e}^{qi} \sum_{j=1}^{i-1} A_j q.
\end{aligned}$$

□

Lemma 5. *Let $\{A_n\}$, $\{w_n\}$ be sequences of nonnegative real numbers satisfying*

$$w_n \le A_n + C \sum_{k=1}^{n-1} (t_n - t_k)^{a-1} w_k \tau \tag{3.6}$$

for $0 < \tau < 1$, $0 < a \leq 1$, $C > 0$, $t_n = n\tau \leq T$. *Then*

$$w_n \leq C\left(A_n + \sum_{k=1}^{n-1} A_k\tau + \sum_{k=1}^{n-1}(t_n - t_k)^{a-1}A_k\tau\right), \quad C = C(a,T).$$

PROOF: Let us put $m = \min\{n \in \mathbb{N};\ na \geq 1\}$. Iterating (3.6) once, we have

$$\begin{aligned} w_n &\leq C\left(A_n + \sum_{k=1}^{n-1}(t_n - t_k)^{a-1}A_k\tau + \sum_{k=1}^{n-1}\sum_{l=1}^{k-1}(t_n - t_k)^{a-1}(t_k - t_l)^{a-1}w_k\tau^2\right) \leq \\ &\leq C\left(A_n + \sum_{k=1}^{n-1}(t_n - t_k)^{a-1}A_k\tau + \sum_{k=1}^{n-1}(t_n - t_k)^{2a-1}w_k\tau\right). \end{aligned}$$

Thus, after $(m-1)$ iterations we obtain

$$w_n \leq C\left(A_n + \sum_{k=1}^{n-1}(t_n - t_k)^{a-1}A_k\tau + \sum_{k=1}^{n-1} w_k\tau\right).$$

The rest of the proof follows from the last inequality and Lemma 4. □

As is standard practice, we need a priori estimates for exact and approximate solutions.

Theorem 2. *Let A be a sectorial operator in a Banach space $\mathbb{X}$ where* $\operatorname{Re}\sigma(A) > \delta_0 > 0$. *Suppose* (1.2), (3.2). *Then for* $0 \leq \gamma \leq \beta$ *we have*

(i) $\|u(t)\|_\gamma \leq C$ *for* $t > 0$,
(ii) $\|u_i\|_\gamma \leq C$ *for* $i \in \mathbb{N}$,
(iii) $\|u(t)\|_{2\alpha} \leq C\ t^{(\beta-2\alpha)_-}$ *for* $t > 0$,

where $u(t)$, u_i *are defined by* (1.3), (3.1), *respectively, and* $(x)_- = \min(0,x)$ *for* $x \in \mathbb{R}$.

PROOF: (i) Using (1.3) one can write

$$\|u(t)\|_\gamma \leq C\left(1 + \int_0^t (t-s)^{-\gamma}\|u(s)\|_\alpha\, ds\right). \tag{3.7}$$

Setting $\gamma = \alpha$ in (3.7) and using Gronwall's lemma, we get $\|u(t)\|_\alpha \leq C$. Hence (3.7) yields (i).

(ii) Let us note that

$$\|T_\tau(t_i)\|_\gamma \leq \|T(t_i)\|_\gamma + \|T_\tau(t_i) - T(t_i)\|_\gamma \leq C\left(t_i^{-\gamma} + \tau^{-\gamma}(i-\gamma)^{-1}\right) \leq C\ t_i^{-\gamma}.$$

The rest can be proved analogously as (i).

(iii) In view of (1.3) we can write

$$A^{2\alpha}u(t) = A^{2\alpha-\beta}T(t)A^\beta v + \int_0^t A^\alpha T(t-s)A^\alpha f(s,u(s))\, ds.$$

Hence

$$\|u(t)\|_{2\alpha} \leq C\ t^{(\beta-2\alpha)_-} + C\int_0^t (t-s)^{-\alpha}\left(1 + \|u(s)\|_{2\alpha}\right) ds.$$

Now, applying Gronwall's lemma we conclude the proof. □

Theorem 3. *Let A be a sectorial operator in a Banach space $\mathbb{X}$ where* $\operatorname{Re}\sigma(A) > \delta_0 > 0$*. Suppose* (1.2), (3.2)*. Then for $0 \le \gamma < \beta$ we have*

$$\|u(t+h)-u(t)\|_\gamma \le C\ h^{\beta-\gamma}, \quad 0 \le t < t+h \le T.$$

Proof: We can write

$$u(t+h)-u(t) = \big(T(h)-I\big)T(t)v + \int_t^{t+h} T(t+h-s)f(s,u(s))\,ds + \\ + \int_0^t \big(T(h)-I\big)T(t-s)f(s,u(s))\,ds = I_1 + I_2 + I_3\,. \tag{3.8}$$

