

Strong shape of the Stone-Čech compactification

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Abstract. J. Keesling has shown that for connected spaces X the natural inclusion $e : X \rightarrow \beta X$ of X in its Stone-Čech compactification is a shape equivalence if and only if X is pseudocompact. This paper establishes the analogous result for strong shape. Moreover, pseudocompact spaces are characterized as spaces which admit compact resolutions, which improves a result of I. Lončar.

Keywords: inverse system, resolution, Stone-Čech compactification, pseudocompact space, shape, strong shape

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1. Introduction.

For every completely regular space X , there is a natural embedding $e : X \rightarrow \beta X$ of X in its Stone-Čech compactification βX . In 1975, K. Morita proved that, for pseudocompact spaces X , the shape $sh(X) = sh(\beta X)$ [11, Theorem 5.2 and Corollary 5.3]. In a survey article of J. Keesling, published in 1980, it is stated that, for connected spaces X , the embedding e is a shape equivalence if and only if X is pseudocompact [4, Theorem 1.2]. In the present paper we prove the analogous result for strong shape. Moreover, we improve a result of I. Lončar [8], who in the class of normal spaces characterized the countably compact spaces as spaces which admit a resolution consisting of metric compacta. Recall that for normal spaces countable compactness and pseudocompactness are equivalent properties (see [2, Theorem 3.10.20 and 3.10.21]).

Theorem 1. *For connected Tychonoff spaces X the following statements are equivalent.*

- (i) *The natural embedding $e : X \rightarrow \beta X$ is a strong shape equivalence.*
- (ii) *The natural embedding $e : X \rightarrow \beta X$ is a shape equivalence.*
- (iii) *X is pseudocompact.*
- (iv) *X admits a resolution $\mathbf{p} : X \rightarrow \mathbf{X}$, which consists of compact polyhedra.*
- (v) *X admits a resolution $\mathbf{p} : X \rightarrow \mathbf{X}$, which consists of compact spaces.*

In the sections which follow we will prove the implications (i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (iii) and (iv) \Rightarrow (i). Only (ii) \Rightarrow (iii) uses the assumption that X is connected. Basic facts about the Stone-Čech compactification can be found in [2] and [12]. For shape theory we use [10] and for strong shape [6], [7].

2. Shape-equivalence of e implies pseudocompactness of X .

(i) \Rightarrow (ii). Recall that in shape theory one defines a shape category \mathbf{Sh} and a shape functor $S : \mathbf{HTop} \rightarrow \mathbf{Sh}$, where \mathbf{HTop} denotes the homotopy category. In particular, this functor assigns to every mapping $f : X \rightarrow Y$ a shape morphism $S([f]) : X \rightarrow Y$, which depends only on the homotopy class $[f]$ of f . When we say that a mapping f is a shape equivalence, we mean that $S([f])$ is an isomorphism of \mathbf{Sh} . Similarly, in strong shape theory one defines a strong shape category \mathbf{SSH} and a strong shape functor $S_1 : \mathbf{HTop} \rightarrow \mathbf{SSH}$. Moreover, there is a functor $S_2 : \mathbf{SSH} \rightarrow \mathbf{Sh}$ and $S_2 S_1 = S$. If we say that a mapping f is a strong shape equivalence, we mean that $S_1([f])$ is an isomorphism of \mathbf{SSH} . Clearly, the implication (i) \Rightarrow (ii) is a consequence of the existence of the functor S_2 .

(ii) \Rightarrow (iii) Assuming that e is a shape equivalence and X is connected, we will show that every mapping $\phi : X \rightarrow \mathbb{R}$ is bounded, i.e. X is pseudocompact. Consider the mapping $\exp : \mathbb{R} \rightarrow S^1 = \{z \in \mathbb{C} : |z| = 1\}$, defined by $\exp(t) = e^{2\pi it}$, $t \in \mathbb{R}$. Since S^1 is compact, the mapping $\exp \circ \phi : X \rightarrow S^1$ admits an extension $g : \beta X \rightarrow S^1$, so that

$$(1) \quad ge = \exp \circ \phi.$$

By the contractibility of \mathbb{R} , ϕ is homotopic to the constant map 0. Therefore, $ge = \exp \circ \phi$ is homotopic to the constant map 1, i.e.