Now we estimate all the addends in (3.8).

$$\|I_1\|_\gamma \le \|\big(T(h)-I\big)A^{\gamma-\beta}T(t)A^\beta v\| \le C\ h^{\beta-\gamma}\,.$$

For the second item we have

$$\|I_2\|_\gamma \le \int_t^{t+h} \|A^\gamma T(t+h-s)\|\ \|f(s,u(s))\|\,ds \le \\ \le C\int_t^{t+h}(t+h-s)^{-\gamma}\,ds \le C\ h^{\beta-\gamma}\,.$$

Estimating the last term we find

$$\|I_3\|_\gamma \le C\ h^{\beta-\gamma}\int_0^t \|A^\gamma T(t-s)f(s,u(s))\|\,ds \le C\ h^{\beta-\gamma}\,,$$

which concludes the proof. □

We are ready now to give the estimate of $(u(t_i)-u_i)$ in the norm of $\mathbb{X}_\gamma$. It is very important for practical reasons to have a positive rate of convergence uniformly for every $i \in \mathbb{N}$.

Theorem 4. *Let A be a sectorial operator in a Banach space $\mathbb{X}$ where* $\operatorname{Re}\sigma(A) > \delta_0 > 0$*. Suppose* (1.2), (3.2)*. Then for $i \in \mathbb{N}$ and $0 \le \gamma < \beta$ we have*

$$\|u(t_i)-u_i\|_\gamma \le C\ \tau^{\beta-\gamma}\,.$$

Proof: (i) Let us rewrite $(u(t_i)-u_i)$ into the following form

$$u(t_i)-u_i = \sum_{j=1}^{7} I_j \tag{3.9}$$

where

$$
\begin{aligned}
I_1 &= \big(T(t_i) - T_\tau(t_i)\big)v \\
I_2 &= \sum_{k=1}^{i-1} T_\tau(t_i - t_k)\big(f(t_{k+1}, u(t_k)) - f(t_{k+1}, u_k)\big)\tau, \\
I_3 &= \sum_{k=1}^{i-1} \big(T(t_i - t_k) - T_\tau(t_i - t_k)\big) f(t_{k+1}, u(t_k))\tau, \\
I_4 &= \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} T(t_i - s)\big(f(s, u(s)) - f(t_{k+1}, u(s))\big)\, ds, \\
I_5 &= \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} T(t_i - s)\big(f(t_{k+1}, u(s)) - f(t_{k+1}, u(t_k))\big)\, ds, \\
I_6 &= \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} \big(T(t_i - s) - T(t_i - t_k)\big) f(t_{k+1}, u(t_k))\, ds, \\
I_7 &= \int_0^\tau T(t_i - s) f(s, u(s))\, ds - T_\tau(t_i) f(\tau, v)\tau.
\end{aligned}
$$

Let us start estimating all the items in (3.9). Using Theorem 1, we have

$$\|I_1\|_\gamma = \|A^{\gamma-\beta}\big(T(t_i) - T_\tau(t_i)\big)A^\beta v\| \le C\ \tau^{\beta-\gamma}\,, \tag{3.10$_1$}$$

$$\|I_1\|_\alpha = \|\big(T(t_i) - T_\tau(t_i)\big)A^\alpha v\| \le C\ \tau t_i^{-1}\,. \tag{3.10$_2$}$$

The second term can be estimated as follows:

$$\|I_2\|_\gamma \le \sum_{k=1}^{i-1} \|A^\gamma T_\tau(t_i - t_k)\big(f(t_{k+1}, u(t_k)) - f(t_{k+1}, u_k)\big)\|\tau.$$

Thus

$$\|I_2\|_\gamma \le C \sum_{k=1}^{i-1} (t_i - t_k)^{-\gamma} \|u(t_k) - u_k\|_\alpha \tau. \tag{3.11}$$

Further

$$A^\gamma I_3 = \sum_{k=1}^{i-1} A^\gamma \big(T(t_i - t_k) - T_\tau(t_i - t_k)\big) f(t_{k+1}, u(t_k))\tau.$$

From this, applying Theorem 1, Theorem 2 we have

$$\|I_3\|_\gamma \le C\ \tau^{1-\gamma} \sum_{k=1}^{i-1} (t_i - t_k - \gamma\tau)^{-1}\tau \le C\ \tau^{\beta-\gamma}\,. \tag{3.12$_1$}$$

Analogously

$$A^\alpha I_3 = \sum_{k=1}^{i-1} \big(T(t_i - t_k) - T_\tau(t_i - t_k)\big) A^\alpha f(t_{k+1}, u(t_k))\tau.$$

Hence

$$\begin{aligned} \|I_3\|_\alpha &\le C\ \tau \sum_{k=1}^{i-1} (t_i - t_k)^{-1} \big(1 + \|u(t_k)\|_{2\alpha}\big)\tau \le \\ &\le C\ \tau \ln \tau^{-1} \big(1 + t_i^{(\beta-2\alpha)-}\big)\,. \end{aligned} \tag{3.12$_2$}$$

For the forth item, one can write

$$\begin{aligned} A^\gamma I_4 &= \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} A^\gamma T(t_i - s)\big(f(s, u(s)) - f(t_{k+1}, u(s))\big)\, ds, \\ \|I_4\|_\gamma &\le C\ \tau \int_0^{t_i} (t_i - s)^{-\gamma}\, ds \le C\ \tau. \end{aligned} \tag{3.13}$$

We rewrite I_5 into the following form

$$A^\gamma I_5 = \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} A^\gamma T(t_i - s)\big(f(t_{k+1}, u(s)) - f(t_{k+1}, u(t_k))\big)\, ds.$$

So, applying [3, T. 3.5.2], we obtain

$$\|I_5\|_\gamma \le C \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} (t_i - s)^{-\gamma} \|u(s) - u(t_k)\|_\alpha\, ds \le$$

$$\le C \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} (t_i - s)^{-\gamma} \int_{t_k}^{s} \|\partial_\xi u(\xi)\|_\alpha\, d\xi\, ds \le C\ \tau \int_\tau^{t_i} (t_i - s)^{-\gamma} s^{-1}\, ds.$$

Hence

$$\|I_5\|_\gamma \le C\ \tau \ln \tau^{-1} t_i^{-\gamma}\,. \tag{3.14}$$

The sixth term is estimated as follows:

$$A^\gamma I_6 = \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} \big(I - T(s - t_k)\big) A^\gamma T(t_i - s) f(t_{k+1}, u(t_k))\, ds,$$

thus

$$\|I_6\|_\gamma \le C\ \tau^{\beta-\gamma} \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} (t_i - s)^{-\beta} \|f(t_{k+1}, u(t_k))\|\, ds \le C\ \tau^{\beta-\gamma}\,. \tag{3.15$_1$}$$

On the other hand

$$A^\alpha I_6 = \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} \big(I - T(s-t_k)\big) T(t_i - s) A^\alpha f(t_{k+1}, u(t_k))\, ds,$$

hence

$$\begin{aligned} \|I_6\|_\alpha &\le C\ \tau \sum_{k=1}^{i-1} \int_{t_k}^{t_{k+1}} (t_i - s)^{-1} \big(1 + \|u(t_k)\|_{2\alpha}\big)\, ds \le \\ &\le C\ \tau \ln \tau^{-1} \big(1 + t_i^{(\beta-2\alpha)-}\big)\,. \end{aligned} \tag{3.15$_2$}$$

For the last addendum in (3.9), one can easily find

$$\|I_7\|_\gamma \le C\ \tau t_i^{-\gamma}\,. \tag{3.16}$$

Collecting all the results from (3.9), $(3.10)_2$, (3.11), $(3.12)_2$, (3.13), (3.14), $(3.15)_2$ and (3.16) for $\gamma = \alpha$ we obtain

$$\|u(t_i) - u_i\|_\alpha \le C\ \Big(\tau + \tau \ln \tau^{-1} + \tau t_i^{-1} + \tau \ln \tau^{-1} t_i^{-\nu} + \sum_{k=1}^{i-1} (t_i - t_k)^{-\alpha} \|u(t_k) - u_k\|_\alpha \tau\Big)$$

for any $0 \le \nu < 1$. Thus Lemma 5 implies

$$\|u(t_i) - u_i\|_\alpha \le C\ \big(\tau + \tau \ln \tau^{-1} + \tau t_i^{-1} + \tau \ln \tau^{-1} t_i^{-\xi}\big) \tag{3.17}$$

for any $0 \le \xi < 1$.
Now using (3.9), $(3.10)_1$, (3.11), $(3.12)_1$, (3.13), (3.14), $(3.15)_1$ and (3.16) we get

$$\|u(t_i) - u_i\|_\gamma \le C\ \Big(\tau^{\beta-\gamma} + \sum_{k=1}^{i-1} (t_i - t_k)^{-\gamma} \|u(t_k) - u_k\|_\alpha \tau\Big)\,. \tag{3.18}$$

At the end, inserting (3.17) into (3.18) we conclude the proof. □

4. Appendix.

In this appendix we present some ideas connected with the previous sections. At the end we give some simple examples in order to demonstrate our results.