$$(2) \quad ge \simeq i \circ 1,$$

where $1 : X \rightarrow \{1\}$ and $i : \{1\} \rightarrow S^1$ is the inclusion mapping. We claim that g , too, is homotopic to a constant mapping. Since S^1 and $\{1\}$ are polyhedra, it suffices to show that $S(g)$ factors in \mathbf{Sh} through $\{1\}$. However, this follows from (2) because

$$S(g) = S(g)S(e)S(e)^{-1} = S(ge)S(e)^{-1} = S(i)S(1)S(e)^{-1}$$

and $S(1)S(e)^{-1} : \beta X \rightarrow \{1\}$. Since $\exp : \mathbb{R} \rightarrow S^1$ is a fibration and g is homotopic to a constant, the homotopy lifting property yields a mapping $\psi : \beta X \rightarrow \mathbb{R}$ such that $g = \exp \circ \psi$. It follows that $\exp \circ \phi = ge = \exp \circ \psi e$ or $\exp \circ (\phi - \psi e) = 1$. We conclude that $(\phi - \psi e)(X) \subseteq \mathbb{Z}$. Since X is connected and \mathbb{Z} is discrete, it follows that $(\phi - \psi e)(X)$ is a single point. The set $\psi(\beta X)$ is compact and therefore bounded in \mathbb{R} . Consequently, $(\psi e)(X) \subseteq \psi(\beta X)$ is also bounded. This implies that $\phi(X)$ is bounded, too.

3. Resolutions.

We now recall the notion of resolution (see [9] and [10]). Let $\mathbf{X} = (X_a, p_{aa'}, A)$ be an inverse system. A morphism of $\mathbf{pro-Top}$ (also called a *mapping of systems*) $\mathbf{p} = (p_a) : X \rightarrow \mathbf{X}$ is a resolution of X , provided it possesses the following two properties:

(R1) For any polyhedron P , open covering \mathcal{V} of P and mapping $f : X \rightarrow P$, there exist an $a \in A$ and a mapping $g : X_a \rightarrow P$, such that the mappings f and gp_a are \mathcal{V} -near, which we denote by $(f, gp_a) \prec \mathcal{V}$.

(R2) For any polyhedron P and open covering \mathcal{V} of P , there exists an open covering \mathcal{V}' of P such that, for any $a \in A$ and mappings $g, g' : X_a \rightarrow P$, which satisfy $(gp_a, g'p_a) \prec \mathcal{V}'$, there exists an $a' \geq a$ such that $(gp_{aa'}, g'p_{aa'}) \prec \mathcal{V}$.

A resolution is said to be cofinite provided each element of the index set A has only finitely many predecessors.

(iii) \Rightarrow (iv) Let X be pseudocompact. Since βX is compact, there exists a cofinite inverse system of compact polyhedra $\mathbf{X} = (X_a, p_{aa'}, A)$ and a collection of mappings $q_a : \beta X \rightarrow X_a, a \in A$, such that $\mathbf{q} = (q_a)$ is an inverse limit of βX . Note that \mathbf{q} is a resolution and therefore satisfies the conditions (R1) and (R2) (see [10, I, 6.1, Theorem 1]). We now define mappings $p_a : X \rightarrow X_a, a \in A$, by putting $p_a = q_a \circ e$. The desired implication will be established if we prove the following lemma.

Lemma 1. $\mathbf{p} = (p_a) : X \rightarrow \mathbf{X}$ is a resolution of X .

In order to prove Lemma 1 we need the following simple fact.

Lemma 2. A pseudocompact subspace Q of a polyhedron P is a compact space.

PROOF OF LEMMA 2: Let K be a simplicial complex whose geometric realization $|K|$ is P . Denote by $|K|_{CW}$ and $|K|_m$ the spaces obtained by endowing K with the CW -topology and the metric topology, respectively. Let $j : |K|_{CW} \rightarrow |K|_m$ denote the identity mapping, which is known to be continuous. Since the continuous image of a pseudocompact space is pseudocompact, we see that $j(Q)$ is a pseudocompact subspace of the metric space $|K|_m$. Therefore, $j(Q)$ is compact (see [2, Theorems 3.10.21 and 5.1.20]) and thus a closed subset of $|K|_m$. This implies that $Q = j^{-1}j(Q)$ is a closed subset of $P = |K|_{CW}$. Since P is paracompact, Q is also paracompact. However, paracompact pseudocompact spaces are compact [loc. cit.] and we conclude that indeed, Q is a compact space. \square

PROOF OF LEMMA 1: Clearly, $p_{aa'}p_{a'} = p_a$, for $a \leq a'$. Therefore, it remains to verify the conditions (R1) and (R2).