Remark 2. The restriction $\beta < 1$ in (3.2) is probably due to our proof-technique. We have not been able to remove it in general but it can be a good pastime to do so. $\square$

Remark 3. The error estimate at time steps described in Theorem 4 can be easily prolonged to the whole interval $\langle 0, T\rangle$. In fact, if we define

$$u_n(t) = u_{i-1} + (t - t_{i-1})\tau^{-1}(u_i - u_{i-1}) \quad t \in \langle t_{i-1}, t_i), \ i \in \mathbb{N},$$

then for $t \in \langle t_{i-1}, t_i)$

$$\begin{aligned} u(t) - u_n(t) = u(t) - u(t_{i-1}) + u(t_{i-1}) - u_{i-1} + \\ + (t - t_{i-1})\tau^{-1}\big(u_i - u(t_i) + u(t_i) - u(t_{i-1}) + u(t_{i-1}) - u_{i-1}\big)\,. \end{aligned}$$

Now, applying Theorems 3 and 4 we can estimate $\|u(t) - u_n(t)\|_\gamma$. $\square$

There arise the following questions in many applications: "How is the minimal regularity of initial data, if we want to derive the error estimate independent of t in some functional space?" "How is the rate of convergence?" The answer, in many concrete cases, can be found using our results and the Sobolev imbedding theorem.

In the following examples, let us suppose that $\Omega \subset \mathbb{R}^N$ is a bounded domain with sufficiently smooth $\partial\Omega$. Let us denote

$$\mathbb{X}_\delta = W^{2\delta,p}(\Omega) \cap \mathring{W}^{\delta,p}(\Omega)$$

for $\delta \geq 0$, $p > 1$, $N \geq 1$, $\mathbb{X} = \mathbb{X}_0 = L_p(\Omega)$. The norm in $\mathbb{X}_\delta$ is equivalent to the one in $W^{2\delta,p}(\Omega)$ where

$$\|w\|^p_{W^{k+s,p}(\Omega)} = \|w\|^p_{W^{k,p}(\Omega)} + \sum_{|\alpha|=k} \int_\Omega \int_\Omega |D^\alpha w(x) - D^\alpha w(y)|^p |x - y|^{-n-ps}\, dx\, dy$$

for $k \in \mathbb{N}$, $0 < s < 1$. Further we denote

$$B = \sum_{i=1}^N a_i \partial_{x_i}, \ a_i \in \mathbb{R}.$$

Example 1.

$$\begin{aligned} \frac{du(t)}{dt} - \triangle\, u(t) &= f(t, u(t)) \ \text{ in } \ \Omega \\ u &= 0 \ \text{ on } \ \partial\Omega, \\ u(0) &= v \in \mathbb{X}_\beta, \quad 0 < \beta < 1, \end{aligned}$$

where f as a real function is global Lipschitz continuous in all variables. Then

$$\|u(t_i) - u_i\|_{L_p(\Omega)} \leq C\, \tau^\beta\,.$$

$\square$

Example 2.

$$\begin{gathered}
\frac{du}{dt} - \triangle u = \sin(Bu) \quad \text{in} \quad \Omega \\
u = 0 \quad \text{on} \quad \partial\Omega, \\
u(0) = v \in \mathbb{X}_\beta \,.
\end{gathered}$$

If $1/2 \le \beta < 1$, $0 \le \gamma < \beta$ then

$$\|u(t_i) - u_i\|_{W^{2\gamma,p}(\Omega)} \cong \|u(t_i) - u_i\|_\gamma \le C\ \tau^{\beta-\gamma} \,.$$

If $N/(2p) < \gamma < \beta < 1$, $1/2 \le \beta$ then

$$\|u(t_i) - u_i\|_{L_\infty(\Omega)} \le C\ \|u(t_i) - u_i\|_\gamma \le C\ \tau^{\beta-\gamma} \,.$$

These results cannot be obtained by the classical proof technique in Rothe's method because of the low regularity of the initial data v (cf. [4], [6]). □

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