Verification of (R1): Let P be a polyhedron, \mathcal{V} an open covering of P and $f : X \rightarrow P$ a mapping. Clearly, $f(X)$ is pseudocompact. By Lemma 1, it follows that $f(X)$ is even compact. Therefore, there exists a compact subpolyhedron $Q \subseteq P$ such that $f(X) \subseteq Q$. If we view f as mapping $f : X \rightarrow Q$, it admits an extension $\tilde{f} : \beta X \rightarrow Q, \tilde{f} \circ e = f$. Applying (R1) for \mathbf{q} to \tilde{f} and $\mathcal{V} \upharpoonright Q$, we obtain an $a \in A$ and a mapping $g : X_a \rightarrow Q \subseteq P$, such that $(gq_a, \tilde{f}) \prec \mathcal{V} \upharpoonright Q$. Therefore, we also have the desired relation $(gp_a, f) \prec \mathcal{V}$.

Verification of (R2): Let P be a polyhedron and \mathcal{V} an open covering of P . Choose a covering \mathcal{V}' of P , by (R2) applied to \mathbf{q} . Let \mathcal{W} be a star-refinement of \mathcal{V} , i.e. $\text{st}(\mathcal{W})$ refines \mathcal{V}' , where $\text{st}(\mathcal{W})$ denotes the covering formed by all the stars $\text{st}(W, \mathcal{W}), W \in \mathcal{W}$. We claim that \mathcal{W} has the properties required by (R2) for \mathbf{p} , i.e. that whenever $a \in A$ and $g, g' : X_a \rightarrow P$ are mappings, for which

$$(3) \quad (gp_a, g'p_a) \prec \mathcal{W},$$

then there exists an $a' \geq a$ such that

$$(4) \quad (gp_{aa'}, g'p_{aa'}) \prec \mathcal{V}.$$

It suffices to show that (3) implies

$$(5) \quad (gq_a, g'q_a) \prec \mathcal{V}',$$

because (4) follows from (5), by the choice of \mathcal{V}' . To verify (5), consider any point $y \in \beta X$. Choose members W_1, W_2 of \mathcal{W} so that

$$(6) \quad gq_a(y) \in W_1, g'q_a(y) \in W_2.$$

Choose an open neighborhood U of y in βX so small that

$$(7) \quad gq_a(U) \subseteq W_1, g'q_a(U) \subseteq W_2.$$

Since $e(X)$ is dense in βX , there exists a point $x \in X$ such that $e(x) \in U$. By (3), there exists a $W \in \mathcal{W}$ such that

$$(8) \quad gp_a(x), g'p_a(x) \in W.$$

Since $e(x), y \in U$, (7) implies

$$(9) \quad gq_a(y), gp_a(x) = gq_ae(x) \in W_1, g'q_a(y), g'p_a(x) = g'q_ae(x) \in W_2.$$

Now, (8) and (9) yield

$$(10) \quad gq_a(y), g'q_a(y) \in \text{st}(W, \mathcal{W}).$$

Since \mathcal{W} is a star-refinement of \mathcal{V}' , there exists a $V' \in \mathcal{V}'$ such that $\text{st}(W, \mathcal{W}) \subseteq V'$ and therefore, $gq_a(y), g'q_a(y) \in V' \in \mathcal{V}'$, which proves (5). \square

(iv) \Rightarrow (v) is obvious.

(v) \Rightarrow (iii). Let $p : X \rightarrow \mathbf{X}$ be a resolution, which consists of compact Hausdorff spaces X_a . We must show that each mapping $f : X \rightarrow \mathbb{R}$ is bounded. Let \mathcal{V} be the open covering of \mathbb{R} , which consists of all intervals of length 1. By (R1), there exist an $a \in A$ and a mapping $g : X_a \rightarrow \mathbb{R}$ such that

$$(11) \quad (f, gp_a) \prec \mathcal{V}.$$

Since $g(X_a)$ is compact, it is contained in a segment $[b, c] \subseteq \mathbb{R}$. However, $gp_a(X) \subseteq g(X)_a$. Therefore, also $gp_a(X) \subseteq [b, c]$. Now (11) shows that

$$(12) \quad f(X) \subseteq [b - 1, c + 1].$$

4. Strong shape.

If $\mathbf{p} : X \rightarrow \mathbf{X}$ and $\mathbf{q} : Y \rightarrow \mathbf{Y}$ are cofinite polyhedral resolutions, then there is a functorial one-to-one correspondence between strong shape morphisms $F : X \rightarrow Y$ and morphism $[\mathbf{f}] : \mathbf{X} \rightarrow \mathbf{Y}$ of the category \mathbf{CPHTop} of coherent prohomotopy. Here $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is a coherent mapping and $[\mathbf{f}]$ is its homotopy class. Therefore, F is an isomorphism of \mathbf{SSH} if and only if $[\mathbf{f}]$ is an isomorphism of \mathbf{CPHTop} . For a given mapping $f : X \rightarrow Y$, there exists a unique morphism $[\mathbf{f}] : \mathbf{X} \rightarrow \mathbf{Y}$ of \mathbf{CPHTop} such that

$$(13) \quad [\mathbf{f}][\mathbf{p}] = [\mathbf{q}][\mathbf{f}].$$

By definition, $S_1([\mathbf{f}])$ is given by $[\mathbf{f}]$. Therefore, in order to prove that a mapping $f : X \rightarrow Y$ is a strong shape equivalence, it suffices to find cofinite polyhedral resolutions \mathbf{p}, \mathbf{q} and an isomorphism $[\mathbf{f}]$ of \mathbf{CPHTop} such that (13) holds. Also recall that there is a functor $C : \mathbf{pro-Top} \rightarrow \mathbf{CPHTop}$. It takes mappings of systems into morphisms of \mathbf{CPHTop} . Now, it is clear that the following assertion holds.

Lemma 3. *Let $f : X \rightarrow Y$ be a mapping and let $\mathbf{p} : X \rightarrow \mathbf{X}$ and $\mathbf{q} : Y \rightarrow \mathbf{Y}$ be cofinite polyhedral resolutions. If $\mathbf{f} : \mathbf{X} \rightarrow \mathbf{Y}$ is an isomorphism of $\mathbf{pro-Top}$ and*

$$(14) \quad \mathbf{f}\mathbf{p} = \mathbf{q}\mathbf{f},$$

then f is a strong shape equivalence.

(iv) \Rightarrow (i). Let X admit a cofinal resolution $\mathbf{p} = (p_a) : X \rightarrow \mathbf{X}$, where each X_a , $a \in A$, is a compact polyhedron. We want to prove that $e : X \rightarrow \beta X$ is a strong shape equivalence by applying Lemma 3. Therefore, we first define a suitable polyhedral resolution for βX . It is of the form $\mathbf{q} : \beta X \rightarrow \mathbf{X}$, i.e. it uses the same inverse system \mathbf{X} . For each $a \in A$, we take for $q_a : \beta X \rightarrow X_a$, $a \in A$, the unique extension of $p_a : X \rightarrow X_a$ to βX , so that

$$(15) \quad p_a = q_a e, \quad a \in A.$$

The extension q_a exists because X_a is compact. Uniqueness of q_a follows from the density of $e(X)$ in βX . This is also the reason why $p_{aa'} p_{a'} = p_a$ implies $p_{aa'} q_{a'} = q_a$, for $a \leq a'$. Consequently, the mappings q_a , $a \in A$, form a mapping of systems $\mathbf{q} : \beta X \rightarrow \mathbf{X}$.

Lemma 4. *$\mathbf{q} : \beta X \rightarrow \mathbf{X}$ is a cofinite polyhedral resolution of βX .*

Once Lemma 4 is proved, we can apply Lemma 3 to the mapping $e : X \rightarrow \beta X$, to the polyhedral resolutions \mathbf{p} and \mathbf{q} and to the identity isomorphism $\mathbf{1} : \mathbf{X} \rightarrow \mathbf{X}$ of $\mathbf{pro-Top}$. In this case (14) becomes

$$(16) \quad \mathbf{1}\mathbf{p} = \mathbf{q}e.$$

It holds, because it reduces to (15).

PROOF OF LEMMA 4: We need only to verify the conditions (R1) and (R2).

Verification of (R1): Let P be a polyhedron, \mathcal{V} an open covering of P and $f : \beta X \rightarrow P$ a mapping. Choose a covering \mathcal{U} of P , which is a star-refinement of \mathcal{V} . By (R1) for \mathbf{p} , there exist an index $a \in A$ and a mapping $g : X_a \rightarrow P$, such that

$$(17) \quad (fe, gp_a) \prec \mathcal{U}.$$

We claim that

$$(18) \quad (f, gq_a) \prec \mathcal{V}.$$

Indeed, consider any point $y \in \beta X$. Choose members U_1, U_2 of \mathcal{U} so that

$$(19) \quad f(y) \in U_1, gq_a(y) \in U_2.$$

Choose an open neighborhood W of y in βX so small that

$$(20) \quad f(W) \subseteq U_1, gq_a(W) \subseteq U_2.$$

Since $e(X)$ is dense in βX , there exists a point $x \in X$ such that $e(x) \in W$. By (17), there exists a $U \in \mathcal{U}$ such that

$$(21) \quad fe(x), gp_a(x) \in U.$$

Since $y, e(x) \in W$, (20) implies

$$(22) \quad f(y), fe(x) \in U_1, gq_a(y), gp_a(x) = gq_ae(x) \in U_2.$$

Now, (21) and (22) imply

$$(23) \quad f(y), gq_a(y) \in \text{st}(U, \mathcal{U}).$$

Since \mathcal{U} is a star-refinement of \mathcal{V} , there exists a $V \in \mathcal{V}$ such that $\text{st}(U, \mathcal{U}) \subseteq V$ and therefore, $f(y), gq_a(y) \in V \in \mathcal{V}$, which proves (18).

Verification of (R2): Let P be a polyhedron and \mathcal{V} an open covering of P . We choose a covering \mathcal{V}' of P , by (R2) applied to \mathbf{p} . We claim that this covering also fulfills the requirements of the condition (R2) for \mathbf{q} . Indeed, let $a \in A$ and let $g, g' : X_a \rightarrow P$ be mappings such that

$$(24) \quad (gq_a, g'q_a) \prec \mathcal{V}'.$$

Then also

$$(25) \quad (gp_a, g'p_a) \prec \mathcal{V}'.$$

Therefore, there exists an $a' \geq a$ such that

$$(26) \quad (gp_{aa'}, g'p_{aa'}) \prec \mathcal{V},$$

which is the desired conclusion. \square

Remark 1. In our theorem one cannot replace the Stone-Čech compactification $e : X \rightarrow \beta X$ by an arbitrary compactification $i : X \rightarrow \tilde{X}$. E.g., if $X = (0, 1) \subseteq \mathbb{R}$ and $\tilde{X} = [0, 1] \subseteq \mathbb{R}$, then the inclusion $i : X \rightarrow \tilde{X}$ is a homotopy equivalence and therefore, also a (strong) shape equivalence. However, X is not pseudocompact. K. Morita showed [11, Corollary 5.3] that every pseudocompact space has the shape of a compact space. This also follows from our theorem. The converse does not hold because \mathbb{R} has compact shape, i.e. the shape of a point, but is not pseudocompact.

Remark 2. Two spaces X and Y can have the same strong shape, $\mathbf{SSh}(X) = \mathbf{SSh}(Y)$, but the strong shape of their Stone-Čech compactifications can be different, $\mathbf{SSh}(\beta X) \neq \mathbf{SSh}(\beta Y)$. E.g., if $X = \mathbb{R}$ and $Y = \{*\}$ is a point, then X and Y are of the same homotopy type and therefore have the same strong shape. On the other hand, already C.H. Dowker showed in [2] that the first Čech cohomology group with integer coefficients $\check{H}^1(\beta\mathbb{R}, \mathbb{Z})$ is an uncountable group (see also [5]). Since the Čech cohomology groups are (strong) shape invariants and $\check{H}^1(\{*\}, \mathbb{Z}) = 0$, it follows that $\mathbf{SSh}(\beta X) \neq \mathbf{SSh}(\beta Y)$. There exist spaces X, Y with $\mathbf{SSh}(X) \neq \mathbf{SSh}(Y)$, but $\mathbf{SSh}(\beta X) = \mathbf{SSh}(\beta Y)$. Such an example is given by $X = \mathbb{R}$ and $Y = \beta\mathbb{R}$.

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